THE NILPOTENT PART OF A SPECTRAL OPERATOR

CHARLES A. MCCARTHY

1. Introduction. Throughout this paper, $\mathfrak{X}$ is a Banach space, T a bounded spectral operator on $\mathfrak{X}$ with scalar part S, nilpotent part N, and resolution of the identity $E(\sigma)$ for σ a Borel set in the complex plane. M is the bound for the norms of the $E(\sigma)$; $|E(\sigma)| \leq M$ for all Borel sets σ. The resolvent function for $T, (\lambda - T)^{-1}$, is denoted by $R(\lambda, T)$. The operator $R(\lambda, T)E(\sigma)$ has an unique analytic extension from the resolvent set of T to the complement of $\bar{\sigma}$, and on the subspace $E(\sigma)\mathfrak{X}$ it is equal to the operator $R(\lambda, T_\sigma)$ where T_σ is the restriction of T to $E(\sigma)\mathfrak{X}$. For material on spectral operators, we refer to the papers on N. Dunford [1], [2]. $\chi_\sigma(\xi)$ is the characteristic function of the Borel set σ: $\chi_\sigma(\xi) = 1$ if $\xi \in \sigma, \chi_\sigma(\xi) = 0$ if $\xi \notin \sigma$. For p a non-negative real number, μ_p is Hausdorff p-dimensional measure [3, pp. 102 *ff.*]; μ_2 is Lebesgue planar measure multiplied by $\pi/4$, and μ_1 restricted to an arc is majorized by arc length.

We assume throughout that there is an integer m for which the resolvent function for T satisfies the mth order rate of growth condition

$$|R(\lambda, T)E(\sigma)| \leq K \cdot d(\lambda, \sigma)^{-m}, \lambda \notin \bar{\sigma}, |\lambda| \leq |T| + 1 ,$$

where $d(\lambda, \sigma)$ is the distance from λ to σ and K is a constant independent of σ. If $\mathfrak{X}$ is Hilbert space, it is known that this growth condition implies $N^m = 0$ [1, p. 337]. In an arbitrary Banach space, this is no longer true; the best that can be done is $N^{m+2} = 0$. If $\mathfrak{X}$ is weakly complete, $N^{m+1}=0$; or if σ is a set of μ_2 measure zero, $N^{m+1}E(\sigma) = 0$. If σ lies in an arc and either $\mathfrak{X}$ is weakly complete or σ has μ_1 measure zero, then $N^m E(\sigma) = 0$. Examples show that we cannot obtain lower indices of nilpotency in general.

2. The fundamental lemma and some easy consequences. If $f(\xi)$ is a bounded, scalar valued Borel function, the operator $\int f(\xi)E(d\xi)$ exists as a bounded operator with norm at most $4M \cdot \sup_\xi |f(\xi)|$ [1, p. 341], so that uniform convergence of a sequence of bounded Borel functions $f_n(\xi)$ implies convergence in the uniform operator topology of the operators $\int f_n(\xi)E(d\xi)$. Thus for a given bounded Borel function $f(\xi)$ and a given positive number η, there exist a finite number of disjoint Borel sets σ_i and points $\xi_i \in \sigma_i$ such that

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$$\left| \int f(\xi)E(d\xi) - \sum_i f(\xi_i)E(\sigma_i) \right| < \eta .$$

Similiarly if A_n are a finite number of bounded operators and $f_n(\xi)$ are bounded Borel functions, for any positive number η, there exist a finite number of disjoint Borel sets σ_i and points $\xi_i \in \sigma_i$ such that

$$\left| \sum_n \int A_n f_n(\xi)E(d\xi) - \sum_i \sum_n A_n f_n(\xi_i)E(\sigma_i) \right| < \eta ;$$

in particular, for an integer k and a positive number η, there exist a finite number of disjoint Borel sets σ_i and points $\xi_i \in \sigma_i$ such that

$$\left| \int (T - \xi)^k E(d\xi) - \sum_i (T - \xi_i)^k E(\sigma_i) \right| < \eta .$$

LEMMA 2.1. *There exist constants M_k such that $|N^k E(\sigma)| \leq M_k \varepsilon^{k+1-m}$ for any choice of ε, $0 < \varepsilon \leq 1$, and Borel set σ of diameter no greater than ε.*

Proof. Pick ε, $0 < \varepsilon \leq 1$, and let σ be any Borel set of diameter no greater than ε. We have [1, p, 338]

$$N^k E(\sigma) = \int_\sigma (T - \xi)^k E(d\xi) .$$

For any positive number η, there is a decomposition of σ into a finite number of disjoint Borel sets $\sigma_i \subset \sigma$, and points $\xi_i \in \sigma_i$ such that

$$\left| \int (T - \xi)^k E(d\xi) - \sum_i (T - \xi_i)^k E(\sigma_i) \right| < \eta .$$

Since σ is of diameter at most ε, there is a circle Γ of diameter 3ε which encloses σ and for which $|\gamma - \xi| \geq \varepsilon$ for all $\gamma \in \Gamma$ and $\xi \in \sigma$. Then

$$(T - \xi_i)^k E(\sigma_i) = \frac{1}{2\pi i} \int_\Gamma (\gamma - \xi\,)^k R(\gamma, T)E(\sigma_i)d\gamma ,$$

so that

$$\sum_i (T - \xi_i)^k E(\sigma_i) = \frac{1}{2\pi i} \int_\Gamma R(\gamma, T) \sum_i (\gamma - \xi_i)^k E(\sigma_i)d\gamma ,$$

which in norm in no greater than

$$(*) \qquad \frac{1}{2\pi} \cdot \sup_{\gamma \in \Gamma} |R(\gamma, T)E(\sigma)| \cdot \sup_{\gamma \in \Gamma} \left| \sum_i (\gamma - \xi_i)^k E(\sigma_i) \right| \cdot \text{length of } \Gamma .$$

The mth order rate of growth condition gives

$$\sup_{\gamma \in \Gamma} |R(\gamma, T)E(\sigma)| \leq K\varepsilon^{-m} .$$

For any $\gamma \in \Gamma$,

$$|\textstyle\sum_i (\gamma - \xi_i)^k E(\sigma_i)| \leq 4M \cdot \max_i |\gamma - \xi_i|^k \leq 4M(2\varepsilon)^k ,$$

so that (*) is no greater than

$$\frac{1}{2\pi} K\varepsilon^{-m} \cdot 4M(2\varepsilon)^k \cdot 6\pi\varepsilon = M_k \varepsilon^{k+1-m} ,$$

where $M_k = 3 \cdot 2^{k+2}KM$, and is independent of η, ε, σ, and the manner in which σ is decomposed. Thus

$$|N^k E(\sigma)| \leq M_k \varepsilon^{k+1-m} + \eta$$

for every positive η, which proves the lemma.

THEOREM 2.2. *Let σ be a Borel set whose Hausdorff p-measure is zero for a given p. Then $N^k E(\sigma) = 0$ where k is an integer and $k \geq p + m - 1$.*

Proof. Since σ has p-measure zero, for every $\varepsilon > 0$, there is a covering of σ by disjoint sets σ_i of diameter ε_i such that $\sum_i \varepsilon_i^p < \varepsilon$. By Lemma 2.1 we have

$$\begin{aligned} |N^k E(\sigma)| &\leq \textstyle\sum_i |N^k E(\sigma_i)| \leq M_k \sum_i \varepsilon_i^{k+1-m} \\ &\leq M_k \textstyle\sum_i \varepsilon_i^{(p+m-1)+1-m} \leq M_k \sum_i \varepsilon_i^p \leq M_k \varepsilon . \end{aligned}$$

Since ε may be chosen arbitrarily small, $N^k E(\sigma) = 0$.

COROLLARY 2.3. $N^{m+2} = 0$.

Proof. Taking σ to be the spectrum of T and $p = 3$, $N^{m+2}E(\sigma(T)) = 0$; but $E(\sigma(T))$ is the identity mapping on $\mathfrak{X}$.

COROLLARY 2.4. *If σ has planar measure zero, then $N^{m+1}E(\sigma) = 0$.*

COROLLARY 2.5. *If σ has μ_1-measure zero, then $N^m E(\sigma) = 0$.*

3. **The case of weakly complete $\mathfrak{X}$.** Let σ be a Borel set in the plane. For any $\varepsilon > 0$, we can cover σ with disjoint Borel sets. σ_i of diameter ε_i, $\varepsilon_i \leq 1$, such that

$$\textstyle\sum_i \varepsilon_i^2 \leq \mu_2(\sigma) + \varepsilon .$$

Thus by Lemma 2.1,

$$|N^{m+1}E(\sigma)| \leq \sum_i |N^{m+1}E(\sigma_i)| \leq M_{m+1} \sum_i \varepsilon_i^2$$
$$\leq M_{m+1}(\mu_2(\sigma) + \varepsilon) .$$

Since ε and σ are arbitrary, we have for all Borel sets σ,

$$|N^{m+1}E(\sigma)| \leq M_{m+1}\mu_2(\sigma) .$$

As a consequence, all the scalar measures $x^*N^{m+1}E(\cdot)x = [(N^*)^{m+1}E^*(\cdot)x^*]x$, $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$, are absolutely continuous with respect to μ_2, and have derivative bounded by $M_{m+1}|x^*||x|$.

Suppose that $f(\xi) = \sum_{p=1}^{P} \alpha_p \chi_{\sigma_p}(\xi)$ is a simple Borel function; α_p are scalar constants and σ_p are disjoint Borel sets. We have

$$\left|\int f(\xi)(N^*)^{m+1}E^*(d\xi)\right| \leq \sum_{p=1}^{P} |\alpha_p(N^*)^{m+1}E^*(\sigma_p)|$$
$$\leq \sum_{p=1}^{P} |\alpha_p| M_{m+1}\mu_2(\sigma_p)$$
$$= M_{m+1}|f|_{L_1(\mu_2)} .$$

Thus if $f_n(\xi)$ are simple Borel functions converging in $L_1(\mu_2)$ to $f(\xi)$, the operators $\int f_n(\xi)(N^*)^{m+1}E^*(d\xi)$ converge in the uniform operator topology to an operator which we denote by $\int f(\xi)(N^*)^{m+1}E^*(d\xi)$; this limit operator has norm bounded by $M_{m+1}|f|L_1(\mu_2)$.

THEOREM 3.1. *If $\mathfrak{X}$ is weakly complete, then $N^{m+1} = 0$.*

Proof. Assume that $N^{m+1} \neq 0$, so that also $(N^*)^{m+1} \neq 0$. We will first obtain a bicontinuous map of an infinite dimensional L_1 space into $\mathfrak{X}^*$. An analogous map into $\mathfrak{X}$ would show then that $\mathfrak{X}$ cannot be reflexive, since the image in $\mathfrak{X}$ of this L_1 space would be a closed nonreflexive subspace of $\mathfrak{X}$; however, the map into $\mathfrak{X}^*$ is needed for the slightly more general case of $\mathfrak{X}$ weakly complete.

Let the Borel set $\sigma, x_0 \in \mathfrak{X}$, and $x_0^* \in \mathfrak{X}^*$ be chosen so that $[(N^*)^{m+1}E^*(\sigma)x_0^*]x_0 \neq 0$, and let the derivative of the measure $[(N^*)^{m+1}E^*(\cdot)x_0^*]x_0$ be denoted by $g(\xi)$. We can then find a subset τ of σ and a constant $a > 0$ such that $\mu_2(\tau) > 0$ and $|g(\xi)| \geq a$ on τ.

Define the map Φ of $L_1(\tau, \mu_2)$ into $\mathfrak{X}^*$ by

$$\Phi(f) = \int_\tau f(\xi)(N^*)^{m+1}E^*(d\xi)x_0^* .$$

Φ is a linear map with bound $M_{m+1}|x_0^*|$. Now take

$$x = \int_\tau [g(\xi)]^{-1}\operatorname{sgn}\overline{f(\xi)}E(d\xi)x_0 ;$$

The norm of x is no greater than $4M \cdot a^{-1} \cdot |x_0|$. But we have

$$\begin{aligned}[\Phi(f)](x) &= \int_\tau f(\xi)[g(\xi)]^{-1} \operatorname{sgn} \overline{f(\xi)} [(N^*)^{m+1} E^*(d\xi) x_0^*] x_0 \\ &= \int_\tau |f(\xi)| [g(\xi)]^{-1} g(\xi) \mu_2(d\xi) \\ &= |f|_{L_1} ,\end{aligned}$$

which shows that

$$|\Phi(f)| \geq |f|_{L_1} \cdot a \cdot (4M|x_0|)^{-1} ,$$

so that Φ is one-to-one and has a continuous inverse.

Now let Ψ be the map of $L_\infty(\tau, \mu_2)$ into $\mathfrak{X}$:

$$\Psi(h) = \int_\tau [g(\xi)]^{-1} h(\xi) E(d\xi) x_0 ,$$

Ψ is a continuous map with bound no greater than $4M \cdot a^{-1}|x_0|$; we will show that Ψ is one-to-one and bicontinuous. We have

$$\begin{aligned}\Phi(f)\Psi(h) &= \int_\tau f(\xi)[g(\xi)]^{-1} h(\xi)[(N^*)^{m+1} E^*(d\xi) x_0^*] x_0 \\ &= \int_\tau f(\xi) h(\xi) \mu_2(d\xi) ,\end{aligned}$$

so that

$$\begin{aligned}\sup_{|f|_{L_1} \leq 1} |\Phi(f)\Psi(h)| &= \sup_{|f|_{L_1} \leq 1} \left| \int_\tau f(\xi) h(\xi) \mu_2(d\xi) \right| \\ &= |h|_{L_\infty} .\end{aligned}$$

But since Φ is bounded,

$$\begin{aligned}\sup_{|f|_{L_1} \leq 1} |\Phi(f)\Psi(h)| &\leq \sup_{\substack{x^* \in X^* \\ |x^*| \leq |\Phi|}} |x^* \Psi(h)| \\ &= |\Phi| \, |\Psi(h)| ,\end{aligned}$$

so that

$$|h|_{L_\infty} \leq |\Phi| \, |\Psi(h)| ;$$

thus Ψ is one-to-one and bicontinuous, The range $\mathfrak{Y}$ of Ψ in $\mathfrak{X}$ is then a closed non weakly complete subspace of $\mathfrak{X}$. But this is impossible, because every closed subspace of a weakly complete Banach space is again weakly complete; the proof of this last remark is as follows.

Let $\mathfrak{X}$ be a weakly complete Banach space, $\mathfrak{Y}$ a closed subspace. Let y_n be a weakly Cauchy sequence in $\mathfrak{Y}$, so that $y^* y_n$ is a Cauchy sequence of numbers for every y^* in Y^*. Since any x^* in X^*, when restricted to $\mathfrak{Y}$, is an element of $\mathfrak{Y}^*$, x^*y_n is a Cauchy sequence of numbers for every x^* in $\mathfrak{X}^*$. Since $\mathfrak{X}$ is weakly complete, there is an x_0 in $\mathfrak{X}$ such that $\lim_{n\to\infty} x^*y_n = x^*x_0$ for every x^* in $\mathfrak{X}^*$; and since $\mathfrak{Y}$ is strongly closed in $\mathfrak{X}$, it is weakly closed, so that x_0 must lie in $\mathfrak{Y}$. Finally since every y^* in $\mathfrak{Y}^*$ is, by the Hahn-Banach theorem, the restriction of an x^* in $\mathfrak{X}^*$, $\lim y^*y_n = y^*x_0$ for every y^* in $\mathfrak{Y}^*$, so that $\mathfrak{Y}$ is weakly complete.

THEOREM 3.2. *If $\mathfrak{X}$ is weakly complete, then $N^mE(\sigma) = 0$ for every set σ of finite μ_1-measure.*

Proof. Follow exactly the same discussion above, replacing the number $m+1$ by m and the measure μ_2 by μ_1.

Note that Theorems 3.1 and 3.2 also hold if $\mathfrak{X}$ is assume to be separable instead of weakly complete, for the image of the L_∞ space in $\mathfrak{X}$ would be a nonseparable closed subspace of $\mathfrak{X}$; but every closed subspace of a separable space is again separable.

4. Examples. In the following examples we will need two computational lemmas.

LEMMA 4.1. *For each real number $p \geq 1$ and Borel set σ,*

$$\int_\tau |\lambda - \xi|^{-(p+2)}\mu_2(d\xi) \leq 8d(\lambda, \sigma)^{-p}, \text{ for all } \lambda \notin \bar{\sigma} .$$

Proof.

$$\begin{aligned}
&\int_\sigma |\lambda - \xi|^{-(p+2)}\mu_2(d\xi) \\
&\quad \leq \int_{|\lambda-\xi|\geq d(\lambda,\sigma)} |\lambda - \xi|^{-(p+2)}\mu_2(d\xi) \\
&\quad = \frac{4}{\pi}\int_0^{2\pi} d\theta \int_{d(\lambda,\sigma)}^{\infty} r^{-(p+2)} r\, dr \qquad\qquad (\lambda - \xi = re^{i\theta}) \\
&\quad \leq 8d(\lambda, \sigma)^{-p} .
\end{aligned}$$

LEMMA 4.2. *For each real number $p \geq 1$ and Borel subset σ of the real line,*

$$\int_\sigma |\lambda - \xi|^{-(p+1)}\mu_1(d\xi) \leq 2^{p+1}\pi d(\lambda, \sigma)^{-p} ,$$

where μ_1 is Lebesgue measure along the line, and λ is any complex number, $\lambda \notin \bar{\sigma}$.

Proof. Let $\lambda = \alpha + i\beta$, α, β real. Then either, (i), $d(\alpha, \sigma) \geq d(\lambda, \sigma)/2$ or, (ii) $|\beta| \geq d(\lambda, \sigma)/2$. In case (i) we have

$$\int_{\sigma} |\lambda - \xi|^{-(p+1)} \mu_1(d\xi) \leq \int_{d(\lambda,\sigma)/2}^{\infty} \eta^{-(p+1)} d\eta \qquad (\lambda - \xi = \eta)$$
$$\leq 2^{p+1} p^{-1} d(\lambda, \sigma)^{-p} .$$

In case (ii) we have

$$\int_{\sigma} |\lambda - \xi|^{-(p+1)} \mu_1(d\xi) \leq \int_{-\infty}^{\infty} |\xi - i\beta|^{-(p+1)} d\xi$$
$$\leq \int_{-\infty}^{\infty} (\xi^2 + \beta^2)^{-\frac{1}{2}(p+1)} d\xi$$
$$\leq 2^{p+1} \pi d(\lambda, \sigma)^{-p} .$$

EXAMPLE 4.3. Let Σ be a disc in the plane with μ_2-measure 1. Let

$$x = L_\infty(\Sigma) \oplus L_2(\Sigma) \oplus \cdots \oplus L_2(\Sigma) \oplus L_1(\Sigma) ,$$

where m copies of $L_2(\Sigma)$ are taken. Let T be the operator $S + N$ where S and N are defined as

$$S[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)]$$
$$= [\xi f(\xi) \oplus \xi g_1(\xi) \oplus \cdots \oplus \xi g_m(\xi) \oplus \xi h(\xi)] ,$$

$$N[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)]$$
$$= [0 \oplus f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi)] .$$

Since Σ has measure 1, any function in L_r is in L_s for all $s \leq r$, and the L_s norm is no greater than the L_r norm; thus N is a bounded operator with norm 1. Also N is a nilpotent for which $N^{m+1} \neq 0$. The operator T is a spectral operator with resolution of the identity

$$E(\sigma)[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)]$$
$$= [f(\xi)\chi_\sigma(\xi) \oplus g_1(\xi)\chi_\sigma(\xi) \oplus \cdots \oplus g_m(\xi)\chi_\sigma(\xi) \oplus h(\xi)\chi_\sigma(\xi)] .$$

The resolvent function is

$$R(\lambda, T)E(\sigma)[f(\xi) \oplus g_1(\xi) \oplus \cdots \oplus g_m(\xi) \oplus h(\xi)]$$
$$= \left[\frac{f(\xi)\chi_\sigma(\xi)}{\lambda - \xi} \oplus \left(\frac{g_1(\xi)\chi_\sigma(\xi)}{\lambda - \xi} + \frac{f(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^2} \right) \oplus \cdots \oplus \right.$$
$$\left(\frac{g_m(\xi)\chi_\sigma(\xi)}{\lambda - \xi} + \cdots + \frac{g_1(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^m} + \frac{f(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^{m+1}} \right)$$
$$\left. \oplus \left(\frac{h(\xi)\chi_\sigma(\xi)}{\lambda - \xi} + \frac{g_m(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)} + \cdots + \frac{g_1(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^{m+1}} + \frac{f(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^{m+2}} \right) \right] .$$

All the terms are clearly of mth order rate of growth except possibly for

$$\text{(a)}\left|\frac{f(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^{m+1}}\right|_{L_2},\quad \text{(b)}\left|\frac{f(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^{m+2}}\right|_{L_1},\ \text{and}\ \text{(c)}\left|\frac{g_1(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^{m+1}}\right|_{L_1}.$$

For (a) we have

$$\left\{\int_\sigma |f(\xi)(\lambda-\xi)^{-(m+1)}|^2\mu_2(d\xi)\right\}^{1/2} \le |f|_{L_\infty}\left\{\int_\sigma |\lambda-\xi|^{-2m-2}\mu_2(d\xi)\right\}^{1/2}$$
$$\le |f|_{L_\infty}\sqrt{8}\,d(\lambda,\sigma)^{-m},$$

for (b) we have

$$\int_\sigma |f(\xi)(\lambda-\xi)^{-(m+2)}|\,\mu_2(d\xi) \le |f|_{L_\infty}\int_\sigma |\lambda-\xi|^{-(m+2)}\mu_2(d\xi)$$
$$\le |f|_{L_\infty}\cdot 8d(\lambda,\sigma)^{-m},$$

and for (c) we have

$$\int_\sigma |g_1(\xi)(\lambda-\xi)^{-(m+1)}|\,\mu_2(d\xi) \le \left\{\int_\sigma |g_1(\xi)|^2\mu_2(d\xi)\right\}^{1/2}\left\{\int_\sigma |\lambda-\xi|^{-2m-2}\mu_2(d\xi)\right\}^{1/2}$$
$$\le |g_1|_{L_2}\cdot\sqrt{8}\cdot d(\lambda,\sigma)^{-m}.$$

Thus each term of the resolvent, and hence the resolvent itself satisfies the mth order rate of growth condition; this shows that Corollary 2.3 cannot be improved.

EXAMPLE 4.4. Let Σ be as in the previous example and let

$$\mathfrak{X} = L_r(\Sigma)\oplus\cdots\oplus L_r(\Sigma)\oplus L_s(\Sigma)$$

where m copies of L_r are taken. r and s are to satisfy $1 < s < r < \infty$ and $rs \le 2(r-s)$. Let $T = S + N$, where S and N are defined in essentially the same way as in the previous example. The resolvent function is given by

$$R(\lambda,T)E(\sigma)[f_1(\xi)\oplus\cdots\oplus f_m(\xi)\oplus g(\xi)]$$
$$=\left[\frac{f_1(\xi)\chi_\sigma(\xi)}{\lambda-\xi}\oplus\cdots\oplus\left(\frac{f_m(\xi)\chi_\sigma(\xi)}{\lambda-\xi}+\cdots+\frac{f_1(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^m}\right)\right.$$
$$\left.\oplus\left(\frac{g(\xi)\chi_\sigma(\xi)}{\lambda-\xi}+\frac{f_m(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^2}+\cdots+\frac{f_1(\xi)\chi_\sigma(\xi)}{(\lambda-\xi)^{m+1}}\right)\right].$$

Each of the terms is clearly of mth order rate of growth except possibly for the L_s norm of $f_1(\xi)(\lambda-\xi)^{-(m+1)}\chi_\sigma(\xi)$, and for this we have

$$\left\{\int_\sigma |f_1(\xi)(\lambda-\xi)^{m+1}|^s\mu_2(d\xi)\right\}^{1/s}$$
$$\le\left\{\int_\sigma |f_1(\xi)|^r\mu_2(d\xi)\right\}^{1/r}\left\{\int |\lambda-\xi|^{-\frac{(m+1)rs}{r-s}}\mu_2(d\xi)\right\}^{\frac{r-s}{rs}}$$
$$\le |f_1|_{L_r}\cdot 8^{\frac{s-r}{rs}}\cdot d(\lambda,\sigma)^{-m-\left(1-\frac{2(r-s)}{rs}\right)}$$
$$\le |f_1|_{L_r}\cdot 8^{\frac{r-s}{rs}}d(\lambda,\sigma)^{-m}$$

Thus the resolvent satisfies the mth order rate of growth condition, and $N^m = 0$. Since $\mathfrak{X}$ is reflexive, this shows that Theorem 3.1 cannot be improved. Note that $\mathfrak{X}$ is also separable.

EXAMPLE 4.5. Let Σ be the interval $[0, 1]$ endowed with μ_1-measure, and let

$$\mathfrak{X} = L_\infty(\Sigma) \oplus \cdots \oplus L_\infty(\Sigma) \oplus L_1(\Sigma)$$

where m copies of L_∞ are taken. Let $T = S + N$ where S and N are defined in essentially the same way as in the previous examples. The resolvent function is given by

$$\begin{aligned} &R(\lambda, T)E(\sigma)[f_1(\xi) \oplus \cdots \oplus f_m(\xi) \oplus g(\xi)] \\ &\quad = \left[\frac{f_1(\xi)\chi_\sigma(\xi)}{\lambda - \xi} \oplus \cdots \oplus \left(\frac{f_m(\xi)\chi_\sigma(\xi)}{\lambda - \xi} + \cdots + \frac{f_1(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^m}\right)\right. \\ &\qquad \left. \oplus \left(\frac{g(\xi)\chi_\sigma(\xi)}{\lambda - \xi} + \frac{f_m(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^2} + \cdots + \frac{f_1(\xi)\chi_\sigma(\xi)}{(\lambda - \xi)^{m+1}}\right)\right] . \end{aligned}$$

Each of the terms is clearly of mth order rate of growth except for the L_1 norm of $f_1(\xi)(\lambda - \xi)^{-(m+1)}\chi_\sigma(\xi)$, and for this we have

$$\begin{aligned} \int_\sigma |f(\xi)(\lambda - \xi)^{-(m+1)}|\mu_1(d\xi) &\leq |f|_{L_\infty} \int_\sigma |\lambda - \xi|^{-(m+1)}\mu_1(d\xi) \\ &\leq |f|_{L_\infty} 2^{m+1}\pi d(\lambda, \sigma)^{-m} . \end{aligned}$$

Thus we have an example of an operator with spectrum in a rectifiable arc which satisfies the mth order rate of growth condition, but for which $N^m \neq 0$.

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YALE UNIVERSITY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY