CHARACTERISTIC DIRECTION FOR EQUATIONS OF MOTION OF NON-NEWTONIAN FLUIDS

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1. Introduction. According to the Reiner-Rivlin theory of non-Newtonian fluids,[1] the stress tensor t^i_j is given in terms of the rate of strain tensor d^i_j by relations of the form

$$t^i_j = -p\delta^i_j + \mathscr{F}_1 d^i_j + \mathscr{F}_2 d^i_k d^k_j \,, \tag{1}$$

where p is an arbitrary hydrostatic pressure, the $\mathscr{F}$'s are essentially arbitrary differentiable functions of

$$\mathrm{II} = -\frac{1}{2} d^i_j d^j_i \,, \qquad \mathrm{III} = \det d^i_j \,, \tag{2}$$

and d^i_j satisfies the incompressibility condition

$$d^i_i = 0 \,. \tag{3}$$

The tensors d^i_j and t^i_j are both symmetric.

It is known [2] that the characteristic directions of the corresponding equations of motion are the unit vectors ν_i satisfying

$$F(\nu_i) \equiv 2U^2 + 2UU^i_i + (U^i_i)^2 - U^i_j U^j_i = 0 \,, \tag{4}$$

where

$$U = \mathscr{F}_1 + \mathscr{F}_2 \mu^i \nu_i \,,$$

$$U^i_j = \mathscr{F}_2 (d^i_j - \nu^i \mu_j) + 2(\mu^i - \nu^i \mu_k \nu^k)\left(\mu^m d_{mj} \frac{\partial \mathscr{F}_1}{\partial \mathrm{III}} - \mu_j \frac{\partial \mathscr{F}_1}{\partial \mathrm{II}} \right)$$

$$+ 2(d^i_m \mu^m - \nu^i \mu_m \mu^m)\left(\mu^n d_{nj} \frac{\partial \mathscr{F}_2}{\partial \mathrm{III}} - \mu_j \frac{\partial \mathscr{F}_2}{\partial \mathrm{II}} \right) ,$$

$$\mu_i = d_{ij} \nu^j \,.$$

Since $F(\nu_i)$ is a continuous function of ν_i on the compact set $\nu_i \nu^i = 1$, a necessary and sufficient condition that no real characteristic directions exist is that $F(\nu_i)$ be of one sign for all unit vectors. Using this fact, we obtain simpler necessary conditions which are shown to be sufficient when $\mathscr{F}_2 \equiv 0$.

2. Necessary conditions. Let d_1, d_2 and d_3 denote the eigenvalues of d^i_j. From (3),

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[1] This theory was proposed independently by Reiner [4] for compressible fluids, by Rivlin [5] for incompressible materials. We treat the latter case.

(5) $$d_1+d_2+d_3=0 .$$

We restrict our attention to unit vectors ν_i which are perpendicular to an eigenvector of d^i_j and note that $F(\nu_i)$, being a continuous function of ν_i, must be of one sign for all unit vectors in order that no real characteristic directions exist. Given any unit vector ν_i perpendicular to an eigenvector e_i corresponding to d_3, we may introduce a rectangular Cartesian coordinate system such that, at a point, ν_i is parallel to the positive x^1-axis and e_i is parallel to the x^3-axis. Then

$$\nu_i=\delta_{i1}, \; d_{13}=d_{23}=d_{1i}d^i_3=d_{21}d^i_3=0 ,$$
$$2d_{12}=(d_1-d_2)\sin 2\phi , \; d_{33}=d_3 ,$$

where ϕ is the angle between ν_i and an eigenvector corresponding to d_1. Making these substitutions in $F(\nu_i)$, given by (4), we obtain, by a routine calculation,

(6) $$F(\nu_i)=2[\mathscr{F}_1-\mathscr{F}_2 d_2]\Big\{\mathscr{F}_1-\mathscr{F}_2 d_3-\frac{1}{2}(d_1-d_2)^2\sin^2 2\phi\Big[\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}} -d_3\frac{\partial\mathscr{F}_2}{\partial \mathrm{II}}+d_3\frac{\partial\mathscr{F}_1}{\partial \mathrm{III}}-d_3^2\frac{\partial\mathscr{F}_2}{\partial \mathrm{III}}\Big]\Big\} ,$$

which must be of one sign for all real angles ϕ. This is clearly true if and only if it is of the same sign for $\phi=0$ and $\phi=\pi/4$. That is, either

(7) $$[\mathscr{F}_1-\mathscr{F}_2 d_2][\mathscr{F}_1-\mathscr{F}_2 d_3]>0$$

and

(8) $$[\mathscr{F}_1-\mathscr{F}_2 d_2]\Big\{\mathscr{F}_1-\mathscr{F}_2 d_3-\frac{1}{2}(d_1-d_2)^2\Big[\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}} -d_3\frac{\partial\mathscr{F}_2}{\partial \mathrm{II}}+d_3\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}}-d_3^2\frac{\partial\mathscr{F}_2}{\partial \mathrm{II}}\Big]\Big\}>0 ,$$

or (7) and (8) hold simultaneously with the inequalities reversed. By similarly analyzing the cases where ν_i is perpendicular to eigenvectors of d^i_j corresponding to d_1 and d_2, we conclude that either

(9) $$[\mathscr{F}_1-\mathscr{F}_2 d_i][\mathscr{F}_1-\mathscr{F}_2 d_j]>0 \qquad (i\neq j),$$

and

(10) $$[\mathscr{F}_1-\mathscr{F}_2 d_j]\Big\{\mathscr{F}_1-\mathscr{F}_2 d_k-\frac{1}{2}(d_i-d_j)^2\Big[\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}} -d_k\frac{\partial\mathscr{F}_2}{\partial \mathrm{II}}+d_k\frac{\partial\mathscr{F}_1}{\partial \mathrm{III}}-d_k^2\frac{\partial\mathscr{F}_2}{\partial \mathrm{III}}\Big]\Big\}>0 \qquad (i, j, k\neq),$$

or

$$[\mathscr{F}_1-\mathscr{F}_2 d_i][\mathscr{F}_1-\mathscr{F}_2 d_j]<0 \qquad (i\neq j), \tag{11}$$

and (10) holds with the inequality reversed. Now (11) cannot hold for all i and j, so this possibility is ruled out. We thus have

THEOREM 1. *A necessary and sufficient condition that no real characteristic directions exist is that* $F(\nu_i)>0$; *in order that there exist no real characteristic directions perpendicular to an eigenvector of* d^i_j, *it is necessary and sufficient that the inequalities* (9) *and* (10) *hold.*

For (9) and (10) to hold, it is necessary and sufficient that either

$$\mathscr{F}_1-\mathscr{F}_2 d_i>0 \tag{12}$$

and

$$\mathscr{F}_1-\mathscr{F}_2 d_k-\frac{1}{2}(d_i-d_j)^2\left[\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}}-d_k\frac{\partial\mathscr{F}_2}{\partial \mathrm{II}}+d_k\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}}-d_k^2\frac{\partial\mathscr{F}_2}{\partial \mathrm{III}}\right]>0 \qquad (i,j,k\neq), \tag{13}$$

or

$$\mathscr{F}_1-\mathscr{F}_2 d_i<0 \tag{14}$$

and

$$\mathscr{F}_1-\mathscr{F}_2 d_k-\frac{1}{2}(d_i-d_j)^2\left[\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}}-d_k\frac{\partial\mathscr{F}_2}{\partial \mathrm{II}}+d_k\frac{\partial\mathscr{F}_1}{\partial \mathrm{III}}-d_k^2\frac{\partial\mathscr{F}_2}{\partial \mathrm{III}}\right]<0 \qquad (i,j,k\neq). \tag{15}$$

3. Equivalent conditions. Let t_i denote the eigenvalues of the stress tensor corresponding to the eigenvalue d_i of d_{mn} so that from (1),

$$t_i=-p+\mathscr{F}_1 d_i+\mathscr{F}_2 d_i^2 .$$

Using (5),

$$\begin{aligned} t_i-t_j&=[\mathscr{F}_1+\mathscr{F}_2(d_i+d_j)](d_i-d_j)\\ &=[\mathscr{F}_1-\mathscr{F}_2 d_k](d_i-d_j) \qquad (i,j,k\neq). \end{aligned} \tag{16}$$

From (2) and (5),

$$\begin{aligned} \mathrm{II}&=-\frac{1}{2}(d_1^2+d_2^2+d_3^2)=-\frac{1}{4}(d_i-d_j)^2-\frac{3}{4}d_k^2,\\ \mathrm{III}&=d_1d_2d_3=\frac{1}{4}d_k[d_k^2-(d_i-d_j)^2] \qquad (i,j,k\neq). \end{aligned} \tag{17}$$

Using (16) and (17) to express t_i-t_j as a function of d_i-d_j and $d_k (i,j,k\neq)$, we calculate

$$(18)\qquad \left.\frac{\partial(t_i-t_j)}{\partial(d_i-d_j)}\right|_{d_k=\text{const.}}$$

$$=\mathscr{F}_1-\mathscr{F}_2 d_k-\frac{1}{2}(d_i-d_j)^2\left[\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}}-d_k\frac{\partial\mathscr{F}_2}{\partial \mathrm{II}}+d_k\frac{\partial\mathscr{F}_1}{\partial \mathrm{III}}-d_k^2\frac{\partial\mathscr{F}_2}{\partial \mathrm{III}}\right].$$

From (12), (13), (14), (15), (16), (18) and Theorem 1, we have

THEOREM 2. *When the eigenvalues of d^i_j are all unequal, a necessary and sufficient condition that there exist no real characteristic direction perpendicular to an eigenvector of d^i_j is that either*

$$(t_i-t_j)/(d_i-d_j)>0 \quad\text{and}\quad \partial(t_i-t_j)/\partial(d_i-d_j)|_{d_k=\text{const.}}>0\ ,$$

or

$$(t_i-t_j)/(d_i-d_j)<0 \quad\text{and}\quad \partial(t_i-t_j)/\partial(d_i-d_j)|_{d_k=\text{const.}}<0 \qquad (i,j,k\neq).$$

When (12) holds, the stress power Φ, given by

$$3\Phi=3t^i_jd^j_i=(t_1-t_2)(d_1-d_2)+(t_2-t_3)(d_2-d_3)+(t_3-t_1)(d_3-d_1)$$

is negative, a possibility which many writers exclude on thermodynamic grounds.

4. The case $\mathscr{F}_2\equiv 0$. When $\mathscr{F}_2\equiv 0$, $\mathscr{F}_1\neq 0$, the characteristic equation (4) has been shown [2] to reduce to

$$(19)\qquad G(\nu_i)\equiv\mathscr{F}_1+A^iB_i=0\ ,$$

where

$$A^i=2(\mu^i-\nu^i\mu_k\nu^k)\ ,$$

$$B_i=\mu^m d_{mi}\frac{\partial\mathscr{F}_1}{\partial \mathrm{III}}-\mu_i\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}}\ .$$

In fact, $F(\nu_i)=2\mathscr{F}_1G(\nu_i)$. When $\mathscr{F}_2=0$, $\mathscr{F}_1=0$, every direction is characteristic, a case which we exclude. Using the Hamilton-Cayley theorem,

$$d^i_jd^j_kd^k_m=\mathrm{III}\delta^i_m-\mathrm{II}d^i_m\ ,$$

we can reduce (19) to the form

$$(20)\qquad G(\alpha,\beta)\equiv\mathscr{F}_1+2(\mathrm{III}-\mathrm{II}\alpha-\beta\alpha)\frac{\partial\mathscr{F}_1}{\partial \mathrm{III}}+2(\alpha^2-\beta)\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}}=0\ ,$$

where

$$(21)\qquad \alpha=\mu_i\nu^i=d_{ij}\nu^i\nu^j\ ,\qquad \beta=\mu^i\mu_i=d^i_kd_{im}\nu^k\nu^m\ .$$

Now (21) is a mapping of the unit sphere $\nu_i \nu^i = 1$ onto a region R in the $\alpha - \beta$ plane. The conditions

$$\frac{\partial G}{\partial \alpha} = -2(\mathrm{II} + \beta)\frac{\partial \mathscr{F}_1}{\partial \mathrm{III}} + 4\alpha \frac{\partial \mathscr{F}_1}{\partial \mathrm{II}} = 0 \ ,$$

$$\frac{\partial G}{\partial \beta} = -2\alpha \frac{\partial \mathscr{F}_1}{\partial \mathrm{III}} - 2\frac{\partial \mathscr{F}_1}{\partial \mathrm{II}} = 0 \ ,$$

$$\pm d^2 G = \pm 4\left[\frac{\partial \mathscr{F}_1}{\partial \mathrm{II}} d\alpha^2 - \frac{\partial \mathscr{F}_1}{\partial \mathrm{III}} d\alpha d\beta\right] \geqq 0 \text{ for all } d\alpha, d\beta \ ,$$

must be satisfied at any interior point of R at which G is a maximum or minimum. These conditions cannot be satisfied unless $\partial \mathscr{F}_1 / \partial \mathrm{II} = \partial \mathscr{F}_1 / \partial \mathrm{III} = 0$, in which case $G(\nu_i)$ is independent of ν_i, and $\mathscr{F}_1 \neq 0$ is then necessary and sufficient that there exist no real characteristics. From the implicit function theorem, values of ν_i corresponding to boundary points of R are such that the equations

$$d\alpha = 2d_{ij}\nu^i d\nu^j \ , \qquad d\beta = 2d^i_k d_{im}\nu^k d\nu^m \ , \qquad 0 = \nu_i d\nu^i$$

do not admit a unique solution for $d\nu^i$ in terms of $d\alpha$ and $d\beta$. We thus have

THEOREM 3. *Maximum and minimum values of $G(\nu_i)$, hence of $F(\nu_i)$, hence of $F(\nu_i)$, occur only at values of ν_i such that the vectors ν_i, $d_{ij}\nu^j$ and $d^k_i d_{km}\nu^m$ are linearly dependent or, equivalently, at values such that the determinant D of these three vectors vanishes.*

Whatever be the unit vector ν_i, we can always choose rectangular Cartesian coordinates such that, at a point, $\nu_i = \delta_{i1}$, $d_{23} = 0$. The condition $D = 0$ then reduces to

$$0 = \begin{vmatrix} 1 & 0 & 0 \\ d_{11} & d_{21} & d_{31} \\ d_{11}^2 + d_{12}^2 + d_{13}^2 & d_{21}(d_{11} + d_{22}) & d_{31}(d_{11} + d_{33}) \end{vmatrix}$$
$$= d_{21} d_{31}(d_{33} - d_{22}) \ .$$

If $d_{21} = 0 (d_{31} = 0)$, $\delta_{i2}(\delta_{i3})$ is an eigenvector of d_{ij}. If $d_{21}d_{31} \neq 0$, $d_{33} = d_{22}$, the vector with components $(0, d_{31}, -d_{21})$ is an eigenvector of d_{ij}, whence follows

THEOREM 4. *The vectors ν_i, $d_{ij}\nu^j$, $d^k_i d_{km}\nu^m$ can be linearly dependent only when ν_i is perpendicular to an eigenvector of d^i_j.*

Theorems 3 and 4 imply that, when $\mathscr{F}_2 \equiv 0$, we will have $F(\nu_i) > 0$ for all unit vectors ν_i if and only if $F(\nu_i) > 0$ for each unit vector ν_i which is perpendicular to an eigenvector of d^i_j. From Theorem 1, we then deduce

THEOREM 5. *When* $\mathscr{F}_2 \equiv 0$, *a necessary and sufficient condition that there exist no real characteristic directions is that the inequalities* (9) *and* (10) *hold.*

REFERENCES

1. M. Baker and J. L. Ericksen, *Inequalities restricting the form of the stress-deformation relations for isotropic elastic solids and Reiner-Rivlin fluids*, J. Wash. Acad. Sci. **44** (1954), 33–35.

2. J. L. Ericksen, *Characteristic surfaces of the equations of motion for non-Newtonian fluids*, ZAMP **4** (1953), 260–267.

3. ———, *A consequence of inequalities proposed by Baker and Ericksen*, J. Wash. Acad. Sci. **45** (1955), 268.

4. M. Reiner, *A mathematical theory of dilatancy*, Amer. J. Math. **67** (1945), 350–362.

5. R. S. Rivlin, *The hydrodynamics of non-Newtonian fluids* I., Proc. Roy. Soc. London (A) **193** (1948), 260–281.

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