ON THE NUMBER OF LATTICE POINTS IN $x^t + y^t = n^{t/2}$

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Introduction. Suppose that t is independent of n, $n > 1$; $t = (2M)/(2N+1)$; $M = 1, 2, 3, \cdots$; $N = 0, 1, 2, \cdots$; $M \geqq N+1$, so that $t > 1$. Let $L_t(n^{t/2})$ be the number of lattice points, (x, y), satisfying $x^t + y^t \leqq n^{t/2}$. Our main objective is the proof of the relation

$$(1.1)\quad S(n) = t/2\ n^{1-t/2}\int_0^n L_t(w^{t/2})w^{t/2-1}dw$$
$$= c_1 n^2 - c_2/\pi n^{(2t-1)/(2t)}\sum_{\alpha=1}^{\infty}\alpha^{-2-1/t}\cos(2\pi\sqrt{n}\,\alpha - \pi/(2t))$$
$$-\frac{2t}{\pi^2\sqrt{t-1}}n^{3/4}\sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty}\frac{\cos(2\pi H\sqrt{n} - \pi/4)}{(\alpha\beta)^{(t-2)/(2t-2)}H^{(3t-1)/(2t-2)}} + O(\sqrt{n})$$

with $t > 1$, $c_1 = \dfrac{2\Gamma^2(1/t)}{(t+2)\Gamma(2/t)}$, $c_2 = \dfrac{2^{(2t-1)/t}t^{1/t}\Gamma(1/t)}{\pi^{(t+1)/t}}$,

$H = (\alpha^{t/(t-1)} + \beta^{t/(t-1)})^{(t-1)/t}$. The case $t = 2$ is known in connection with the classical problem of the lattice points in a circle [4, pp. 221, 235].

By choosing t as specified above the analysis is less bulky than it would be if we considered the slightly more general problem of $L_T(n^{T/2})$ corresponding to the curve $|x|^T + |y|^T = n^{T/2}$ with real $T > 0$. Expressions and estimates for $L_T(n^{T/2})$ have been obtained by Bachmann [1, pp. 447–450], Cauer [2], and van der Corput [3]. In particular van der Corput [3] found that

$$(1.2)\quad L_T(n^{T/2}) = c_1'n - 8T^{(1-T)/T}n^{(T-1)/(2T)}\int_0^{\infty}g_1(\sqrt{n} - x)x^{(1-T)/T}dx + O(n^{1/3}),\ T > 3;$$
$$= c_1'n - 8\sum_{j=1}^{\infty}(-1)^{j+1}\binom{1/T}{j}\zeta(-jT)n^{(1-jT)/2} + O(n^{1/3}),\ 0 < T \leqq 3,\ T \neq 1;$$

where

$$c_1' = \frac{2\Gamma^2(1/T)}{T\Gamma(2/T)},$$

$g_1(x) = x - [x] - 1/2$, $[x]$ is the integral part of x, $\zeta(s)$ is the Riemann zeta function and $\binom{a}{b}$ is the binomial coefficient. From (1.2) it follows that

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$$(1.3)\quad L_T(n^{T/2}) = c_1'n + O(n^{(T-1)/(2T)}),\ L_T(n^{T/2}) = c_1'n + \Omega(n^{(T-1)(2T)}\ ,\quad T > 3\ .$$

These results in (1.3) and analogous results can be obtained from (1.1) also. Our methods fail to establish the analogue of (1.1) for $0 < t < 1$.

2. First auxiliary result. We first obtain the result

$$(2.1)\quad S(n) = n^2 \sum_{\alpha=-\infty}^{\infty} \sum_{\beta=-\infty}^{\infty} \iint_{x^t+y^t\leq 1} (1 - x^t - y^t)\cos 2\pi\sqrt{n}\,(\alpha x + \beta y)dxdy\ ,\qquad t > 1\ .$$

In § 4 we prove that the double series is absolutely convergent.

We have [4, p. 205]

$$(2.2)\quad \int_0^W L_t(w)dw = \int_0^W \sum_{j^t+k^t\leq w}\sum 1\, dw = \sum_{j^t+k^t\leq W}\sum \int_{j^t+k^t}^W dw$$
$$= \sum_{j^t+k^t\leq W}\sum (W - j^t - k^t) = \sum_{-W^{1/t}\leq j\leq W^{1/t}}\ \sum_{-(W-j^t)^{1/t}\leq k\leq (W-j^t)^{1/t}} (W - j^t - k^t)\ .$$

To this we apply the Poisson summation formula [4, p. 204] to obtain

$$(2.3)\quad \int_0^W L_t(w)dw = \sum_{\alpha=-\infty}^{\infty} \int_{-W^{1/t}}^{W^{1/t}} \cos 2\pi\alpha x \sum_{-(W-x^t)^{1/t}\leq k\leq (W-x^t)^{1/t}} (W - x^t - k^t)dx$$
$$= \sum_{\alpha=-\infty}^{\infty} \int_{-W^{1/t}}^{W^{1/t}} \cos 2\pi\alpha x \sum_{\beta=-\infty}^{\infty} \int_{-(W-x^t)^{1/t}}^{(W-x^t)^{1/t}} \cos 2\pi\beta y\cdot (W-x^t-y^t)dydx.$$

Integrating by parts and applying the second mean value theorem for integrals, we have, for the inner integral,

$$\frac{t}{\pi\beta}\int_0^{(W-x^t)^{1/t}} \sin 2\pi\beta y \cdot y^{t-1}dy = \frac{t(W-x^t)^{(t-1)/t}}{\pi\beta}\int_\xi^{(W-x^t)^{1/t}} \sin 2\pi\beta y dy\ ,$$

where $0 \leqq \xi < (W - x^t)^{1/t}$, so that the sum over β is uniformly convergent in x. Hence we can interchange the order of operations in $\int dx \sum_\beta$ in (2.3) to obtain

$$(2.4)\quad \int_0^W L_t(w)dw = \sum_{\alpha=-\infty}^{\infty} \sum_{\beta=-\infty}^{\infty} \iint_{x^t+y^t\leq W} \cos 2\pi\alpha x \cos 2\pi\beta y \cdot (W - x^t - y^t)dxdy\ .$$

By symmetry we can replace $\cos 2\pi\alpha x \cos 2\pi\beta y$ by $\cos 2\pi(\alpha x + \beta y)$. If also we set $w = z^{t/2}$, $x = W^{1/t}r$, $y = W^{1/t}s$, $W = n^{t/2}$, we reduce (2.4) to

$$(2.5)\quad t/2\int_0^n L_t(z^{t/2})z^{t/2-1}dz = n^{t/2+1} \sum_{\alpha=-\infty}^{\infty} \sum_{\beta=-\infty}^{\infty} \iint_{r^t+s^t\leq 1} (1-r^t-s^t) \cos 2\pi\sqrt{n}(\alpha r \beta s)drds$$

and then (2.1) follows upon multiplication of each side by $n^{(2-t)/t}$.

3. Second auxiliary result. For $t > 1$, we shall obtain from (2.1) the identity

$$(3.1)\quad S(n) = T_1 + T_2 + T_3 + T_4 + T_5$$

where

$$T_1 = c_1 n^2,\ c_1 = \frac{2\Gamma^2(1/t)}{(t+2)\Gamma(2/t)};$$

$$T_2 = c_2 n^{5/4-1/(2t)} \sum_{\alpha=1}^{\infty} \alpha^{-3/2-1/t} J_{3/2+1/t}(2\pi\sqrt{n}\,\alpha),$$

where

$$c_2 = \frac{2^{(2t-1)/t} t^{1/t} \Gamma(1/t)}{\pi^{(t+1)/t}},$$

and $J_r(x)$ is the ordinary Bessel function of order r;

$$T_3 = c_3 n^2 \sum_{\alpha=1}^{\infty} \int_0^1 f(x, t) \cos 2\pi\sqrt{n}\,\alpha x dx,\quad c_3 = \frac{16t}{t+1},$$

and $f(x, t) = (1 - x^t)^{(t+1)/t} - (t/2)^{(t+1)/t}(1 - x^2)^{(t+1)/t}$;

$$T_4 = -\frac{2t}{\pi\sqrt{t-1}} n \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty} \frac{J_0(2\pi H\sqrt{n})}{(\alpha\beta)^{(t-2)/(2t-2)} H^{t/(t-1)}},\ H = (\alpha^{t/(t-1)} + \beta^{t/(t-1)})^{(t-1)/t};$$

$$T_5 = \frac{2t}{\pi^2} n \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty} \frac{1}{\alpha\beta} \int_{-\infty}^{\infty} G(u, \alpha, \beta) \cos 2\pi H\sqrt{n}\, v(u, \alpha, \beta) \cdot v'(u, \alpha, \beta) du,$$

where

$$v(u, \alpha, \beta) = H^{-1} A_0^{-1/t}(u),\quad A_i(u) = (-1)^i \alpha^{-t}(P\alpha - u)^{t-i} + \beta^{-t}(Q\beta + u)^{t-i},$$

$$P = \frac{\alpha^{1/(t-1)}}{\alpha^{t/(t-1)} + \beta^{t/(t-1)}},\quad Q = \frac{\beta^{1/(t-1)}}{\alpha^{t/(t-1)} + \beta^{t/(t-1)}},$$

$$G(u, \alpha, \beta) = \frac{A_{-1}(u)A_1(u) - A_0^2(u)}{v'(u, \alpha, \beta)A_0^2(u)} - a_{-1}(\alpha, \beta)\operatorname{sgn} u[1 - v^2(u, \alpha, \beta)]^{-1/2},$$

$$a_{-1}(\alpha, \beta) = \frac{(\alpha\beta)^{t/(2t-2)}}{\sqrt{t-1}(\alpha^{t/(t-1)} + \beta^{t/(t-1)})}.$$

In the proof of (3.1) we make use of the following result on Bessel functions [5, p. 366],

$$(3.2)\quad \int_0^1 (1 - x^2)^{m-1/2} \cos Kx dx = \sqrt{\pi}\, 2^{m-1} K^{-m} \Gamma(m + 1/2) J_m(K)\quad m > -1/2.$$

First, it is convenient to break up the double sum in (2.1) as follows,

$$S(n) = \sum_{\alpha=0}\sum_{\beta=0} + \sum_{\substack{\alpha=-\infty\\ \alpha\neq 0}}^{\infty}\sum_{\beta=0} + \sum_{\alpha=0}\sum_{\substack{\beta=-\infty\\ \beta\neq 0}}^{\infty} + \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty} + \sum_{\alpha=-\infty}^{-1}\sum_{\beta=-\infty}^{-1} + \sum_{\alpha=-\infty}^{-1}\sum_{\beta=1}^{\infty} + \sum_{\alpha=1}^{\infty}\sum_{\beta=-\infty}^{-1} . \tag{3.3}$$

By symmetry this can be written as

$$\begin{aligned} S(n) &= n^2 \iint_{x^t+y^t\leq 1} (1 - x^t - y^t)dxdy \\ &+ 4n^2 \sum_{\alpha=1}^{\infty} \iint_{x^t+y^t\leq 1} (1 - x^t - y^t)\cos 2\pi\sqrt{n}\,\alpha x dxdy \\ &+ 4n^2 \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty} \iint_{x^t+y^t\leq 1} (1 - x^t - y^t)\cos 2\pi\sqrt{n}(\alpha x + \beta y)dxdy \\ &= S_1 + S_2 + S_3 . \end{aligned} \tag{3.4}$$

S_1 can be evaluated in terms of gamma functions to obtain

$$S_1 = \frac{2\Gamma^2(1/t)}{(t+2)\Gamma(2/t)} n^2 = c_1 n^2 . \tag{3.5}$$

Let I_2 denote the integral in S_2. Then

$$\begin{aligned} I_2 &= 4\int_0^1 \cos 2\pi\sqrt{n}\,\alpha x dx \int_0^{(1-x^t)^{1/t}} (1 - x^t - y^t)dy \\ &= \frac{4t}{t+1}\int_0^1 (1 - x^t)^{(t+1)/t} \cos 2\pi\sqrt{n}\,\alpha x dx \\ &= \frac{4t}{t+1}\left(\frac{t}{2}\right)^{(t+1)/t} \int_0^1 (1 - x^2)^{(t+1)/t} \cos 2\pi\sqrt{n}\,\alpha x dx \\ &+ \frac{c_3}{4}\int_0^1 f(x, t)\cos 2\pi\sqrt{n}\,\alpha x dx \end{aligned} \tag{3.6}$$

by the definition of $f(x, t)$ in (3.1). Applying (3.2) to (3.6) we have

$$S_2 = 4n^2 \sum_{\alpha=1}^{\infty} I_2 = T_2 + T_3 . \tag{3.7}$$

Let I_3 denote the integral in S_3. Then by symmetry

$$I_3 = 2 \iint_{\substack{x^t+y^t\leq 1\\ \alpha x+\beta y\geq 0}} (1 - x^t - y^t)\cos 2\pi\sqrt{n}(\alpha x - \beta y)dxdy . \tag{3.8}$$

The transformation

$$x = Hv(P - u/\alpha) ,\ y = Hv(Q + u/\beta) \tag{3.9}$$

transforms $x^t + y^t = 1$ into

$$v = H^{-1}A_0^{-1/t}(u) \tag{3.10}$$

where H, P, Q, and $A_i(u)$ are defined in (3.1). The transformation (3.9) is one to one for $\alpha x + \beta y \geqq 0$ and the absolute value of the Jacobian is

$$J\left(\frac{x, y}{v, u}\right) = \frac{H^2 v}{\alpha\beta} . \tag{3.11}$$

The graph of (3.10) resembles that of $v = 1/(1 + u^2)$ except that the curve is not symmetric to the v axis unless $t = 2$. The curve has a relative maximum at (0, 1).

Applying (3.9) to (3.8) we transform $x^t + y^t \leqq 1$ and $\alpha x + \beta y \geqq 0$ into $v \leqq H^{-1}A_0^{-1/t}(u)$ and $v \geqq 0$ respectively, so that (3.8) becomes

$$I_3 = \frac{2H^2}{\alpha\beta}\int_{-\infty}^{\infty} du \int_0^{v(u)} [1 - H^t v^t A_0(u)] v \cos 2\pi H\sqrt{n}\, v dv . \tag{3.12}$$

Upon integration by parts with respect to v, the integrated terms vanish and we obtain

$$\begin{aligned} I_3 &= -\frac{H}{\pi\sqrt{n}\,\alpha\beta}\int_{-\infty}^{\infty} du \int_0^{v(u)} [1 - (t+1)H^t v^t A_0(u)] \sin 2\pi H\sqrt{n}\, v dv \\ &= -\frac{H}{\pi\sqrt{n}\,\alpha\beta}\int_0^1 \sin 2\pi H\sqrt{n}\, v dv \int_{u_-(v)}^{u_+(v)} [1 - (t+1)H^t v^t A_0(u)] du \end{aligned} \tag{3.13}$$

where $u_+(v)$ and $u_-(v)$ refer to the first and second quadrant branches of (3.10) respectively. Since

$$\begin{aligned} A_i(u) &= (-1)^i \alpha^{-t}(P\alpha - u)^{t-i} + \beta^{-t}(Q\beta + u)^{t-i}, \\ &\qquad \frac{d}{du} A_i(u) = (t - i) A_{i+1}(u) , \end{aligned} \tag{3.14}$$

we can write (3.13) as

$$\begin{aligned} I_3 = &-\frac{H}{\pi\sqrt{n}\,\alpha\beta}\int_0^1 [u_+(v) - H^t v^t A_{-1}(u_+(v))] \sin 2\pi H\sqrt{n}\, v dv \\ &-\frac{H}{\pi\sqrt{n}\,\alpha\beta}\int_0^1 [-u_-(v) + H^t v^t A_{-1}(u_-(v))] \sin 2\pi H\sqrt{n}\, v dv . \end{aligned} \tag{3.15}$$

By the change of variable (3.10) this can be written as

$$I_3 = \frac{H}{\pi\sqrt{n}\,\alpha\beta}\int_{-\infty}^{\infty}\left[u - \frac{A_{-1}(u)}{A_0(u)}\right] \sin 2\pi H\sqrt{n}\, v(u) \cdot v'(u) du . \tag{3.16}$$

From (3.14) we obtain

$$u - \frac{A_{-1}(u)}{A_0(u)} = \frac{P\alpha^{1-t} - Q\beta^{1-t}}{\alpha^{-t} + \beta^{-t}} + O\left(\frac{1}{u}\right) \tag{3.17}$$

for large u, so that upon integrating by parts again we obtain

$$I_3 = \frac{t}{2\pi^2 n\alpha\beta} \int_{-\infty}^{\infty} F(u) \cos 2\pi H\sqrt{n}\, v(u) \cdot v'(u)du \tag{3.18}$$

where

$$F(u) = F(u, \alpha, \beta) = \frac{A_{-1}(u)A_1(u) - A_0^2(u)}{v'(u)A_0^2(u)} . \tag{3.19}$$

The function $a_{-1} \operatorname{sgn} u[1 - v^2(u)]^{-1/2}$ is an asymptotic equivalent of $F(u)$ in the neighborhood of $(0, 1)$, even though $v(0) = 1$ and $v'(0) = 0$, if $a_{-1} = a_{-1}(\alpha, \beta)$ is determined from

$$\begin{aligned} a_{-1} &= \lim_{u \to 0+} F(u)\sqrt{1 - v^2(u)} = \lim_{u \to 0+} \frac{\sqrt{1 - v^2}}{-v'} \\ &= \lim_{u \to 0+} \frac{vv'(1 - v^2)^{-1/2}}{v''} = \frac{1}{v''(0)} \lim_{u \to 0+} \frac{v'}{\sqrt{1 - v^2}} \\ &= \frac{-1}{v''(0)a_{-1}} = \frac{1}{\sqrt{|v''(0)|}} . \end{aligned} \tag{3.20}$$

From (3.10) and (3.14) we obtain

$$v''(u) = - H^{-1}A_0^{-(1+2t)/t}(u)[-(t + 1)A_1^2(u) + (t - 1)A_0(u)A_2(u)] \tag{3.21}$$

from which a_{-1}, as given in (3.1), can be determined.

We now write (3.18) as

$$\begin{aligned} I_3 &= \frac{ta_{-1}}{2\pi^2 n\alpha\beta} \int_{-\infty}^{\infty} \operatorname{sgn} u[1 - v^2(u)]^{-1/2} \cos 2\pi H\sqrt{n}\, v(u) \cdot v'(u)du \\ &+ \frac{t}{2\pi^2 n\alpha\beta} \int_{-\infty}^{\infty} [F(u) - a_{-1} \operatorname{sgn} u[1 - v^2(u)]^{-1/2}] \cos 2\pi H\sqrt{n} v(u) \cdot v'(u)du \\ &= -\frac{ta_{-1}}{\pi^2 n\alpha\beta} \int_0^1 (1 - v^2)^{-1/2} \cos 2\pi H\sqrt{n}\, vdv \\ &+ \frac{t}{2\pi^2 n\alpha\beta} \int_{-\infty}^{\infty} G(u) \cos 2\pi H\sqrt{n}\, v(u) \cdot v'(u)du \end{aligned} \tag{3.22}$$

where $G(u) = G(u, \alpha, \beta)$ is defined in (3.1). Applying (3.2) to (3.22) we obtain

$$S_3 = 4n^2 \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} I_3 = T_4 + T_5 . \tag{3.23}$$

Collecting the results of (3.4), (3.5), (3.7), and (3.23), we have (3.1).

4. Convergence investigations. We next prove that the double series in (2.1) is absolutely convergent. We write (3.18) as

$$I_3 = \frac{t}{2\pi^2 n\alpha\beta}\Big(\int_{-\infty}^{0} + \int_{0}^{\sigma} + \int_{\sigma}^{P\alpha} + \int_{P\alpha}^{\infty}\Big) \tag{4.1}$$
$$= \frac{t}{2\pi^2 n\alpha\beta}(I_4 + I_5 + I_6 + I_7)$$

where $0 < \sigma < P\alpha$.

First we consider

$$I_7 = \int_{P\alpha}^{\infty} F(u) \cos 2\pi H\sqrt{n}v(u) \cdot v'(u)du \,. \tag{4.2}$$

By (3.14) and (3.19) we have

$$F(u) = \frac{H(P\alpha-u)^{t-1}(Q\beta+u)^{t-1}}{(\alpha\beta)^t A_3^{1-1/t}(u)A_1(u)} = \frac{-H[\alpha^{-t}(u-P\alpha)^t+\beta^{-t}(u+Q\beta)^t]^{(1-t)/t}}{\alpha^t(u - P\alpha)^{1-t} + \beta^t(u + Q\beta)^{1-t}} \tag{4.3}$$

From (4.3) we find that

$$\frac{dF(u)}{du} = \frac{(1-t)H}{(\alpha\beta)^t}\Big(\frac{(u - P\alpha)^t}{\alpha^t} + \frac{(u + Q\beta)^t}{\beta^t}\Big)^{(1-2t)/t} \tag{4.4}$$
$$\times \Big(\frac{-\beta^{2t}(u - P\alpha)^{2t-1} + \alpha^{2t}(u + Q\beta)^{2t-1}}{(u - P\alpha)^t(u + Q\beta)^t[\alpha^t(u - P\alpha)^{1-t} + \beta^t(u + Q\beta)^{1-t}]^2}\Big) .$$

From (4.3) and (4.4) we derive certain information about the graph of F(u), namely,

$$\begin{aligned} &F(u) > 0,\ F'(u) < 0,\ 0 < u < P\alpha\,;\\ &F'(P\alpha) = \infty,\ 1 < t < 2\,;\ F'(P\alpha) = 0,\ 2 < t\,;\\ &F(u) < 0,\ P\alpha < u < \infty\,;\\ &F'(u) = 0,\ u = u_1,\ P\alpha < u_1 < \infty,\ \beta > \alpha\,;\\ &F'(u) < 0,\ P\alpha < u < \infty,\ \beta \leqq \alpha\,. \end{aligned} \tag{4.5}$$

The point (u_1, v_1) is a relative minimum and from (4.3) and (4.4) we find that

$$u_1 = \frac{Q\beta\alpha^{(2t)/(2t-1)} + P\alpha\beta^{(2t)/(2t-1)}}{\beta^{(2t)/(2t-1)} - \alpha^{(2t)/(2t-1)}}\,, \tag{4.6}$$
$$v_1 = F(u_1) = -H(\alpha^{t/(2t-1)} + \beta^{t/(2t-1)})^{-(2t-1)/t} .$$

Thus by (4.5) and the second mean value theorem for integrals we have, for $\beta > \alpha$, and $P\alpha \leqq \xi_1 < u_1 < \xi_2 \leqq \infty$,

$$I_7 = \int_{P\alpha}^{u_1} + \int_{u_1}^{\infty} = F(u_1)\int_{\xi_1}^{u_1} + F(u_1)\int_{u_1}^{\xi_2} \tag{4.7}$$

$$= F(u_1)\int_{\xi_1}^{\xi_2} \cos 2\pi H\sqrt{n}\, v(u) \cdot v'(u)du$$
$$= O\{F(u_1)H^{-1}n^{-1/2}\} = O\{(\alpha^{t/(2t-1)} + \beta^{t/(2t-1)})^{-(2t-1)/t}n^{-1/2}\}$$
$$= O\{(n\alpha\beta)^{-1/2}\}$$

by the inequality $x^2 + y^2 \geqq 2xy, x > 0, y > 0$. Similarly, for $\beta \leqq \alpha$, and $P\alpha \leqq \xi_3 < \infty$, we have

$$(4.8)\qquad I_7 = \int_{P\alpha}^{\infty} = F(\infty)\int_{\xi_3}^{\infty} \cos 2\pi H\sqrt{n}\, v(u) \cdot v'(u)du$$
$$= O\{F(\infty)H^{-1}n^{-1/2}\} = \{(\alpha^{-t} + \beta^{-t})^{(1-t)/t}(\alpha^t + \beta^t)^{-1}n^{-1/2}\}$$
$$= O\{(\alpha\beta)^{t-1}(\alpha^t + \beta^t)^{-(2t-1)/t}n^{-1/2}\} = O\{(n\alpha\beta)^{-1/2}\} .$$

We next consider I_5 in (4.1). By (4.3) we can write

$$(4.9)\qquad I_5 = \int_0^{\sigma} F_1(u) \cos 2\pi H\sqrt{n}\, v(u)du$$

where

$$(4.10)\qquad -F_1(u) = \frac{(pq)^{t-1}}{(\alpha\beta)^t A_0^2(u)} , \ p = P\alpha - u, q = Q\beta + u ,$$
$$= \frac{A_0^{-2/t}(u)}{\alpha\beta} \cdot \frac{[(pq)/(\alpha\beta)]^{t-1}}{[(p/\alpha)^t + (q/\beta)^t]^{2(t-1)/t}} < \frac{A_0^{-2/t}(u)}{\alpha\beta} \cdot \frac{1}{2^{2(t-1)/t}}$$
$$< \frac{A_0^{-2/t}(0)}{\alpha\beta 2^{2(t-1)/t}} = O\left(\frac{H^2}{\alpha\beta}\right) .$$

Therefore

$$(4.11)\qquad I_5 = O\left(\frac{H^2}{\alpha\beta}\int_0^{\sigma} du\right) = O\left(\frac{H^2\sigma}{\alpha\beta}\right) .$$

Turning next to I_6 in (4.1), we note that by the first line of (4.5) we can use the second mean value theorem to write, for some ξ_4 satisfying $\sigma < \xi_4 \leqq P\alpha$,

$$(4.12)\qquad I_6 = F(\sigma)\int_{\sigma}^{\xi_4} \cos 2\pi H\sqrt{n}\, v(u) \cdot v'(u)du = O\left(\frac{F(\sigma)}{H\sqrt{n}}\right) .$$

To examine the question of the order of $F(u)$ in $0 < u < P\alpha$ we use (4.3) with, $p = P\alpha - u, q = Q\beta + u$, and write

$$(4.13)\qquad F(u) = \frac{H}{(\alpha\beta)^{1/2}} \cdot \frac{[(pq)/(\alpha\beta)]^{(t-1)/2}}{[(p/\alpha)^t + (q/\beta)^t]^{(t-1)t}} \cdot$$
$$\cdot \frac{1}{-(\beta/\alpha)^{t/2}(p/q)^{(t-1)/2} + (\alpha/\beta)^{t/2}(q/p)^{(t-1)/2}}$$
$$\leqq \frac{H}{(\alpha\beta)^{1/2}} \cdot \frac{1}{2^{(t-1)/2}} \cdot \frac{1}{F_2(u)} .$$

Since $F_2(0)=0$ and

$$(4.14)\quad F_3(u)=\frac{dF_2(u)}{du}=\frac{t-1}{2}\left[\left(\frac{\beta}{\alpha}\right)^{t/2}\left(\frac{p}{q}\right)^{(t-3)/2}\frac{1}{q^2}+\left(\frac{\alpha}{\beta}\right)^{t/2}\left(\frac{q}{p}\right)^{(t-3)/2}\frac{1}{p^2}\right],$$

we have, by the mean value theorem,

$$(4.15)\quad F(u)\leqq\frac{H\lambda}{(\alpha\beta)^{1/2}}\cdot\frac{(t-1)/2}{F_3(u_3)u},\ \lambda=\frac{2^{(5-t)/2}}{t-1},\ 0<u<P\alpha,\ 0<u_3<P\alpha\ .$$

Setting $p_3=P\alpha-u_3$, $q_3=Q\beta+u_3$, we obtain

$$(4.16)\quad F(u)\leqq\frac{H\lambda}{(\alpha\beta)^{1/2}}\cdot\frac{p_3q_3}{[(\beta/\alpha)^{t/2}(p_3/q_3)^{(t-1)/2}+(\alpha/\beta)^{t/2}(q_3/p_3)^{(t-1)/2}]u}$$
$$\leqq\frac{H\lambda}{(\alpha\beta)^{1/2}}\cdot\frac{p_3q_3}{[(\beta/\alpha)^{t/2}+(\alpha/\beta)^{t/2}]u}\leqq\frac{H\lambda}{(\alpha\beta)^{1/2}}\cdot\frac{4}{[(\beta/\alpha)^{t/2}+(\alpha/\beta)^{t/2}]u}$$
$$=O\left\{\frac{H(\alpha\beta)^{(t-1)/2}}{(\alpha^t+\beta^t)u}\right\}.$$

Hence combining (4.11), (4.12), and (4.16), we obtain

$$(4.17)\quad I_5+I_6=O\left(\frac{H^2\sigma}{\alpha\beta}\right)+O\left(\frac{(\alpha\beta)^{(t-1)/2}}{(\alpha^t+\beta^t)\sigma n^{1/2}}\right)=O\left(\frac{H(\alpha\beta)^{(t-3)/4}n^{-1/4}}{(\alpha^t+\beta^t)^{1/2}}\right).$$

In the further analysis of I_5+I_6 we use the inequalities,

$$(4.18)\qquad 1+x^m<(1+x)^m,\ 0<x<1,\ m>1\ ,$$

$$(4.19)\qquad (x+1)^m<2^{m-1}(x^m+1),\ x>1,\ m>1\ .$$

In (4.17) suppose $1<t\leqq2$. Since $H=(\alpha^{t/(t-1)}+\beta^{t/(t-1)})^{(t-1)/t}$ and $t/(t-1)>t$, we have by (4.18), $H<(\alpha^t+\beta^t)^{1/t}$, and therefore, for $1<t\leqq2$, we have

$$(4.20)\quad \frac{H(\alpha\beta)^{(t-3)/4}}{(\alpha^t+\beta^t)^{1/2}}<\frac{(\alpha^t+\beta^t)^{(2-t)/(2t)}}{(\alpha\beta)^{(3-t)/4}}<\frac{(\alpha+\beta)^{(2-t)/2}}{(\alpha\beta)^{(t-1)/4}(\alpha\beta)^{(2-t)/2}}<\frac{2^{(2-t)/2}}{(\alpha\beta)^{(t-1)/4}}\ .$$

Hence from (4.17) and (4.20) we have, for $1<t\leqq2$,

$$(4.21)\qquad I_5+I_6=O\{(\alpha\beta)^{-(t-1)/4}n^{-1/4}\}\ .$$

If $t>2$ is (4.17), then $t>t/(t-1)$ and so by (4.19) we have

$$(4.22)\qquad \frac{H(\alpha\beta)^{(t-3)/4}}{(\alpha^t+\beta^t)^{1/2}}=\frac{(\alpha^{t/(t-1)}+\beta^{t/(t-1)})^{(t-1)/t}}{(\alpha^t+\beta^t)^{1/t}}\cdot\frac{(\alpha\beta)^{(t-3)/4}}{(\alpha^t+\beta^t)^{(t-2)/(2t)}}$$
$$<2^{(t-2)/t}\cdot\frac{(\alpha\beta)^{(t-3)/4}}{(\alpha^t+\beta^t)^{(t-2)/(2t)}}$$
$$<\frac{2^{(t-2)/t}(\alpha\beta)^{(t-3)/4}}{2^{(t-2)/(2t)}(\alpha\beta)^{(t-2)/4}}=\frac{2^{(t-2)/(2t)}}{(\alpha\beta)^{1/4}}\ .$$

Hence from (4.17) and (4.22) we have, for $t > 2$,

$$I_5 + I_6 = O\{(n\alpha\beta)^{-1/4}\} . \tag{4.23}$$

By (3.10) $v(-u, \alpha, \beta) = v(u, \alpha, \beta)$ so that an estimate for $I_5 + I_6 + I_7$ holds also for I_4 in (4.1). By this fact, and the results of (4.7), (4.8), (4.21), and (4.23), it now follows that for S_3, defined by (3.4), (3.23), and (4.1), we have,

$$\begin{aligned} S_3 &= \frac{2tn}{\pi^2} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{1}{\alpha\beta} \int_{-\infty}^{\infty} F(u) \cos 2\pi H\sqrt{n}\, v(u) \cdot v'(u) du \\ &= O(n^{3/4}),\ t > 1 , \end{aligned} \tag{4.24}$$

the double series being absolutely convergent.

Integrating by parts and applying the second mean value theorem, we have, from (3.6), for $x_1 = [(t-1)/t]^{1/t}$,

$$\begin{aligned} S_2 &= \frac{16t}{t+1} n^2 \sum_{\alpha=1}^{\infty} \int_0^1 (1 - x^t)^{(t+1)/t} \cos 2\pi\sqrt{n}\, \alpha x dx \\ &= \frac{8t}{\pi} n^{3/2} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha} \int_0^1 (1 - x^t)^{1/t} x^{t-1} \sin 2\pi\sqrt{n}\, \alpha x dx \\ &= \frac{8t}{\pi} n^{3/2} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha} \left\{ \int_0^{x_1} + \int_{x_1}^1 \right\} \\ &= \frac{8t}{\pi} n^{3/2} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha} \left\{ (1 - x_1^t)^{1/t} x_1^{t-1} \int_{\xi_5}^{x_1} \sin 2\pi\sqrt{n}\, \alpha x dx \right. \\ &\qquad \left. + (1 - x_1^t)^{1/t} x_1^{t-1} \int_{x_1}^{\xi_6} \sin 2\pi\sqrt{n}\, \alpha x dx \right\} \\ &= \frac{8t}{\pi} n^{3/2} \sum_{\alpha=1}^{\infty} O\left(\frac{1}{\sqrt{n}\, \alpha^2} \right) = O(n),\ t > 1 , \end{aligned} \tag{4.25}$$

the series being absolutely convergent. The absolute convergence of the double series in (2.1) now follows from the results leading to (4.24) and (4.25).

5. Proof of (1.1). Finally we deduce (1.1) from (3.1). We make use of the asymptotic expansion for the general Bessel function, namely [5, p. 368],

$$J_m(K) = \sqrt{\frac{2}{\pi K}} \cos\left(K - \frac{m\pi}{2} - \frac{\pi}{4}\right) + O(K^{-3/2}) , \tag{5.1}$$

for large K and m independent of K.

By (5.1) and the absolute convergence of the sum we have

$$(5.2)\quad T_2 = c_2 n^{5/4-1/(2t)} \sum_{\alpha=1}^{\infty} \alpha^{-3/2-1/t}\left\{\frac{\cos[2\pi\sqrt{n}\,\alpha - \pi(1+1/(2t))]}{(\pi^2\sqrt{n}\,\alpha)^{1/2}} + O(n^{-3/4}\alpha^{-3/2})\right\}$$

$$= -\frac{c_2}{\pi} n^{1-1/(2t)} \sum_{\alpha=1}^{\infty} \alpha^{-2-1/t} \cos(2\pi\sqrt{n}\,\alpha - \pi/(2t)) + O(n^{1/2-1/(2t)}) .$$

In $T_3\, f'(0, t) = 0$ and $f^{(k)}(1, t) = 0$, $k = 0, 1, 2$. Hence if we integrate by parts twice the integrated terms vanish and we have left

$$(5.3)\qquad T_3 = -\frac{4t}{\pi^2(t+1)} n \sum_{\alpha=1}^{\infty} \frac{1}{\alpha^2} \int_0^1 f''(x, t) \cos 2\pi\sqrt{n}\,\alpha x dx .$$

$f''(x, t)$ is continuous in $0 \leqq x \leqq 1$ and independent of n and α and so it has a finite number, independent of n and α, of relative and absolute extrema whose values are also independent of n and α. Hence dividing the interval of integration into pieces in which $f''(x, t)$ is monotonic, we obtain by the second mean value theorem, for appropriate $\xi_j, \xi'_j, \xi'_{j+1}$ in the interval from 0 to 1, the ξ's depending on n and α, the result,

$$(5.4)\quad T_3 = -\frac{4t}{\pi^2(t+1)} n^{3/4} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha^2} \sum_j f''(\xi_j, t) \int_{\xi'_j}^{\xi'_{j+1}} \cos 2\pi\sqrt{n}\,\alpha x dx = O(\sqrt{n}) .$$

Applying (5.1) to T_4 we obtain

$$T_4 = -\frac{2t}{\pi^2\sqrt{t-1}} n^{3/4} \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty} \frac{\cos(2\pi H\sqrt{n} - \pi/4)}{(\alpha\beta)^{(t-2)/(2t-2)} H^{(3t-1)/(2t-2)}}$$
$$- \frac{2tn}{\pi\sqrt{t-1}} \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty} \frac{O\{(\alpha^{t/(t-1)} + \beta^{t/(t-1)})^{-(3t-3)/(2t)} n^{-3/4}\}}{(\alpha\beta)^{(t-1)/(2t-2)}(\alpha^{t/(t-1)} + \beta^{t/(t-1)})} .$$

Since

$$(\alpha^{t/(t-1)} + \beta^{t/(t-1)})^{-(5t-3)/(2t)} \leqq 2^{-(5t-3)/(2t)}(\alpha\beta)^{-(5t-3)/(4t-4)} ,$$

the double series are absolutely convergent so that

$$(5.5)\quad T_4 = -\frac{2t}{\pi^2\sqrt{t-1}} n^{3/4} \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty} \frac{\cos(2\pi H\sqrt{n} - \pi/4)}{(\alpha\beta)^{(t-2)/(2t-2)} H^{(3t-1)/(2t-2)}} + O(n^{1/4}) .$$

Next we consider T_5. We have shown that $-T_4$ and S_3 are absolutely convergent double series for $t > 1$ and hence so is their term by term sum which is identical with T_5. We break up the interval of integration in T_5 into a finite number, independent of n, α, β, of subintervals in which $G(u, \alpha, \beta)$ is monotonic and write

$$(5.6)\quad T_5 = \frac{2tn}{\pi^2} \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty} \frac{1}{\alpha\beta} \sum_j \int_{\xi_j}^{\xi_{j+1}} G(u, \alpha, \beta) \cos 2\pi H\sqrt{n}\,v(u) \cdot v'(u) du .$$

Now $G(u, \alpha, \beta)$ is continuous in each $\xi_j \leqq u \leqq \xi_{j+1}$. The only doubt arises, at $u = 0$ where $v'(u) = (1 - v^2)^{1/2} = 0$, and at $u = \infty$ where $v'(u) = 0$. But, using the definitions in (3.1) and evaluating an indeterminate form, we obtain

$$(5.7) \qquad G(0+, \alpha, \beta) = \frac{-HA_{-1}(0)}{A_0^{1-1/t}(0)} - \lim_{u\to 0+}\left(\frac{1}{v'(u)} + \frac{a_{-1}}{\sqrt{1 - v^2(u)}}\right)$$
$$= \frac{-HA_{-1}(0)}{A_0^{1-1/t}(0)} + O\left(\frac{\alpha^{t/(t-1)} - \beta^{t/(t-1)}}{\alpha^{t/(t-1)} + \beta^{t/(t-1)}}\right)$$

which is bounded. On the other hand, by (4.3),

$$(5.8) \qquad G(\infty, \alpha, \beta) = -H(\alpha\beta)^{t-1}(\alpha^t + \beta^t)^{(1-2t)/t} - a_{-1}\,,$$

which is also bounded.

Applying the second mean value theorem to (5.6) we obtain

$$(5.9) \quad T_5 = \frac{2tn}{\pi^2}\sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty}\frac{1}{\alpha\beta}\sum_j G(\zeta_j', \alpha, \beta)\int_{\zeta_j}^{\zeta_{j+1}} \cos 2\pi H\sqrt{n}\, v(u)\cdot v'(u)du$$

for appropriate ζ_j', ζ_j, ζ_{j+1} in the interval from ξ_j to ξ_{j+1}. Further we have

$$(5.10) \qquad T_5 = \frac{2tn}{\pi^2}\sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty}\frac{1}{\alpha\beta}\sum_j G(\zeta_j', \alpha, \beta)\frac{O(1)}{H\sqrt{n}}$$
$$= \frac{2t\sqrt{n}}{\pi^2}\sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty}\frac{1}{\alpha\beta H}\sum_j G(\zeta_j', \alpha, \beta)O(1)$$
$$= O(\sqrt{n})$$

by the absolute convergence of the double series.

The relation (1.1) now follows from (3.1), (3.5), (5.2), (5.4), (5.5), and (5.10).

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