THE SUSPENSION OF THE GENERALIZED PONTRJAGIN COHOMOLOGY OPERATIONS

EMERY THOMAS

1. The main theorem. In a previous paper [9] I have defined a sequence of new cohomology operations, called the *generalized Pontrjagin operations*. These operations use as coefficient groups the summands of a certain type of graded ring: namely, *a ring with divided powers* (defined by H. Cartan in [1]), which is termed a Γ-ring in [9]. Let $A = \sum_k A_k$ be a ring with divided powers such that each summand A_k is a cyclic group of infinite or prime power order; we termed such rings *p-cyclic* in [9]. Then, the Pontrjagin operations are functions

$$\mathfrak{P}_t\colon H^{2n}(X;\ A_{2k}) \longrightarrow H^{2tn}(X;\ A_{2tk}) \quad (k,\ n > 0;\ t = 0,\ 1,\ \cdots)$$

where $H^q(Y,\ B;\ G)$ denotes the qth (singular) cohomology group of the pair $(Y,\ B)$ with coefficients in the group G.

Let C be a cohomology operation relative to integers r, s and coefficient groups G, Π. That is, C is a natural transformation

$$C\colon H^r(Y,\ B;\ G) \longrightarrow H^s(Y,\ B;\ \Pi)\ .$$

With each operation C we associate a second operation, $S(C)$, called the *suspension* of C. $S(C)$ is a natural transformation

$$H^{r-1}(Y,\ B;\ G) \longrightarrow H^{s-1}(Y,\ B;\ \Pi)\ ;$$

its definition is given in § 3.

The purpose of this note is to determine $S(\mathfrak{P}_t)$, where $\mathfrak{P}_t$ is the generalized Pontrjagin operation. In order to state our result concerning $S(\mathfrak{P}_t)$, we need an additional cohomology operation, the Postnikov square (see [3], [10]). This was defined in [9], but only for a restricted class of coefficient groups. In this paper we will define the Postnikov square as a cohomology operation

$$\mathfrak{p}\colon H^q(Y,\ B;\ A_{2k}) \longrightarrow H^{2q+1}(Y,\ B;\ A_{4k})\ , \qquad (q,\ k > 0)$$

where A_{2k} is an even summand of a p-cyclic ring with divided powers.

We now may state the main result of the paper.

THEOREM I. *For any cohomology operation C, let $S(C)$ denote the suspension of the operation C. Then,*

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(i) $$S(\mathfrak{P}_2) = \mathfrak{p}$$

(ii) $$S(\mathfrak{P}_t) = 0\ , \qquad (t > 2)$$

where 0 *denotes the zero cohomology operation.*

The proof of Theorem I is given in § 5. In § 2 we define the operation $\mathfrak{p}$, while in § 3 we give the definition of the suspension. In § 4 we discuss relative cohomology operations, while in § 6 we give some additional properties of the operation $\mathfrak{p}$. In particular, we show that $S(\mathfrak{p}) = 0$. Finally, the last section gives the theorem, $\delta S(C) = C\delta$, for any operation C.

I would like to thank Professor N. E. Steenrod for the valuable suggestions made to me at the time of revising the paper. In particular the definition of the suspension in § 3 and Theorem 7.1 are due to him.

2. The definition of the Postnikov square. The definition of the Postnikov square, $\mathfrak{p}$, is obtained by first defining a "model operation", p, which uses only a restricted category $\mathscr{C}$ of coefficient groups. The category $\mathscr{C}$ is defined as follows: let $Z_r = Z/rZ$ $(r = 0, 1, \cdots)$, where Z = integers = Z_0. Denote by $\mathscr{C}$ the category of all groups of the form Z_θ, where θ is zero or a power of a prime. For each group Z_θ in $\mathscr{C}$ we have defined a p-cyclic ring with divided powers,

$G(Z_\theta) = G_0(Z_\theta) + \cdots + G_t(Z_\theta) + \cdots$ (direct sum) (see [9; 1.17]).

In particular,

$$G_2(Z_\theta) = \begin{cases} Z_\theta\ , & \text{if } \theta \text{ is zero or odd} \\ Z_{2\theta}\ , & \text{if } \theta \text{ is a power of 2.} \end{cases}$$

We define a generator for $G_2(Z_\theta)$ by

$$g_2(1_\theta) = \begin{cases} 1_\theta\ , & \text{if } \theta \text{ is zero or odd} \\ 1_{2\theta}\ , & \text{if } \theta \text{ is a power of 2} \end{cases}$$

where $1_r = 1 \bmod r$ $(r = 0, 1, \cdots)$. The group $G_2(Z_\theta)$ will be the coefficient domain for the operation $\mathfrak{p}$. As remarked in [9; § 2], once we have defined the operation $\mathfrak{p}$ for the category of *regular cell complexes*, the definition easily extends to the category of all topological spaces. Hence, in what follows we restrict attention to regular cell complexes, which we will simply term *complexes*.

Let K be a complex and L a subcomplex of K. Let Z_θ be a group in the category $\mathscr{C}$; that is, θ is zero or a power of a prime. We define an operation

$$p\colon H^q(K, L; Z_\theta) \longrightarrow H^{2q+1}(K, L; G_2(Z_\theta))$$

as follows. Let $u \in H^q(K, L; Z_\theta)$; let β be the homomorphism from Z_θ to $G_2(Z_\theta)$ given by $\beta(1_\theta) = \theta g_2(1_\theta)$. Define

$$(2.1)\qquad p(u) = \beta_*(u \cup \delta_* u)\ .$$

Here, δ_* is the Bockstein coboundary operator associated with the exact sequence

$$0 \longrightarrow Z \xrightarrow{\theta} Z \longrightarrow Z_\theta \longrightarrow 0\ ,$$

and the cup-product is taken relative to the natural pairing $Z_\theta \otimes Z \approx Z_\theta$.

It is easily seen that this agrees with the usual definition of the operation p (see [3] and [10]). For let $\bar{u} \in C^q(K, L; Z)$ be a cochain representing u; that is, $\delta\bar{u} = \theta\bar{v}$, for some cochain $\bar{v} \in C^{q+1}(K, L; Z)$. Then, a cocycle representing $\beta_*(u \cup \delta_* u)$ is given by $\bar{u} \cup \delta\bar{u}$, which coincides with the definition given in [10].

In [9; 8.14] we defined a function w which goes from $H^q(K; Z_\theta)$ to $H^{2q+1}(K; Z_\theta)$. This function can be extended to the relative case, following the method given in § 4. When this is done it is easily shown that

$$(2.2)\qquad p(u) = \beta_* w(u)\ ,$$

a result we will need later.

The Postnikov square, $\mathfrak{p}$, is defined using the operation p as follows: let $u \in H^q(K, L; A_{2k})$, where A_{2k} is an even summand of a p-cyclic ring with divided powers. By hypothesis, A_{2k} is a cyclic group whose order is infinite or a power of a prime. Thus, there is an integer θ such that A_{2k} is isomorphic to Z_θ, where $Z_\theta \in \mathscr{C}$. Let ν be an isomorphism from A_{2k} to Z_θ. Then, by 3.1 in [9], for each non-negative integer r we have defined a homomorphism ζ_r mapping $G_r(Z_\theta)$ to A_{2rk}, which is an extension of ν^{-1}. We define the operation $\mathfrak{p}$ by

$$(2.3)\qquad \mathfrak{p}(u) = \zeta_2^* p \nu_*(u)\ ;$$

that is, $\mathfrak{p}$ is the composition of the following functions:

$$H^q(K,\ L;\ A_{2k}) \xrightarrow{\nu_*} H^q(K,\ L;\ Z_\theta) \xrightarrow{p}$$
$$H^{2q+1}(K,\ L;\ G_2(Z_\theta)) \xrightarrow{\zeta_2^*} H^{2q+1}(K,\ L;\ A_{4k})\ .$$

We show the independence of this definition from the particular choice of the isomorphism ν (and hence ζ_2). This is a consequence of the fact that

(2.4) LEMMA. $\qquad p\alpha_* = G_2(\alpha)_* p\ ,$

where α is a homomorphism from Z_θ to a group Z_τ in $\mathscr{C}$, and $G_2(\alpha)$ is the homomorphism from $G_2(Z_\theta)$ to $G_2(Z_\tau)$ induced by the functor G (see [9; 1.23]).

Using 2.2, the proof of 2.4 is entirely similar to that given for 5.22 in [9] and is omitted here. From 2.4 the proof of the independence of the definition of $\mathfrak{p}$ follows along exactly the same lines as 3.5 and 3.6 in [9]; we omit the details.

3. Suspension of cohomology operations. The definition of the suspension used here is due to N. E. Steenrod[1]. Let I denote the unit interval, $[0, 1]$, and $\dot{I}$ the subspace $\{0\} \cup \{1\}$. The group $H^1(I, \dot{I}; Z)$ is cyclic infinite; let v be a fixed generator. For each space X and coefficient group G define a function ϕ from $H^q(X; G)$ to $H^{q+1}(I \times X, \dot{I} \times X; G)$ by

$$\phi(u) = v \times u \ . \tag{3.1}$$

We use singular cohomology for X, and the natural pairing $Z \otimes G \approx G$ for the cross-product. In § 7 we prove the following lemma.

(3.2) LEMMA. *The function ϕ is an isomorphism mapping $H^q(X; G)$ onto $H^{q+1}(I \times X, \dot{I} \times X; G)(q > 0)$.*

Consider now any cohomology operation C, which is defined on relative cohomology groups; say, C maps $H^r(X, A; G)$ to $H^s(X, A; \Pi)$ for each pair (X, A). Define an absolute cohomology operation, $S(C)$, which maps $H^{r-1}(Y; G)$ to $H^{s-1}(Y; \Pi)$, for each space Y, by

$$S(C)(u) = \phi^{-1} C\phi(u) \qquad (u \in H^{r-1}(Y; G)) \ . \tag{3.3}$$

Using the method described in § 4 we may extend $S(C)$ to an operation defined on relative cohomology groups, an operation which we continue to denote by $S(C)$. We wish to apply this construction to the operation $\mathfrak{P}_\iota$; as defined in [9], this is just an absolute operation. Thus, to use Definition 3.3 we must first extend the definition of $\mathfrak{P}_\iota$ to the relative case.

4. Relative cohomology operations. Let $O(q, r; G, \Pi)$ denote the set of absolute cohomology operations relative to dimensions q, r and coefficient groups G, Π; that is, if $C \in O(q, r; G, \Pi)$, then C: $H^q(X; G) \to H^r(X, \Pi)$ for each space X. As is well-known the set $O(q, r; G, \Pi)$ is in 1–1 correspondance with the group $H^r(K; \Pi)$, where K is an Eilenberg-MacLane space of type (G, q). The correspondance is obtained by assigning $C(\iota)$ to ι, where ι is the fundamental class in $H^q(K; G)$. Choose now a base point $e \in K$, and let α^*: $H^*(K, e; \Lambda) \approx H^*(K; \Lambda)$ be the isomorphism induced by the inclusion $K \subset (K, e)$. For any CW-complex X and subcomplex A, the homotopy classes of maps $(X, A) \to (K, e)$

[1] This definition has the advantage that it can be used in the case of cohomology with local coefficients.

are in one-to-one correspondance with $H^q(X, A; G)$. Thus we define a relative cohomology operation, C', associated with an absolute operation, C, as follows:

$$C'(u) = f^*\alpha^{*-1} C(\iota) , \tag{4.1}$$

where $u \in H^q(X, A; G)$ and f is a map $(X, A) \to (K, e)$ such that

$$f^*\alpha^{*-1}(\iota) = u .$$

With the operation C' defined, one is then interested in whether the properties of C extend to the operation C'. We now prove a general lemma which essentially asserts that all the properties of C' do carry over to C'.

Let $O(q_1, \cdots, q_n, r; G_1, \cdots, G_n, \Pi)$ denote the group of absolute cohomology operations, T, in n variables; that is, if $u_i \in H^{q_i}(X; G_i)$ $(i = 1, \cdots, n)$, then, $T(u_1, \cdots, u_n) \in H^r(X; \Pi)$. The operation T extends to a relative operation, T', using the method just given for operations of a single variable. Suppose now we are given absolute cohomology operations

$$\begin{aligned} &C \in O(q_1, \cdots, q_n, r; G_1, \cdots, G_n, \Pi) , \\ &E \in O(s_1, \cdots, s_p, r; \Pi_1, \cdots, \Pi_p, \Pi) , \\ \text{and}\quad &D_i \in O(q_1, \cdots, q_n, s_i; G_1, \cdots, G_n, \Pi_i) \\ &\qquad\qquad (i = 1, 2, \cdots, p). \end{aligned}$$

Let C', E' D'_i, be the corresponding relative operations.

(4.2) PROPOSITION. *Suppose that for each space X and cohomology classes $u_i \in H^{q_i}(X; G_i)(i = 1, \cdots, n)$, we have*

$$C(u_1, \cdots, u_n) = E(D_1(u_1, \cdots, u_n), \cdots, D_p(u_1, \cdots, u_n)) .$$

Then, for each pair (X, A) and classes $u'_i \in H^{q_i}(X, A; G_i)$ $(i = 1, \cdots, n)$, we have

$$C'(u'_1, \cdots, u'_n) = E'(D'_1(u'_1, \cdots, u'_n), \cdots, D'_p(u'_1, \cdots, u'_n)) .$$

We give the proof at the end of this section, first illustrating the theorem by giving several corollaries.

(4.3) COROLLARY 1. *Let $C \in O(q, s; R, S)$, $D_i \in O(q_i, s_i; R, S)$ $(i = 1, 2)$, where R, S are rings, $q = q_1 + q_2$, and $s = s_1 + s_2$. Suppose that*

$$C(u_1 \cup u_2) = D_1(u_1) \cup D_2(u_2)$$

for all classes $u_i \in H^{q_i}(X; R)$. Then,

$$C'(u'_1 \cup u'_2) = D'_1(u'_1) \cup D'_2(u'_2) \,,$$

for all classes $u'_i \in (H^{q_i}(X, A; R)$.

Proof. Let $E_R \in O(q_1, q_2, q; R, R, R)$ and $E_S \in O(s_1, s_2, s; S, S, S)$ be the respective cup-products. Let F be the composite operation $C \circ E_R$. Using Proposition 4.2 we see that $F' = C' \circ E'_R$. But since $F(u_1, u_2) = E_S(D_1(u_1), D_2(u_2))$, again using 4.2 we see that

$$F'(u'_1, u'_2) = E'_S(D'_1(u_1), D_2(u'_2)) \,;$$

that is,

$$C'(u'_1 \cup u'_2) = D'_1(u'_1) \cup D'_2(u'_2) \,,$$

as was to be shown.

Let C, D_1, D_2 be the same operations as in Corollary 1. Then,

(4.4) COROLLARY 2. $C'(u'_1 \times u'_2) = D'_1(u'_1) \times D'_2(u'_2)$, *where* $u_i \in H^{q_i}(X_i, A_i; R)$ $(i = 1, 2)$.

Proof. Let $p_1\colon (X_1 \times X_2, A_1 \times X_2) \to (X_1, A_1)$, $p_2\colon (X_1 \times X_2, X_1 \times A_2) \to (X_2, A_2)$ be projections. Then,

$$u'_1 \times u'_2 = p_1^*(u'_1) \cup p_2^*(u'_2) \,.$$

Thus,

$$\begin{aligned} C'(u'_1 \times u'_2) &= C'(p_1^* u'_1 \cup p_2^* u'_2) = D'_1(p_1^* u'_1) \cup D'_2(p_2^* u'_2) \\ &= p_1^*(D'_1 u'_1) \cup p_2^*(D'_2 u'_2) = (D'_1 u'_1) \times (D'_2 u'_2) \,. \end{aligned}$$

Here we have used Corollary 1 and the naturality of the cohomology operations involved.

To apply this to the operations $\mathfrak{P}_t$, recall the way in which these operations were defined (see § 3 in [9]). We defined a set of "model operations", P_t, which used as coefficient groups only the groups of the category $\mathscr{C}$ (see § 2). The operations $\mathfrak{P}_t$ were then defined by composing the operation P_t with coefficient group homomorphisms; that is, precisely the same pattern as followed in Definition 2.3. Thus, the operations $\mathfrak{P}_t$ are defined in the relative case by simply applying the method given in this section to the operations P_t.

Let P'_t be the relative operation obtained from P_t. We note several facts needed later.

(4.5) LEMMA. *Let* $u_i \in H^{q_i}(X_i, A_i; Z_\theta)$ $(i = 1, 2)$, *where* $Z_\theta \in \mathscr{C}$. *Then*

(1) $$P'_t(u_1 \times u_2) = P'_t(u_1) \times P'_t(u_2) \ (t \text{ odd}) \,.$$

If $t=2$ *and* θ *is a power of* 2, *then,*

$$(2) \qquad P'_2(u_1 \times u_2) = P'_2(u_1) \times P'_2(u_2) + \nu_*[Sq_1(u_1) \times \mu_* w(u_2) + \mu_* w(u_1) \times Sq_1(u_2)] .$$

Here, ν is the homomorphism of Z_2 to $G_2(Z_\theta)$ given by $\nu(1_2) = \theta g_2(1_\theta)$, and μ is the factor homomorphism $Z_v \to Z_2$. The functions Sq and w are defined respectively in 9.6 and 8.14 of [9].

Proof. The first statement is a consequence of Corollary 4.3 and the fact that the absolute operations P_t satisfy this formula[2]. Equation 4.5(2) was remarked in [9; § 13] for the absolute operations P_t, and the case dim u_i odd. But it follows from 8.12 in [9] that 4.5(2) holds in general. In fact Theorem 8.11 in [9] can be obtained at once from equation 4.5(2). The extension of the equation to the relative operation P'_t, follows then from application of Proposition 4.2.

Combining Proposition 4.2 and 8.2 of [9] we also obtain

(4.6) LEMMA. *Let t be an an integer where $t = p_k \cdots p_1$ (p_i prime). Let $u \in H^{2q}(X, A; Z)$. ($Z_t \in \mathscr{C}$). Then,*

$$P'_t(u) = P'_{p_k} \circ \cdots \circ P'_{p_1}(u).$$

Since it is in fact the relative operation, P'_t, we will work with, from now on we drop the prime, writing only P_t for both the relative and absolute operation.

Proof of Proposition 4.2. Let $Y = K(G_1, q_1) \times \cdots \times K(G_n, q_n)$, where each $K(G_i, q_i)$ is on Eilenberg-MacLane space of type (G_i, q_i). Let π_j: $Y \to K(G_j, q_j)$ $(j = 1, \cdots, n)$, be the projection map and set $\bar{\iota}_j = \pi_j^*(\iota_j)$, where ι_j is the characteristic class in $H^{q_j}(K(G_j, q_j); G_j)$. Let e_j be a base point in $K(G_j, q_j)$ and set $Y' = (K(G_1, q_1), e_1) \times \cdots \times (K(G_n, q_n), e_n)$. Let ι'_j, $\bar{\iota}'_j$ be the equivalent of ι_j and $\bar{\iota}_j$. Then, Proposition 4.2 follows at once from the following three lemmas (we keep the same notation as used in Proposition 4.2)

$$(4.7) \qquad C(u_1, \cdots, u_n) = E(D_1(u_1, \cdots, u_n), \cdots, D_p(u_1, \cdots, u_n))$$

if and only if

$$C(\bar{\iota}_1, \cdots, \bar{\iota}_n) = E(D_1(\bar{\iota}_1, \cdots, \bar{\iota}_n), \cdots, D_p(\bar{\iota}, \cdots, \bar{\iota}_n)) .$$

$$(4.8) \qquad C'(u'_1, \cdots, u'_n) = E'(D'_1(u'_1, \cdots, u'_n), \cdots, D'_p(u'_1, \cdots, u'_n))$$

if and only if

$$C'(\bar{\iota}'_1, \cdots, \bar{\iota}'_n) = E'(D'_1(\bar{\iota}_1, \cdots, \bar{\iota}'_n), \cdots, D'_p(\bar{\iota}'_1, \cdots, \bar{\iota}'_n))$$

[2] The operations $\mathfrak{P}_t$ are easily defined for odd dimensional classes: see [**9**; § 7].

(4.9) *If* $C(\bar{\iota}_1, \cdots, \bar{\iota}_n) = E(D_1(\bar{\iota}, \cdots, \bar{\iota}_n), \cdots, D_p(\bar{\iota}_1, \cdots, \bar{\iota}_n))$

then,

$$C'(\bar{\iota}'_1, \cdots, \bar{\iota}'_n) = E'(D'_1(\bar{\iota}'_1, \cdots, \bar{\iota}'_n), \cdots, D'_p(\bar{\iota}'_1, \cdots, \bar{\iota}'_n)) .$$

We give only the proof of Lemma 4.7, the others being entirely similar. Assume first we are given classes $u_i \in H^{q_i}(X; G_i)$ $(i = 1, \cdots, n)$. Let f_j: $X \to K(G_j;\ q_j)$ be mappings such that $f_j^*(\iota_j) = u_j$. Set $f = f_1 \times \cdots \times f_n$: $X \to Y$. Then, by naturality, one has

(4.10) (a) $\quad C(u_1, \cdots, u_n) = f^*C(\bar{\iota}_1, \cdots, \bar{\iota}_n)$,

(b) $\quad D_i(u_1, \cdots, u_n) = f^*D_i(\bar{\iota}_1, \cdots, \bar{\iota}_n) \qquad (i = 1, \cdots, p).$

Suppose now that

$$C(\bar{\iota}_1, \cdots, \bar{\iota}_n) = E(D_1(\bar{\iota}_1, \cdots, \bar{\iota}_n), \cdots, D_p(\bar{\iota}_1, \cdots, \bar{\iota}_n)) .$$

Then, by 4.10,

$$C(u_1, \cdots, u_n) = f^*E(D_1(\bar{\iota}_1, \cdots, \bar{\iota}_n), \cdots, D_p(\bar{\iota}_1, \cdots, \bar{\iota}_n)) .$$

But E is natural with respect to mappings. Therefore,

$$\begin{aligned} & f^*E(D_1(\bar{\iota}_1, \cdots, \bar{\iota}_n), \cdots, D_p(\bar{\iota}_1, \cdots, \bar{\iota}_n)) \\ &= E(f^*D_1(\bar{\iota}_1, \cdots, \bar{\iota}_n), \cdots, f^*D_p(\bar{\iota}_1, \cdots, \bar{\iota}_n)) \\ &= E(D_1(u_1, \cdots, u_n), \cdots, D_p(u_1, \cdots, u_n)) , \end{aligned}$$

again by 4.10, which completes the proof of this assertion. The proof in the opposite direction is trivial.

5. The proof of Theorem I. Recall that the operation $\mathfrak{P}_t$ is defined by means of the model operations P_t and coefficient group homomorphisms. But it is clear that the isomorphism ϕ, defined in 3.1, commutes with coefficient group homomorphisms. Thus, it suffices to prove Theorem I with $\mathfrak{P}_t$ replaced by P_t, the operation $\mathfrak{p}$ replaced by p, and the group A_{2k} taken to be a group in the category $\mathscr{C}$, say $A_{2k} = Z_\theta$.

Assume first that t is an odd prime p. Since ϕ is an isomorphism, the proof of Theorem I (ii) consists simply in showing

$$P_p\phi(u) = 0 , \qquad u \in H^r(X;\ Z_\theta).$$

But this is immediate; for

$$P_p\phi(u) = P_p(v \times u) = P_p(\bar{v} \times u) = P_p(\bar{v}) \times P_p(u) ,$$

by Lemma 4.5(1). Here, $\bar{v}$ is a generator of $H^1(I,\ \dot{I};\ Z_3)$. However, $P_p(\bar{v}) = 0$, by dimensionality considerations. Thus, $P_p\phi(u) = 0$; and hence, $S(P_p) = 0$.

Now, suppose that t is any integer > 1 which is not a power of 2; say, $t = mp$, where p is an odd prime. Then, by Lemma 4.6

$$P_t\phi(u) = P_m \circ P_p\phi(u) = P_m(0) = 0 .$$

Consequently,

$$S(P_t) = 0 .$$

Thus, we have proved Theorem I(ii) for the case t is not a power of 2. Before concluding the proof of part (ii), we must prove part (i). Let the classes u and v be as above, where u has coefficients in the group Z_θ. If θ is zero or odd, then by Proposition 7.4 in [9], we have

$$P_2(v \times u) = P_2(\bar{v} \times u) = (\bar{v} \times u)^2 = \pm \bar{v}^2 \times u^2 = 0 ,$$

since $\bar{v}^2 = 0$. Thus, in this case $S(P_2) = 0$. Suppose now that θ is a power of 2.

Let η be the factor map $Z \to Z_\theta$. Then, $v \times u = (\eta_* v) \times u$, where the right hand side uses the pairing $Z_\theta \otimes Z_\theta \approx Z_\theta$. Thus, using Lemma 4.5(2), we have

$$\begin{aligned} P_2(v \times u) = P_2(\eta_* v \times u) &= P_2(\eta_* v) \times P_2(u) \\ &+ \nu_*[Sq_1(\eta_* v) \times \mu_* w(u) + \mu_* w(\eta_* v) \times Sq_1(u)] . \end{aligned}$$

Now, $P_2(\eta_* v) = 0$, $w(\eta_* v) = 0$ by dimensionality considerations. Also, since $\eta_* v$ is a 1-dimensional class, $Sq_1(\eta_* v) = \xi_* v$, where ξ is the natural map $Z \to Z_2$ (see Steenrod [4; 12.6]). Thus,

$$P_2(v \times u) = \nu_*[\xi_* v \times \mu_* w(u)] . \tag{5.1}$$

Consider the following commutative diagram:

$$\begin{tikzcd} Z \otimes Z_\theta \arrow[r, "1\otimes\beta"] \arrow[d, "\xi\otimes\mu"'] & Z \otimes G_2(Z_\theta) \arrow[d, "\approx"', "\omega"] \\ Z_2 \otimes Z_2 \overset{\omega'}{\approx} Z_2 \arrow[r, "\nu"] & G_2(Z_\theta) \end{tikzcd} ,$$

where β is the homomorphism of Z_θ to $G_2(Z_\theta)$ given by $\beta(1_\theta) = \theta g_2(1_\theta)$ (see 2.1). Then, from 5.1,

$$\begin{aligned} P_2(v \times u) &= \nu_* \omega'_* (\xi \otimes \mu)_* [v \otimes w(u)] \\ &= \omega_*(1 \otimes \beta)_*[v \otimes w(u)] \\ &= v \times \beta_* w(u) \\ &= v \times p(u) , \text{ by } 2.2 . \end{aligned}$$

Therefore,

$$P_2\phi(u) = P_2(v \times u) = v \times p(u) = \phi p(u) .$$

That is,

$$S(P_2) = p .$$

This proves part (i) of Theorem I. To complete the proof of the theorem we must show that

$$P_{2^r}\phi(u) = 0 , \qquad (r > 1).$$

But by part (i) of Theorem I and Lemma 4.6, we have

$$\begin{aligned} P_{2^r}\phi(u) &= P_{2^{r-1}} P_2\phi(u) = P_{2^{r-1}} \phi p(u) \\ &= P_{2^{r-2}} P_2\phi p(u) = P_{2^{r-2}} \phi p(p(u)) = 0 . \end{aligned}$$

Here, we use property 6.6 of the function p, which is proved independently in the next section. This completes the proof of Theorem I.

6. The properties of the operation $\mathfrak{p}$. We give here the main properties of the Postnikov square, $\mathfrak{p}$.

(6.1) THEOREM. *Let X be a space, and let $A = \sum_k A_k$ be a p-cyclic ring with divided powers. Suppose that $u \in H^q(X; A_{2k})$ $(q, k > 0)$. Then,*[3]

(6.2) $\qquad \mathfrak{p}(u) = 0$, *if order A_{2k} is odd or infinite,*

(6.3) $$2\mathfrak{p}(u) = 0 ,$$

(6.4) $\qquad \mathfrak{p}$ *is a homomorphism,*

(6.5) $\qquad$ *if order $A_{2k} = 2^i$ $(i > 1)$ and $2u = 0$, then $\mathfrak{p}(u) = 0$,*

(6.6) $$\mathfrak{p}(\mathfrak{p}(u)) = 0 ,$$

(6.7) $$f^*\mathfrak{p}(u) = \mathfrak{p}f^*(u) ,$$

(6.8) $$\alpha_*\mathfrak{p}(u) = \mathfrak{p}\alpha_*(u) ,$$

where f^* is induced by a map f from a space Y to X, and α_* is induced by a homomorphism α from A to a p-cyclic ring with divided powers A'.

The proof of Theorem 6.1 falls into 2 parts. Suppose first that we have proved 6.2 through 6.7 with the operation $\mathfrak{p}$ replaced by the operation p, and the coefficient group A_{2k} restricted to be a group in the category $\mathscr{C}$. Then, the proof of 6.2–6.7 for the general case of the

[3] With the exception of 6.5 and 6.6, these properties are noted by J. H. C. Whitehead in [**10**].

function $\mathfrak{p}$ follows at once, using definition 2.3; that is, $\mathfrak{p} = \zeta_2{}^* p \nu_*$. In particular, 6.2–6.5 are simple consequences of the fact that $\zeta_2{}^*$ and ν_* are homomorphisms; 6.6 follows from 6.3 and 6.5, and 6.7 follows from the fact that f^* commutes with all coefficient group homomorphisms. Finally, to prove 6.8 for the operation $\mathfrak{p}$, one uses 2.4 and exactly the same argument as that used to prove I(9) in § 4 of [9]. Thus, we are left with proving 6.2 through 6.7 for the operation p. Let $u \in H^q(K; Z)$, where $Z_\theta \in \mathscr{C}$. Then,

(i) $$p(u) = 0\ , \quad \textit{if } \theta \textit{ is zero or odd.}$$

This follows at once from 2.1. For if θ is zero or odd, the homomorphism β is zero.

(ii) $$2p(u) = 0$$

This again is immediate from 2.1; for it is always the case that $2\beta = 0$.

(iii) $$p \textit{ is a homomorphism}$$

In § 5 we showed that the operation p is the suspension of the operation P_2. But by 7.4 in [6], all operations which are suspensions are homomorphisms.

(iv) $$\textit{If } \theta = 2^i\ (i > 1), \textit{ and } 2u = 0, \textit{ then } p(u) = 0.$$

Since $2u = 0$, we may use Lemma 13.3 of [9]: namely, there are classes $x \in H^{q-1}(K; Z_2)$ and $y \in H^q(K; Z_2)$ such that

$$u = \lambda_* \delta_*(x) + \nu_*(y)\ ,$$

where δ_* is the coboundary associated with the exact sequence

$$0 \longrightarrow Z \xrightarrow{2} Z \longrightarrow Z_2 \longrightarrow 0\ ,$$

λ is the natural factor map $Z \to Z_\theta$, and ν maps Z_2 to Z_θ by $\nu(1_2) = (\theta/2)1_\theta$ (recall that $\theta = 2^i$, $i > 1$). Hence, by (iii) above,

$$\begin{aligned} p(u) &= p\lambda_*\delta_*(x) + p\nu_*(y) \\ &= G_2(\lambda)_* p\delta_*(x) + G_2(\nu)_* p(y) \\ &= G_2(\nu)_* p(y)\ , \end{aligned}$$

by 2.4 and (i) above, since $\delta_*(u)$ has integer coefficients. Now,

$$G_2(\nu)_* p(y) = G_2(\nu)_* \beta_* w(u)\ ,$$

by 2.2. We show that $p(u) = 0$ by showing that

$$G_2(\nu)\beta = 0\ .$$

From Definition 2.1 we recall that β maps Z_2 to $G_2(Z_2)$ by $\beta(1_2) = 2g_2(1_2)$. Hence, using 1.21 and 1.24 in [9],

$$\begin{aligned} G_2(\nu)\beta(1_2) &= 2G_2(\nu)g_2(1_2) = 2g_2(\nu 1_2) \\ &= 2g_2((\theta/2)1_\theta) = 2(\theta^2/4)g_2(1\ ) = (\theta^2/2)1_{2\theta} = 0 \ . \end{aligned}$$

For, $\theta^2/2 = 2^{2i}/2 = 2^{2i-1}$; and, $2\theta = 2^{i+1}$. But by hypothesis, $i \geqq 2$; thus $2i - 1 \geqq i + 1$.

(v) $$p(p(u)) = 0$$

This follows at once from (ii) and (iv) above.

(vi) $$f^*p(u) = pf^*(u) \ .$$

This is simply a special case of Theorem 3.6 of [7]. This, then completes the proof of Theorem 6.1.

We consider one more property of the operation $\mathfrak{p}$: namely, its behaviour with respect to suspension. We continue to denote by $S(C)$ the suspension of a cohomology operation C.

(6.9) PROPOSITION. *$S(\mathfrak{p}) = 0$, where 0 denotes the trivial cohomology operation.*

Proof. By the same reasoning as given in § 5, it sufficies to prove Proposition 6.9 with $\mathfrak{p}$ replaced by the operation p, and the coefficient group A_{2k} taken to be a group in the category $\mathscr{C}$, say $A_{2k} = Z$. Thus, we need simply show that $p\phi(u) = 0$, where $u \in H^q(L; Z_\theta)$. Now by Nakaoka [2] we have[4]:

$$p(v_1 \times v_2) = P_2(v_1) \times p(v_2) + p(v_1) \times P_2(v_2) \ ,$$

for classes $v_i \in H^{q_i}(X_i, A_i; Z_\theta)$ $(i = 1, 2)$.
Thus,

$$p\phi(u) = p(\bar{v} \times u) = P_2(\bar{v}) \times p(u) + p(\bar{v}) \times P_2(u) = 0 \ ,$$

since $P_2(\bar{v}) = p(\bar{v}) = 0$ by dimensionality considerations. Here, $\bar{v}$ is the image of v in $H^1(I, \dot{I}; Z_\theta)$. Hence, $S(p) = 0$, as was to be proved.

7. The relation $\delta S(C) = C\delta$. We give here a theorem, whose proof is due to N. E. Steenrod.

(7.1) THEOREM. *Let C be a cohomology operation, and let δ be the relative cohomology coboundary operator. Then,*

[4] Nakaoka only proves this for the case dim v_1, v_2 even; but the result is true in general, as is easily shown using Definition 2.1.

$$\delta S(C) = C\delta ,$$

where $S(C)$ *is the suspension of* C.

We sketch the proof; let X be a space and $A \subset X$ a subspace. Let X' denote the mapping cylinder of the inclusion map $A \subset X$. That is, unite $I \times A$ and X by identifying $1 \times A$ with A in X. Let $A' = 0 \times A$. The inclusions

$$(X',\ A') \longrightarrow (X',\ I \times A) \longleftarrow (X,\ A)$$

induce isomorphisms of the cohomology sequence of $(X,\ A)$ and $(X',\ A')$ with local cosfficients. Thus, we may discuss the behaviour of the coboundary δ in the cohomology sequence of the pair $(X',\ A')$.

Consider the following hexagonal diagram (see [8], page 42):

$$\tag{7.2}\begin{tikzcd}
 & & H^q(I \times X) \arrow[dl, "n_1^*"'] \arrow[dr, "n_0^*"] \arrow[d, "j^*"] & & \\
 & H^q(0 \times X) & & (H^q(1 \times X) & \\
 & & H^q(\dot I \times X) \arrow[ul, "d_1^*"] \arrow[ur, "d_0^*"'] \arrow[dd, "\delta"] & & \\
H^q(\dot I \times X,\ 1 \times X) \arrow[uur, "k_1^*"] \arrow[urr, "i_1^*"] \arrow[drr, "\delta_1"] & & & & H^q(\dot I \times X,\ 0 \times X) \arrow[uul, "k_0^*"'] \arrow[ull, "i_0^*"'] \arrow[dll, "\delta_0"'] \\
 & & H^{p+}(I \times X,\ \dot I \times X) & &
\end{tikzcd}$$

Here all homomorphisms other than δ, δ_1, and δ_2 are induced by inclusions. Standard arguments, using exactness and homotopy equivalence, show that the arrows around the peripheries are isomorphisms. We agree to identify $H^q(X)$ with $H^q(0 \times X)$ by sending $u \rightarrow e \times u$, where e is the unit of $H^0(0;\ Z)$. At the end of this section we will use diagram 7.2 to prove the following lemma:

(7.3) LEMMA. *Let* ϕ *be the function defined in* 3.1. *Then,*

$$\phi = \delta_1 k_1^{*-1} ,$$

where k_1^*, δ_1 *are the functions defined in diagram* 7.2

Notice that this proves Lemma 3.2; for the functions δ_1, k_1^* are isomorphisms. Now let $g^*\colon H^{q+1}(X',\ A' \cup X) \rightarrow H^{q+1}(I \times A,\ \dot I \times A)$ be induced by the inclusion. Using the fact that $\dot I$ is a strong deformation retract of a neighborhood of $\dot I$ in I (see [8]; Chapter 1, 11.6), together with excision, one shows that g^* is an isomorhism onto.

(7.4) LEMMA. *The following diagram is commutative, where* f^* *is induced by the inclusion*

$$\begin{array}{ccc} H^{q+1}(I \times A,\ \dot{I} \times A) & \xrightarrow{g^{*-1}} & H^{q+1}(X',\ A' \cup X) \\ \uparrow{\scriptstyle\phi} & & \downarrow{\scriptstyle f^*} \\ H^q(A') & \xrightarrow{\delta} & H^{q+1}(X',\ A') \ . \end{array}$$

Thus $\delta = f^* g^{*-1}\phi$.

This is a consequence of Lemma 7.3 and commutativity relations in a slightly enlarged diagram. We omit the details.

The proof of Theorem 7.1 is an immediate consequence of Lemma 7.4. For let $u \in H^q(A')$. Then, by this lemma,

$$C\delta(u) = Cf^* g^{*-1}\phi(u) \ .$$

Using the naturality of the operation C, we have

$$Cf^* g^{*-1}\phi(u) = f^* g^{*-1} C\phi(u) \ .$$

But by Definition 3.1, $C\phi = \phi S(C)$.
Thus,

$$C\delta(u) = f^* g^{*-1}\phi S(C)(u) = \delta S(C)(u) \ ,$$

again using Lemma 7.4. This completes the proof of Theorem 7.1.

Proof of Lemma 7.3. We apply diagram 7.2 to the case $X = \varnothing$, $q = 0$, and coefficient group = integers. Then, the unit class of $H^0(\dot{I}; Z)$ can be represented as a sum $v_0 + v_1$, where

$$v_0 = i_1^* k_1^{*-1} d_1^*(v_0 + v_1),\ \ v_1 = i_0^* k_0^{*-1} d_0^*(v_0 + v_1).$$

Thus,

$\delta(v_0) = -\ \delta(v_1) = v =$ a generator of $H^1(I,\ \dot{I};\ Z)$. Therefore, by Definition 3.1,

$$\phi(u) = v \times u = (\delta v_0) \times u \ .$$

But by the axioms for the cross-product, we may write

$$(\delta v_0) \times u = \delta(v_0 \times u) \ .$$

Furthermore, we have

$$v_0 = i_1^* k_1^{*-1}(e) \ ,$$

where $e = d_1^*(v_0 + v_1) =$ unit of $H^0(0;\ Z)$. Thus,

$$\begin{aligned} \delta(v_0 \times u) &= \delta(i_1^* k_1^{*-1}(e) \times u) \\ &= \delta i_1^* k_1^{*-1}(e \times u) = \delta_1 k_1^{*-1}(e \times u) \ . \end{aligned}$$

Here we have used the naturality of the cross-product and the commutativity of diagram 7.2. If we now identify $H^q(X)$ with $H^q(0 \times X)$ by sending $u \rightarrow e \times u$, we then have

$$\phi(u) = \delta(v_0 \times u) = \delta_1 k_1^{*-1}(u) ,$$

as was asserted.

REFERENCES

1. H. Cartan, *Seminaire* H. Cartan 1954/55, Paris (mimeographed).
2. M. Nakaoka, *Note on cohomological operations*, J. Inst. Polytech., Osaka City Univ., 4 (1953), 51-58.
3. M. Postnikov, *The classification of continuous mappings of a 3-dimensional polyhedron iuto a simply connected polyhedron of arbitrary dimension*, C. R. (Doklady) Acad. Sci. U. R. S. S., **64**(1949), 461-462.
4. N. Steenrod, *Products of cocycles and extensions of mappings*, Ann. Math., **48** (1947), 290-320.
5. ———, *Homology groups of symmetric groups and reduced operations*, Proc. Nat. Acad. Sci. U. S. A., **39** (1953), 213-223.
6. ———, *Cohomology operations*, Symposium Int. Topologie algebraica, Univ. de Mexico (1957), (to appear).
7. ———, *Cohomology operations derived from the symmetric group*, Comment. Math. Helv., **31** (1957), 195-218.
8. N. Steenrod and E. Eilenberg, *Foundations of Algebraic Topology*, Princeton University Press, 1952.
9. E. Thomas, *The generalized Pontrjagin cohomology operations and rings with divided powers*, Memoir Number 27 A. M. S. (1957).
10. J. H. C. Whitehead, *On the theory of obstructions*, Ann. Math., **54** (1951), 68-84.

UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA