# EIGENVALUES OF THE UNITARY PART OF A MATRIX

ALFRED HORN AND ROBERT STEINBERG

**1. Introduction.** It is well known that every matrix $A$ (square and with complex entries) has a polar decomposition $A = P_1U_1 = U_2P_2$, where $U_i$ are unitary and $P_i$ are unique positive semi-definite Hermitian matrices. If $A$ is non-singular then $U_1 = U_2 = U$, where $U$ is also unique. In this case we call $U$ the unitary part of $A$. The eigenvalues of $P_1$ are the same as those of $P_2$.

In [2] the following problem was solved. Given the eigenvalues of $P_1$, what is the exact range of variation of the eigenvalues of $A$ ? The answer shows that a knowledge of the eigenvalues of $P_1$ puts restrictions only on the moduli of the eigenvalues of $A$. In this paper we are going to consider the corresponding question for the unitay part $U$ of $A$. In turns out that a knowledge of the eigenvalues of $U$ restricts only the arguments of the eigenvalues of $A$.

Before stating the result, we need some definitions. An ordered pair of $n$-tuples $(\lambda_i)$, $(\alpha_i)$ of complex numbers is said to be *realizable* if there exists a non-singular matrix $A$ of order $n$ with eigenvalues $\lambda_i$ such that the unitary part of $A$ has eigenvalues $\alpha_i$. If $(\gamma_j)$ is an $n$-tuple of complex numbers of modulus 1, and if two of the $\gamma_j$ are of the form $e^{ib}$, $e^{ic}$ with $0 < b - c < \pi$ and $0 \leqq d \leqq (b-c)/2$, then the operation of replacing $e^{ib}$, $e^{ic}$ by $e^{i(b-d)}$, $e^{i(c+d)}$ is called a *pinch* of $(\gamma_j)$. In other words, a pinch of $(\gamma_j)$ consists in choosing two of the $\gamma_j$ which do not lie on the same line through 0 and turning them toward each other through equal angles.

If $(a_i)$, $(b_i)$ are $n$-tuples of real numbers, and if $(a'_i)$, $(b'_i)$ are their rearrangements in non-decreasing order, then we write $(a_i) \prec (b_i)$ when $\sum_r^n a'_i \leqq \sum_r^n b'_i$, $r = 2, \cdots, n$ and $\sum_1^n a'_i = \sum_1^n b'_i$. It is easily seen that the conditions are equivalent to the conditions $\sum_1^r a'_i \geqq \sum_1^r b'_i$, $r = 1, \cdots, n-1$, and $\sum_1^n a'_i = \sum_1^n b'_i$.

Our main theorem is the following.

THEOREM 1. *Let $(\lambda_i)$, $(\alpha_i)$ be $n$-tuples of complex numbers such that $\lambda_i \neq 0$ and $|\alpha_i| = 1$. Then the following statements are equivalent:*

(1) *the pair $(\lambda_i)$, $(\alpha_i)$ is realizable;*

(2) *$(\alpha_i)$ can be reduced to $(\lambda_i/|\lambda_i|)$ by a finite sequence of pinches;*

(3) *$\prod_1^n \alpha_i = \prod_1^n (\lambda_i/|\lambda_i|)$, and exactly one of the following hold:*

(a) *there is a line through 0 containing all the $\alpha_i$ and $(\lambda_i/|\lambda_i|)$ is a rearrangement of $(\alpha_i)$;*

(b) *there is no line through 0 containing all $\alpha_i$ but there is*

Received September 26, 1958.

*a closed half plane $H$ with 0 on its boundary containing all $\alpha_i$, and, if we choose a branch of the argument function which is continuous in $H - \{0\}$, then* $(\arg \lambda_i) \prec (\arg \alpha_i)$;

(c) *there is no closed half plane with 0 on its boundary which contains all $\alpha_i$.*

The proof of Theorem 1 will be given at the end of the paper.

**2. Definitions and preliminary results.** Two matrices $A$ and $B$ are said to be *congruent* if there exists a non-singular matrix $X$ such that $B = X^*AX$. A *triangular* matrix is a matrix such that all entries below the main diagonal are 0. If $P$ is a positive definite matrix, then $P^{1/2}$ denotes the unique positive definite matrix whose square is $P$. We will use the symbol diag $(a_1, \cdots, a_n)$ to denote the diagonal matrix with diagonal elements $a_1, \cdots, a_n$.

LEMMA 1. *If $\lambda_i \neq 0$ and $|\alpha_i| = 1$, then the pair $(\lambda_i)$, $(\alpha_i)$ is realizable if and only if there exists a matrix $A$ with eigenvalues $\lambda_i$ which is congruent to $D = \mathrm{diag}\,(\alpha_1, \cdots, \alpha_n)$.*

*Proof.* We use the fact that for any two matrices $B$ and $C$, $BC$ and $CB$ have the same eigenvalues. If $(\lambda_i)$, $(\alpha_i)$ is realizable, there exists a unitary matrix $U$ with eigenvalues $\alpha_i$ and a positive definite matrix $P$ such that $PU$ has eigenvalues $\lambda_i$. Let $V$ be a unitary matrix such that $U = V^*DV$. Then $PU$ has the same eigenvalues as $P^{1/2}V^*DVP^{1/2}$, which is congruent to $D$. Conversely, if $X^*DX$ has eigenvalues $\lambda_i$, then so does $A = XX^*D$, and $D$ is the unitary part of $A$ since $XX^*$ is positive definite.

LEMMA 2. *If $(\lambda_i)$, $(\alpha_i)$ is realizable and $\rho_i > 0$ for each $i$, then $(\rho_i\lambda_i)$, $(\alpha_i)$ is realizable.*

*Proof.* Suppose $D = \mathrm{diag}\,(\alpha_1, \cdots, \alpha_n)$ is congruent to a matrix $A$ with eigenvalues $\lambda_i$. Then $A$ is congruent to a triangular matrix $B$ with diagonal elements $\lambda_i$. If $X = \mathrm{diag}\,(\rho_1^{1/2}, \cdots, \rho_n^{1/2})$, then $X^*BX$ obviously has eigenvalues $\rho_i\lambda_i$ and is congruent to $D$.

LEMMA 3. *If $(\lambda_i)$, $(\alpha_i)$ is realizable and $z$ is any complex number of modulus 1, then $(z\lambda_i)$, $(z\alpha)$ is realizable.*

LEMMA 4. *If $(\mu_1, \mu_2)$ results from $(\lambda_1, \lambda_2)$ by a pinch and $T$ is a triangular matrix with diagonals elements $\lambda_1$, $\lambda_2$, then $T$ is congruent to a matrix with eigenvalues $\mu_1$, $\mu_2$.*

*Proof.* By multiplication by a suitable constant, we may suppose that $\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$, and $\mu_1 = e^{i\phi}$, $\mu_2 = e^{-i\phi}$, where $0 \leqq \phi \leqq \theta < \pi/2$. It suffices to find a positive matrix $P$ such that $PT$ has eigenvalues $e^{\pm i\phi}$. Suppose

$$T = \begin{pmatrix} e^{i\theta} & a \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Let

$$P = \begin{pmatrix} x & \bar{y} \\ y & x \end{pmatrix},$$

where $x \geqq 1$, $|y|^2 = x^2 - 1$ and $ya = |a|(x^2-1)^{1/2}$. Since $P$ has determinant 1, we need only choose $x$ so that the trace of $PT$ is $2\cos\phi$. The trace of $PT$ is $f(x) = xe^{i\theta} + xe^{-i\theta} + ya = 2x\cos\theta + |a|(x^2-1)^{1/2}$. When $x = 1$, this is $2\cos\theta$, and for $x \geqq 1$, $f(x)$ increases to infinity.

LEMMA 5. *If $(\alpha_i)$ can be reduced to $(\lambda_i/|\lambda_i|)$ by a finite number of pinches, then $(\lambda_i)$, $(\alpha_i)$ is realizable.*

*Proof.* By Lemma 2 we may assume $|\lambda_i| = 1$. We need only prove the following: if $(\lambda_i)$, $(\alpha)$ is realizable, if $|\lambda_i| = 1$ and if $(\mu_i)$ is a pinch of $(\lambda_i)$, then $(\mu_i)$, $(\alpha_i)$ is realizable. We may suppose that the pinch consists in replacing $\lambda_1$, $\lambda_2$ by $\mu_1$, $\mu_2$. By hypothesis there exists a triangular matrix $A$ with eigenvalues $\lambda_i$ which is congruent to diag $(\alpha_1, \cdots, \alpha_n)$. By Lemma 4 there exists a two rowed non-singular matrix $Z$ such that

$$B = Z^*\begin{pmatrix} \lambda_1 & a_{12} \\ 0 & \lambda_2 \end{pmatrix} Z$$

has eigenvalues $\mu_1$, $\mu_2$. Here $a_{12}$ is the (1, 2) entry of $A$. If we set

$$Y = \begin{pmatrix} Z & 0 \\ 0 & I \end{pmatrix},$$

where $I$ is the identity matrix of order $n - 2$, then

$$Y^*AY = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where $D$ is triangular with diagonal elements $\lambda_3, \cdots, \lambda_n$. But this last matrix obviously has eigenvalues $(\mu_1, \mu_2, \lambda_3, \cdots, \lambda_n) = (\mu_1, \cdots, \mu_n)$.

LEMMA 6. *If $(a_1, \cdots, a_k) \prec (b_1, \cdots, b_k)$ and $(c_1, \cdots, c_p) \prec (d_1, \cdots, d_p)$ then $(a_1, \cdots, a_k, c_1, \cdots, c_p) \prec (b_1, \cdots, b_k, d_1, \cdots, d_p)$.*

*Proof.* A proof is given in [1; 63].

LEMMA 7. *If $A$ is a matrix such that $(Ax, x) \neq 0$ and $0 < \arg(Ax, x) < \pi$ for all $x \neq 0$, then $A$ is congruent to a unitary matrix.*

*Proof.* Let $H = (A + A^*)/2$, $K = (A - A^*)/2i$. Then $A = H + iK$, and $H$, $K$ are Hermitian. Since $(Ax, x) = (Hx, x) + i(Kx, x)$, the hypothesis implies that $(Kx, x) > 0$ for all $x \neq 0$, so that $K$ is positive definite. Therefore by [3; 261] $H$ and $K$ are simultaneously congruent to real diagonal matrices. Hence $A = H + iK$ is congruent to a diagonal unitary matrix.

LEMMA 8. *If $A$ is congruent to a unitary matrix $U$ with eigenvalues $\alpha_i$, and if $0 < \arg \alpha_1 < \cdots < \arg \alpha_n < \pi$, then $(Ax, x) \neq 0$ for all $x \neq 0$ and*

$$\arg \alpha_j = \inf_{\substack{\dim S \\ = j}} \sup_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x) = \sup_{\substack{\dim S \\ = n-j+1}} \inf_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x)$$

*where $S$ ranges over subspaces of $n$-dimensional complex Euclidean space.*

*Proof.* Let $(u_i)$ be an ortho-normal sequence of eigenvectors of $U$ corresponding to $(\alpha_i)$. If $A = X^*UX$, then $(Ax, x) = \sum_1^n \alpha_i |(Xx, u_i)|^2$. If $S$ is the space spanned by $X^{-1}u_1, \cdots, X^{-1}u_j$, then

$$\sup_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x) = \arg \alpha_j .$$

Now let $S$ be any subspace of dimension $j$. Let $M$ be the space spanned by $X^{-1}u_j, \cdots, X^{-1}u_n$. Then there exists a non-zero vector $x$ in $M \cap S$. But

$$\arg(Ax, x) \geqq \inf_{y \neq 0} \arg \sum_j^n \alpha_i |(y, u_i)|^2 = \arg \alpha_j .$$

Therefore

$$\sup_{\substack{x \in S \\ x \neq 0}} \arg(Ax, x) \geqq \arg \alpha_j .$$

The proof of the second statement is analogous.

Lemma 8 is of course the analogue of the minimax principle for Hermitian matrices. The generalization due to Wielandt [4] also has an analogue for unitary matrices, which we mention without proof since it will not be used.

*If $A$ and $U$ satisfy the hypotheses of Lemma 8 and $1 \leqq i_1 < \cdots < i_k \leqq n$, then*

$$\arg \alpha_{i_1} + \cdots + \arg \alpha_{i_k} = \inf_{\substack{M_1 \subset \cdots \subset M_k \\ \dim M_p = i_p}} \sup_{x_p \in M_p} (\arg \beta_1 + \cdots + \arg \beta_k)$$

*where* $(x_1, \cdots, x_k)$ *ranges over linearly independent sequences of vectors, and the* $\beta_j$ *are the eigenvalues of the matrix of order* $k$ *whose* $(i, j)$ *entry is* $(Ux_i, x_j)$. *The number* $\arg \beta_1 + \cdots + \arg \beta_k$ *depends only on the subspace generated by* $x_1, \cdots, x_k$.

LEMMA 9. *If* $(\lambda_i), (\alpha_i)$ *is realizable and* $0 \leqq \arg \alpha_1 \leqq \cdots \leqq \arg \alpha_n \leqq \pi$, *then* $(\arg \lambda_i) \prec (\arg \alpha_i)$.

*Proof.* By Lemma 1, $\lambda_i$ are the eigenvalues of $X^*DX$, where $X$ is non-singular and $D = \text{diag}(\alpha_1, \cdots, \alpha_n)$. Since the eigenvalues of $X^*DX$ vary continuously with the $\alpha_i$, we need only prove the theorem for the case where $0 < \arg \alpha_1$, $\arg \alpha_n < \pi$. We proceed by induction on $n$. The statement being obvious when $n = 1$, suppose $n > 1$ and the theorem holds for matrices of order $n - 1$. Let $A$ be a triangular matrix with eigenvalues $\lambda_i$ which is congruent to $D$. Suppose the $\lambda_i$ are arranged so that $\arg \lambda_1 \leqq \cdots \leqq \arg \lambda_n$. Let $B$ be the principal minor of $A$ formed from the first $n - 1$ rows and columns of $A$. If $x = (x_1, \cdots, x_{n-1})$ is a vector with $n - 1$ components and $y = (x_1, \cdots, x_{n-1}, 0)$ then $(Bx, x) = (Ay, y)$. Therefore for any such $x \neq 0$, $(Ax, x) \neq 0$ and

$$0 < \arg \alpha_1 \leqq \arg (Ay, y) = \arg (Bx, x) \leqq \arg \alpha_n < \pi ,$$

by Lemma 8, since $A$ is congruent to $D$.

By Lemma 7, $B$ is congruent to a unitary matrix $V$. Let the eigenvalues of $V$ be $\beta_i$, where $\arg \beta_1 \leqq \cdots \leqq \arg \beta_{n-1}$. Since the quadratic form $(Bx, x)$ associated with $B$ is a restriction of the quadratic form associated with $A$, it follows from Lemma 8 that $\arg \alpha_{j+1} \geqq \arg \beta_j \geqq \arg \alpha_j$, $j = 1, \cdots, n - 1$. Also by the induction hypothesis $(\arg \lambda_1, \cdots, \arg \lambda_{n-1}) \prec (\arg \beta_1, \cdots, \arg \beta_{n-1})$. Therefore

$\arg \lambda_1 + \cdots + \arg \lambda_r \geqq \arg \beta_1 + \cdots + \arg \beta_r \geqq \arg \alpha_1 + \cdots + \arg \alpha_r$, $r = 1, \cdots, n - 1$

and

$$\begin{aligned} \arg \alpha_2 + \cdots + \arg \alpha_n &\geqq \arg \lambda_1 + \cdots + \arg \lambda_{n-1} \\ &\geqq \arg \alpha_1 + \cdots + \arg \alpha_{n-1} . \end{aligned}$$

Hence

$$-\pi < \arg \lambda_n - \arg \alpha_n \leqq \sum_1^n (\arg \lambda_i - \arg \alpha_i) \leqq \arg \lambda_n - \arg \alpha_1 < \pi .$$

But

$$\prod_1^n \lambda_i = |\det X|^2 \cdot \prod_1^n \alpha_i .$$

Therefore

$$\sum_1^n \arg \lambda_i = \sum_1^n \arg \alpha_i \ .$$

The proof is complete.

LEMMA 10. *If* $(\beta_i)$, $(\alpha_i)$ *are n-tuples of complex numbers of modulus* 1 *which lie on a line through* 0, *and if* $(\beta\ )$, $(\alpha_i)$ *is realizable, then* $(\beta_i)$ *must be a rearrangement of* $(\alpha_i)$.

*Proof.* By Lemma 3 we may suppose that the $\alpha_i$ and $\beta_i$ are all real. Let $A$ be a matrix with eigenvalues $\beta_i$ which is congruent to $\mathrm{diag}\,(\alpha_1, \cdots, \alpha_n)$. Then $A$ is Hermitian and therefore $A$ is also congruent to $\mathrm{diag}\,(\beta_1, \cdots, \beta_n)$. But by Lemma 1 it follows that $(\alpha_i)$, $(\beta_i)$ is realizable. Therefore by Lemma 9 we have $(\arg \beta_i) \prec (\arg \alpha_i) \prec (\arg \beta_i)$, from which the present theorem follows immediately.

LEMMA 11. *Suppose* $(\beta_i)$, $(\alpha_i)$ *are n-tuples of complex numbers of modulus* 1 *such that* $\prod_1^n \beta_i = \prod_1^n \alpha_i$. *Then there exist determinations of* $\arg \alpha_i$, $\arg \beta_i$ *such that*

$$\max\ \arg \alpha_i - \min\ \arg \alpha_i \leqq 2\pi$$

and

$$(\arg \beta_i) \prec (\arg \alpha_i)\ .$$

*Proof.* The statement is obvious for $n = 1$. Suppose $n > 1$ and it holds for $n$-1-tuples. If any of the $\beta_i$ is equal to any of the $\alpha_i$, say $\beta_1 = \alpha_1$, then by the induction hypothesis, we can find determinations of the remaining $\arg \alpha_i$, $\arg \beta_i$ as stated. If we now choose a value of $\arg\ \alpha_1$ which lies between $\mu$ and $\mu + 2\pi$, where $\mu = \min_{i>1} \arg \alpha_i$, and set $\arg \beta_1 = \arg \alpha_1$, then the conditions of our theorem will be satisfied, by Lemma 6. So henceforth we may assume that $\beta_i \neq \alpha_j$ for all $i$, $j$.

As another special case, suppose the $\alpha_i$ are all equal, say to 1. If we assign arguments to the $\beta_i$ such that $0 < \arg \beta_i < 2\pi$, then $\sum_1^n \arg \beta_i = 2\pi k$, where $k$ is some positive integer $< n$. We need only assign arguments to the $\alpha_i$ such that exactly $k$ of them have argument $2\pi$ and the remaining ones have argument 0.

Now assume the previous two cases do not occur. The $\alpha_i$ divide the unit circle into arcs. At least one of them must contain more than one of the $\beta_i$, for if not the $\alpha_i$ would be all distinct and each of the $n$ arcs determined by them would contain exactly one of the $\beta_i$. We could then assign arguments to arrangements of the $\alpha_i$, $\beta_i$ so that

$$\arg \alpha_1 < \arg \beta_1 < \arg \alpha_2 < \cdots < \arg \alpha_n < \arg \beta_n < \arg \alpha_1 + 2\pi\ .$$

But then $0 < \sum_1^n \arg \beta_i - \sum_1^n \arg \alpha_i < 2\pi$, contradicting the hypothesis $\prod_1^n \alpha_i = \prod_1^n \beta_i$.

Let $C$ be an arc containing more than one of the $\beta_i$. By changing subscripts, we may assume that the endpoints of $C$ when described counterclockwise are $\alpha_1$ and $\alpha_2$. Let $\beta_1$ be one of the $\beta_i$ in $C$ which is nearest to $\alpha_1$ and $\beta_2$ be one of the $\beta_i$ with subscript $\neq 1$ which is nearest to $\alpha_2$. Note that $\beta_1$ may equal $\beta_2$, but $\alpha_1 \neq \alpha_2$. As will be seen from the following argument, we may assume the subarc $\alpha_1\beta_1$ of $C \leqq$ the subarc $\beta_2\alpha_2$ of $C$, (all arcs are described counterclockwise). Let $\beta_1' = \alpha_1$ and let $\beta_2'$ be the point in $\beta_2\alpha_2$ such that $\beta_2\beta_2' = \alpha_1\beta_1 = \delta$. By the first case of the proof, we may assign arguments to $\beta_1', \beta_2', \beta_3, \cdots, \beta_n$ and $\alpha_1, \cdots, \alpha_n$ so that

(1) $\max \arg \alpha_i - \min \arg \alpha_i \leqq 2\pi$

and

(2) $(\arg \beta_1', \arg \beta_2', \arg \beta_3, \cdots, \arg \beta_n) \prec (\arg \alpha_1, \cdots, \arg \alpha_n)$.

If $\arg \alpha_1$ happens to be the largest of $\arg \alpha_i$, and therefore $\arg \alpha_2$ is the smallest of $\arg \alpha_i$, then none of $\beta_1', \beta_2', \beta_3, \cdots, \beta_n$ can lie in the interior of $C$. Therefore $\beta_2' = \alpha_2$, and if we decrease $\arg \alpha_1$ and $\arg \beta_1$ by $2\pi$, then (1) and (2) will still hold. Thus we may assume $\arg \alpha_1 < \arg \alpha_2$, and therefore $\arg \beta_1' < \arg \beta_2'$. Now assign to $\beta_1$ the argument $\beta_1' + \delta$ and to $\beta_2$ the argument $\arg \beta_2' - \delta$. Since

$$(\arg \beta_1' + \delta,\ \arg \beta_2' - \delta) \prec (\arg \beta_1',\ \arg \beta_2')\ ,$$

we have by Lemma 6,

$$\begin{aligned}(\arg \beta_1, \cdots, \arg \beta_n) &\prec (\arg \beta_1', \arg \beta_2', \arg \beta_3, \cdots, \arg \beta_n) \\ &\prec (\arg \alpha_1, \cdots, \arg \alpha_n)\ .\end{aligned}$$

This completes the proof.

LEMMA 12. *If* $(\beta_i)$, $(\alpha_i)$ *are n-tuples of complex numbers of modulus* 1 *which can be assigned arguments such that*

$$\begin{aligned}&\arg \alpha_1 \leqq \cdots \leqq \arg \alpha_n \leqq \arg \alpha_1 + 2\pi\ ,\\ &\arg \beta_1 \leqq \cdots \leqq \arg \beta_n\ ,\\ &(\arg \beta_i) \prec (\arg \alpha_i)\ ,\end{aligned}$$

*and*

$$\arg \alpha_{i+1} - \arg \alpha_i < \pi,\ i = 1, \cdots, n-1\ ,$$

*then a finite number of pinches will reduce* $(\alpha_i)$ *to* $(\beta_i)$.

*Proof*. We proceed by induction on $n$. When $n = 2$, we have $\arg \alpha_1 \leqq \arg \beta_1 \leqq \arg \beta_2 \leqq \arg \alpha_2$, $\arg \alpha_1 + \arg \alpha_2 = \arg \beta_1 + \arg \beta_2$ and $\arg \alpha_2 - \arg \alpha_1 < \pi$. Therefore $\arg \beta_1 - \arg \alpha_1 = \arg \alpha_2 - \arg \beta_2$ and so

$(\beta_1, \beta_2)$ is a pinch of $(\alpha_1, \alpha_2)$.

Suppose $n > 2$ and the theorem holds for all $m$-tuples, $m < n$. Let

$$\delta = \min_{1 \leqq p \leqq n-1} \sum_1^p (\arg \beta_i - \arg \alpha_i) .$$

There exists $k$ such that $\sum_1^k \arg \beta_i - \sum_1^k \arg \alpha_i = \delta$. It is easy to verify that

$$(\arg \beta_1, \cdots, \arg \beta_k) \prec (\arg \alpha_1 + \delta, \arg \alpha_2, \cdots, \arg \alpha_k)$$

and

$$(\arg \beta_{k+1}, \cdots, \arg \beta_n) \prec (\arg \alpha_{k+1}, \cdots, \arg \alpha_{n-1}, \arg \alpha_n - \delta) .$$

Also

$$\arg \alpha_1 + \delta \leqq \arg \beta_1 \leqq \arg \beta_n \leqq \arg \alpha_n - \delta .$$

By the induction hypothesis, we can reduce $(\alpha_1 e^{i\delta}, \alpha_2, \cdots, \alpha_k)$ to $(\beta_1, \cdots, \beta_k)$ and $(\alpha_{k+1}, \cdots, \alpha_{n-1}, \alpha_n e^{-i\delta})$ to $(\beta_{k+1}, \cdots, \beta_n)$ by a finite number of pinches. We need only show that $(\alpha_1, \cdots, \alpha_n)$ can be reduced to $(\alpha_1 e^{i\delta}, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n e^{-i\delta})$ by a finite number of pinches. This will follow from the next lemma if we consider only the distinct $\alpha_i$.

If the $\alpha_i$ all coincide, then so do the $\beta_i$ and the statement of our theorem is trivial.

LEMMA 13. *If $(\alpha_i)$ is an $m$-tuple of numbers of modulus 1 with assigned arguments such that*

$$\arg \alpha_1 < \cdots < \arg \alpha_m \leqq \arg \alpha_1 + 2\pi$$

*and*

$$\arg \alpha_{i+1} - \arg \alpha_i < \pi, \ i = 1, \cdots, m-1 ,$$

*and if $\delta$ is a positive number such that $\arg \alpha_1 + \delta \leqq \arg \alpha_m - \delta$, then $(\alpha_i)$ can be reduced to $(\alpha_1 e^{i\delta}, \alpha_2, \cdots, \alpha_{m-1}, \alpha_m e^{-i\delta})$ by a finite number of pinches.*

*Proof.* This is obvious for $m = 2$. Assume $m > 2$ and the lemma holds for $m - 1 -$ tuples. If

$$\begin{aligned} \eta = \min(&\arg \alpha_2 - \arg \alpha_1, \ \pi - (\arg \alpha_3 - \arg \alpha_2), \cdots, \\ &\pi - (\arg \alpha_m - \arg \alpha_{m-1})) , \end{aligned}$$

and $0 < \varepsilon < \eta$, then each sequence in the following list is a pinch of the preceeding sequence:

$$\alpha_1, \cdots, \alpha_m$$

$$\alpha_1 e^{i\varepsilon},\ \alpha_2 e^{-i\varepsilon},\ \alpha_3,\ \cdots,\ \alpha_m$$
$$\alpha_1 e^{i\varepsilon},\ \alpha_2,\ \alpha_3 e^{-i\varepsilon},\ \cdots,\ \alpha_m$$
$$\cdots$$
$$\alpha_1 e^{i\varepsilon},\ \alpha_2,\ \cdots,\ \alpha_{m-2},\ \alpha_{m-1} e^{-i\varepsilon},\ \alpha_m$$
$$\alpha_1 e^{i\varepsilon},\ \alpha_2,\ \cdots,\ \alpha_{m-1},\ \alpha_m e^{-i\varepsilon}\ .$$

Note that $\arg\alpha_1 + \varepsilon$ need not be $\leqq \arg\alpha_2 - \varepsilon$, and $\arg\alpha_2$ need not be $\leqq \arg\alpha_3 - \varepsilon$, etc.

We may repeat this cycle of $m$ pinches $k-1$ more times to pass from

$$\alpha_1 e^{i\varepsilon},\ \alpha_2,\ \cdots,\ \alpha_{m-1},\ \alpha_m e^{-i\varepsilon} \text{ to } \alpha_1 e^{ki\varepsilon},\ \alpha_2,\ \cdots,\ \alpha_{m-1},\ \alpha_m e^{-ki\varepsilon}$$

as long as $\arg\alpha_1 + k\varepsilon \leqq \arg\alpha_2$, since

$$\arg\alpha_2 + p\varepsilon - \arg\alpha_1 > \arg\alpha_2 - \arg\alpha_1$$

and

$$\pi - (\arg\alpha_n - p\varepsilon - \arg\alpha_{m-1}) > \pi - (\arg\alpha_n - \arg\alpha_{m-1})$$

for $p < k$. Therefore if $\delta \leqq \arg\alpha_2 - \arg\alpha_1$, we need only choose $\varepsilon = \delta/k$, where $k$ is an integer so large that $\delta/k < \eta$. If $\delta > \arg\alpha_2 - \arg\alpha_1$, choose $\varepsilon = (\arg\alpha_2 - \arg\alpha_1)/k$, where $k$ is so large that $\varepsilon < \eta$. Then $(\alpha_1, \cdots, \alpha_m)$ is reduced to $(\alpha_2, \alpha_2, \cdots, \alpha_{m-1}, \alpha_m e^{-ik\varepsilon})$ by the above sequence of pinches. By the induction hypothesis, $(\alpha_2, \alpha_3, \cdots, \alpha_{m-1}, \alpha_m e^{-ik\varepsilon})$ can by a finite number of pinches be reduced to $(\alpha_1 e^{i\delta}, \alpha_3, \cdots, \alpha_{m-1}, \alpha_m e^{-i\delta})$. (The fact that $\alpha_m e^{-ik\varepsilon}$ might be equal to one of the $\alpha_j$ is clearly unimportant.) Therefore $(\alpha_1, \cdots, \alpha_m)$ can be reduced to $(\alpha_1 e^{i\delta}, \alpha_2, \cdots, \alpha_{m-1}, \alpha_m e^{-i\delta})$, and the proof is complete.

## 3. Proof of Theorem 1.

$(2) \to (1)$: This is the statement of Lemma 5.

$(1) \to (3)$: If $(\lambda_i)$, $(\alpha_i)$ is realizable, then by Lemma 1 there exists a matrix $A$ and a non-singular matrix $X$ such that $A = X^* \operatorname{diag}(\alpha_1, \cdots, \alpha_n) X$ and $A$ has eigenvalues $\lambda_i$. Therefore $\prod\lambda_i = \prod\alpha_i \cdot |\det X|^2$ and hence $\prod\lambda_i/|\lambda_i| = \prod\alpha_i$. If the $\alpha_i$ lie on a line through 0, then $(\lambda_i/|\lambda_i|)$ is a rearrangement of $(\alpha_i)$ by Lemmas 2 and 10. If the $\alpha_i$ lie in a closed half plane through 0, then by Lemma 3 we may assume they lie in the upper half plane. By Lemma 9 it follows that $(\arg\lambda_i) \prec (\arg\alpha_i)$.

$(3) \to (2)$: In case (a), the statement is obvious. In case (c), Lemma 11 and the fact that the $\alpha_i$ do not lie in any closed half plane with 0 on its boundary show that the hypotheses of Lemma 12 are satisfied by arrangements of $(\lambda_i/|\lambda_i|)$, $(\alpha_i)$. In case (b), the hypotheses of

Lemma 12 also are satisfied by arrangements of $(\lambda_i/|\lambda_i|)$, $(\alpha_i)$. Thus an application of Lemma 12 completes the proof.

## REFERENCES

1. G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge, 1952.

2. A. Horn, *On the eigenvalues of a matrix with prescribed singular values*, Proc. Amer. Math. Soc., **5** (1954), 4-7.

3. R. R. Stoll, *Linear algebra and matrix theory*, New York, 1952.

4. H. Wielandt, *An extremum property of sums of eigenvalues*, Proc. Amer. Math. Soc., **6** (1944), 106-110.

UNIVERSITY OF CALIFORNIA, LOS ANGELES