

# A NOTE ON POLYNOMIAL AND SEPARABLE GAMES

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**1. Introduction.** A two-person zero-sum game  $\Gamma$  is called polynomial-like or *separable* if its payoff function is of the form

$$M(x, y) = \sum_{i=1}^n f_i(x)g_i(y),$$

where  $x$  and  $y$  are elements of any strategy sets  $X$  and  $Y$ . Important special cases of separable games are those in which  $X$  and  $Y$  are bounded (usually compact) subsets of Euclidean spaces and  $M$  is a polynomial in the coordinates of  $x$  and  $y$ . These latter are called *polynomial games*.

It is a basic and fairly elementary fact concerning separable games [1], that, if optimal strategies exist, then these can always be chosen to be finite mixed strategies. We consider here the inverse question: Given a pair of finite mixed strategies, does there exist a separable (respectively, polynomial) game whose unique optimal strategies are the given pair? In case either  $X$  or  $Y$  is *finite* the answer is known to be in the negative. We here show, however, that.

**THEOREM 1.** *If  $X$  and  $Y$  are metric spaces containing infinitely many points and  $\mu$  and  $\nu$  are any finite mixed strategies on  $X$  and  $Y$  respectively, then there is a payoff  $M$ , bounded continuous and separable on  $X \times Y$ , such that the associated game has  $\mu$  and  $\nu$  as unique optimal strategies.*

**COROLLARY.** *If  $X$  is a metric space containing infinitely many points and  $\mu$  is any finite mixed strategy on  $X$ , then there is a skew-symmetric payoff  $M$ , bounded continuous and separable on  $X \times X$  such that the associated symmetric game has  $\mu$  as the unique optimal strategy.*

For the case of polynomial games we show:

**THEOREM 2.** *If  $X$  and  $Y$  are bounded subsets of Euclidean spaces whose closures contain infinitely many cluster points, then for any finite mixed strategies  $\mu$  and  $\nu$  there exists a polynomial payoff function  $M$  such that the associated game has  $\mu$  and  $\nu$  as its unique optimal strategies.*

(An analogous corollary holds here, also.)

Concerning Theorem 2, we remark that Glicksberg and Gross, [2],

have shown that any pair of mixed strategies can be the unique solution of a continuous game on the unit square. For finite mixtures, however, their construction is complicated, involving consideration of four special cases, and the payoff function is not a polynomial, nor even separable. The rather simple construction involved in our proof of Theorem 2 shows that their result still holds under the much stronger requirement that the payoff be a polynomial.

Finally, we credit Dresher, Karlin and Shapley, [1], for their rather exhaustive study of the structure of solutions of separable and polynomial games. However, their results do not include the theorems proved in this note. Indeed, one of the above authors has pointed out that the construction of the next section provides a *counter-example* to one of the conclusions of a structure theorem in [1], and fortunately (for mathematics) an error in the proof of that part of the theorem<sup>1</sup> was subsequently uncovered.

**2. Polynomial games with prescribed unique solutions.** This section contains the proof of Theorem 2. Let  $X$  and  $Y$  be sets satisfying the hypothesis of the theorem (We pause to note that boundedness of  $X$  and  $Y$  is required to insure integrability, since polynomials may otherwise be unbounded.). Let  $\mu$  be the mixed strategy which assigns the weight  $\mu_i$  to the point  $x_i$  of  $X$ ,  $i = 1, \dots, m$ , where  $\sum \mu_i = 1$ . Similarly, let  $\nu$  assign the weight  $\nu_j$  to the point  $y_j$  in  $Y$ ,  $j = 1, \dots, n$  where  $\sum \nu_j = 1$ .

The set of points  $\{x_1, \dots, x_m\}$ , the *spectrum* of  $\mu$ , will be denoted by  $\sigma(\mu)$ . Similarly,  $\sigma(\nu)$  will denote the spectrum of  $\nu$ .

We now define the following set of polynomials :

$$f_0(x) = \prod_{x' \in \sigma(\mu)} |x - x'|^2,$$

$$f_i(x) = \prod_{x' \in \sigma(\mu) - \{x_i\}} \frac{|x - x'|^2}{|x_i - x'|^2}, \quad i = 1, \dots, m,$$

where  $|x - x'|$  is the usual Euclidean distance from  $x$  to  $x'$ .

It is clear that the above functions are polynomials; however, aside from continuity, the only properties of them which we shall use are the following :

$$f_0(x) \geq 0 \text{ for all } x \in X \text{ and } f_0(x) = 0$$

if and only if  $x \in \sigma(\mu)$ .

$$f_i(x) \geq 0 \text{ for all } x \in X \text{ and } f_i(x) = 0$$

if and only if  $x \in \sigma(\mu) - \{x_i\}$ , ( $i = 1, \dots, m$ ).

<sup>1</sup> Theorem 6, fourth inequality pp. 175-176 of [1].

$$f_i(x_i) = 1, i = 1, \dots, m.$$

In a precisely analogous manner we define the polynomials  $g_0$  and  $g_j, j = 1, \dots, n$ , on the set  $Y$ .

Next, let  $\alpha_0, \alpha_1, \dots, \alpha_n$  be  $n + 1$  distinct cluster points of  $\bar{X}$  (the closure of  $X$ ) which do not meet  $\sigma(\mu)$  (these exist by hypothesis), and define polynomials  $\phi$  and  $\phi_j, j = 0, \dots, n$  on  $X$  via

$$\begin{aligned} \phi(x) &= \prod_{k=0}^n |x - \alpha_k|^2, \\ \phi_j(x) &= \prod_{\substack{k \neq j \\ 0 \leq k \leq n}} |x - \alpha_k|^2, \end{aligned} \quad j = 0, \dots, n.$$

The only properties of these functions we shall use are that they are all non-negative, that  $\phi$  vanishes only on the  $\alpha_k$ , and that  $\phi_j$  vanishes only on  $\alpha_k$  with  $k \neq j$ .

Finally, let  $\beta_0, \dots, \beta_m$  be  $m + 1$  distinct cluster points of  $\bar{Y}$  which do not meet  $\sigma(\nu)$ , and define polynomials  $\psi$  and  $\psi_i$  on  $Y$  analogous to the functions  $\phi$  and  $\phi_j$  above.

We now define the desired payoff  $M$  by

$$\begin{aligned} M(x, y) &= f_0(x)\phi(x)(g_0(y)\psi_0(y) + \sum_{j=1}^n (g_j(y) - \nu_j)\phi_j(y)) \\ (1) \quad &- g_0(y)\psi(y)(f_0(x)\psi_0(y) + \sum_{i=1}^m (f_i(x) - \mu_i)\psi_i(y)) \\ &- (f_0(x)\phi(x))^2 + (g_0(y)\psi(y))^2. \end{aligned}$$

We show first that  $\mu$  and  $\nu$  are optimal strategies. If we compute  $M(x, \nu)$  (in the usual extension), we obtain

$$(2) \quad M(x, \nu) = -(f_0(x)\phi(x))^2 \leq 0.$$

To see this, it is sufficient to observe that, according to the properties noted above,  $\int g_j d\nu = \nu_j$  and  $g_0$  and  $\psi$  vanish on  $\sigma(\nu)$ . Similarly, we obtain

$$M(\mu, y) = (g_0(y)\psi(y))^2 \geq 0.$$

Thus  $\mu$  and  $\nu$  are optimal and 0 is the value of the game.

It follows also from (2) above that if  $\mu'$  is any optimal strategy for player  $I$ , then the spectrum of  $\mu'$  is contained in the zeros of  $f_0\phi$ . Thus any optimal  $\mu'$  has weight only on the pure strategies  $x_i$  and  $\alpha_j$ , and similarly any optimal  $\nu$  for player  $II$  restricts its weight to  $\{y_j\} \cup \{\beta_i\}$ .

We now show that  $\nu$  is the only optimal strategy for player  $II$ . For suppose  $\nu'$  is optimal. Then, in the expression for  $M(x, \nu)$ , the

second and fourth terms in (1) drop out in view of the remark of the preceding paragraph and the payoff becomes

$$M(x, \nu') = f_0(x)\varphi(x) \left[ \varphi_0(x) \int g_0 d\nu' + \sum_{j=1}^n \varphi_j(x) \int (g_j(y) - \nu_j) d\nu' - f_0(x)\varphi(x) \right].$$

For  $x$  close to  $\alpha_0$  the expression in brackets above approaches  $\varphi_0(\alpha_0) \int g_0 d\nu'$ , and since  $\varphi_0(\alpha_0)$  is positive we must have  $\int g_0 d\nu' = 0$ . Otherwise the  $x$ -player by choosing  $x$  sufficiently close to  $\alpha_0$  could achieve a positive payoff, contradicting the optimality of  $\nu'$ . Next, since  $\int g_0 d\nu' = 0$ ,  $\nu'$  must concentrate all of its mass on the zeros of  $g_0$ , that is, on the points  $y_j$ . Finally, if  $\nu'_k \neq \nu_k$  for some index  $k$  then the  $x$ -player could again achieve a positive payoff by choosing  $x$  sufficiently close to  $x_k$ . It follows that  $\nu' = \nu$  as asserted.

Thus, Theorem 2 is established.

**3. Metric space games—construction of payoff.** This section is dedicated to the construction of the payoff required for the establishment of Theorem 1 and its corollary, which will be proved in the final section. The construction and method of proof are quite similar to those used in proving Theorem 2; however, to preserve continuity of presentation, we shall paraphrase identical details.

Therefore, let  $X$  and  $Y$  be the respective spaces according to hypothesis,  $\mu$  and  $\nu$  the respective finite probability measures on them.  $\mu$  and  $\nu$  will be described with the same notations used previously. Finally, let  $\rho$  and  $\rho'$  denote the associated metrics of  $X$  and  $Y$  respectively. Then, without further ado, we initiate our construction.

The basis of our construction hinges on the fact that any infinite metric space contains a sequence of disjoint neighborhoods. To see this for  $X$ , say, there is no loss in generality in assuming that  $X$  has a cluster point, for otherwise we are guaranteed a sequence by the discrete topology induced by  $\rho$  and the infiniteness of  $X$ . Therefore, let  $x^*$  denote a cluster point of  $X$ . First, choose  $\alpha_1 \neq x^*$ , and, for  $i > 1$  choose  $\alpha_i$  so that  $0 < \rho(x^*, \alpha_i) < \rho(x^*, \alpha_{i-1})/2$ . Then, as our sequence of neighborhoods,  $\{N_{\alpha_i}\}$ , we set

$$N_{\alpha_i} = \{x \mid \rho(x, \alpha_i) < r_i\}, \quad i = 1, 2, \dots,$$

where  $r_i = \rho(x^*, \alpha_i)/3$ . It is easy to verify, using the triangle inequality, that these neighborhoods are disjoint.

Therefore, let  $\{N_{\alpha_i}\}$  denote a sequence of disjoint neighborhoods contained in  $X$  (spheres of radius  $r_i$  centered at  $\alpha_i$ ). Define functions  $\phi_j$ ,  $j = 0, \dots, n$ , as follows:

$$\phi_j(x) = \begin{cases} \frac{r_i - \rho(x, \alpha_i)}{r_i} \cdot \frac{1}{i} & \text{if } x \in N_{\alpha_i} \text{ for some } i \text{ (at most one) and} \\ & i \equiv j \pmod{(n + 1)} \\ 0 & \text{otherwise.} \end{cases}$$

One verifies that  $\phi_j$  is a bounded continuous function on  $X$  into the non-negative reals, and which, moreover, satisfies

$$(6) \quad \phi_j(\alpha_i) = \begin{cases} \frac{1}{i} & \text{if } i \equiv j \pmod{(n + 1)} \\ 0 & \text{otherwise.} \end{cases}$$

Next, let the function  $\phi$  be given by

$$(7) \quad \phi(x) = \prod_{i=1}^m \rho(x, x_i), \quad x \in X,$$

(where, as previously,  $\{x_i\} = \sigma(\mu)$ ). There is no question about continuity here. We note merely that

$$(8) \quad \phi(x) \begin{cases} = 0 & \text{if } x \in \sigma(\mu) \\ > 0 & \text{otherwise.} \end{cases}$$

Finally, we define functions  $f_j$ ,  $j = 0, \dots, m$ , as follows :

$$(9) \quad f_0(x) = \phi(x),$$

and, for  $j \in \{1, \dots, m\}$ , set

$$(10) \quad f_j(x) = \prod_{i \neq j} \frac{\rho(x, x_i)}{\rho(x_j, x_i)}.$$

Here, again, continuity is immediate, and we note merely that

$$(11) \quad f_j(x_i) = \delta_{ij}, \quad i, j = 1, \dots, m,$$

where  $\delta$  is Kronecker's delta. Moreover, to insure boundedness of these functions, if such is not the case, we need only replace  $\rho$  by the function  $\rho/(1 + \rho)$  in the formulas (7) and (10) without affecting subsequent arguments.

The remainder of our construction involves defining certain bounded continuous functions on  $Y$  into the non-negative reals. To accomplish this we merely repeat the foregoing construction with the replacements:

$$\begin{array}{ll} \text{“ } X \text{ ”} & \rightarrow \text{“ } Y \text{ ”}, & \text{“ } m \text{ ”} & \rightarrow \text{“ } n \text{ ”}, \\ \text{“ } \rho \text{ ”} & \rightarrow \text{“ } \rho' \text{ ”}, & \text{“ } n \text{ ”} & \rightarrow \text{“ } m \text{ ”}, \\ \text{“ } x \text{ ”} & \rightarrow \text{“ } y \text{ ”}, & \text{“ } \phi \text{ ”} & \rightarrow \text{“ } \psi \text{ ”}, \\ \text{“ } \alpha \text{ ”} & \rightarrow \text{“ } \beta \text{ ”}, & \text{“ } f \text{ ”} & \rightarrow \text{“ } g \text{ ”}, \\ \text{“ } r \text{ ”} & \rightarrow \text{“ } r' \text{ ”}, & \text{“ } \mu \text{ ”} & \rightarrow \text{“ } \nu \text{ ”}. \end{array}$$

In terms of these functions, then, and using the convention  $\mu_0 = \nu_0 = 0$ , we define our bounded continuous polynomial-like payoff  $M$  as follows :

$$\begin{aligned}
 M(x, y) = & -\psi(y) \sum_{j=0}^m (f_j(x) - \mu_j)\phi_j(y) \\
 & + \phi(x) \sum_{j=0}^n (g_j(y) - \nu_j)\phi_j(x) \\
 & - \phi(x) \sum_{j=0}^n \phi_j(x)^2 + \psi(y) \sum_{j=0}^m \psi_j(y)^2,
 \end{aligned}$$

$(x, y) \in X \times Y$ . This completes our construction.

**4. Verification of solution and proof of uniqueness.** To verify that  $(\mu, \nu)$  is a solution, we calculate first the expectation  $M(\mu, y)$  :

$$M(\mu, y) = \psi(y) \sum_{j=0}^m \psi_j(y)^2 \geq 0, \quad \text{all } y \in Y.$$

To see this, we note that the remaining sums vanish by virtue of (8), (9), and (11), i.e.  $\phi$  vanishes on  $\sigma(\mu)$  and  $\int f_j d\mu = \mu_j, j = 0, \dots, m$ . Similarly,

$$(12) \quad M(x, \nu) = -\phi(x) \sum_{j=0}^n \phi_j(x)^2 \leq 0, \quad \text{all } x \in X.$$

Thus,  $(\mu, \nu)$  is a solution and 0 is the value of the game.

To show uniqueness for the first player, let  $\mu'$  denote an optimal strategy for him. From the non-negativity of the functions  $\phi, \phi_j$  in (12), we see that  $\int \phi \phi_j^2 d\mu' = 0$  for all  $j \in \{0, \dots, n\}$  and hence that  $\int \phi \phi_j d\mu' = 0$ ; for otherwise, by (12), a counter strategy is provided by  $\nu$ . Thus, if  $\mu'$  is optimal, we have

$$(13) \quad M(\mu', y) = -\psi(y) \sum_{j=0}^m (\mu - \mu_j)\psi_j(y) + \psi(y) \sum_{j=0}^m \psi_j(y)^2,$$

where we have written  $\mu'_j = \int f_j d\mu', j = 0, \dots, m$ . Next, suppose  $\mu'_0 = \int f_0 d\mu' \neq 0$  (and hence, positive). Choose as possible counters a subsequence of the  $\beta$ 's,  $\{\beta_{n_i}\}$  such that  $n_i \equiv 0 \pmod{m+1}$ . Then, by virtue of the minimizer's counterpart of (6), (13) becomes

$$\begin{aligned}
 (14) \quad M(\mu', \beta_{n_i}) &= -\psi(\beta_{n_i})\mu'_0 \cdot \frac{1}{n_i} + \psi(\beta_{n_i}) \frac{1}{n_i^2} \\
 &= \frac{\psi(\beta_{n_i})}{n_i} \left( -\mu'_0 + \frac{1}{n_i} \right), \quad i = 1, 2, \dots
 \end{aligned}$$

Since  $\psi$  vanishes only on a finite set and is positive elsewhere, we see that the expression above can be made negative for  $i$  sufficiently large. Hence  $\mu'_0 = 0$ , and it follows from (8) and (9) that  $\sigma(\mu') \subseteq \sigma(\mu)$ , i.e. any optimal  $\mu'$  must restrict its spectrum to the set  $\{x_1, \dots, x_m\}$ . Thus, finally, to establish uniqueness, we need only show that the corresponding weights are equal. Let  $\mu'_i$  denote the weight on  $x_i$  placed by  $\mu'$ . Substituting in our payoff  $M$  we obtain (noting  $\mu'_0 = 0$ ),

$$(15) \quad M(\mu', y) = -\psi(y) \sum_{j=1}^m (\mu'_j - \mu_j) \psi_j(y) + \psi(y) \sum_{j=0}^m \psi_j(y)^2.$$

Now suppose  $\mu'_k \neq \mu_k$  for some  $k \in \{1, \dots, m\}$ . Then, since

$$\sum_{j=1}^m \mu'_j = \sum_{j=1}^m \mu_j = 1,$$

we would have some  $j = j_0 \in \{1, \dots, m\}$  such that  $\mu'_{j_0} > \mu_{j_0}$ . But, by choosing the subsequence  $\{\beta_{n_i}\}$  with  $n_i \equiv j_0 \pmod{m+1}$ , by the identical argument used before, we would find a counter rendering the expectation (15) negative. Hence  $\mu'_j = \mu_j$  and thus  $\mu' = \mu$ . Uniqueness for the minimizer can be established in a similar manner, as is clear. So Theorem 1 is proved.

Finally, to establish the corollary, we need only make the appropriate identifications in our payoff to ensure that  $M(x, y) = -M(y, x)$ .

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#### REFERENCES

1. M. Dresher, S. Karlin, and L. S. Shapley, *Polynomial games*, Ann. of Math. Study No. **24** (Princeton 1950), 161-180.
2. I. Glicksberg and O. Gross, *Notes on games over the square*, Ann. of Math. Study No. **28** (Princeton 1953), 173-182.

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