ERROR BOUNDS FOR AN APPROXIMATE SOLUTION TO THE VOLTERRA INTEGRAL EQUATION

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In 1945 Michal [2] obtained several results which he asserted were useful for approximating the solution to the Volterra integral equation. These results were concerned with certain equations in Fréchet differentials having as their unique solutions the resolvent kernel and the exact solution to the Volterra integral equation of the second kind. Michal treated the resolvent kernel $S[K \mid x, t]$ and the solution $y[K \mid x]$ as functions[1] of the given kernel $K(x, t)$, the setting being the Banach spaces

$$T = \{G(x, t) \mid G(x, t) \text{ is real and continuous on } a \leq t \leq x \leq b\}$$

and

$$I = \{g(x) \mid g(x) \text{ is real and continuous on } a \leq x \leq b\}$$

with the norms

$$\begin{aligned} \| G(x, t) \| &= \max \mid G(x, t) \mid \quad (a \leq t \leq x \leq b)\ , \\ \| g(x) \| &\equiv \max \mid g(x) \mid \qquad (a \leq x \leq b)\ , \end{aligned} \tag{1}$$

respectively. In another work [3, pp. 16-17] Michal showed that the solution $y[K \mid x]$ can be expressed by a Taylor-type expansion in Fréchet differentials of $y[K \mid x]$ about an arbitrary $K_0(x, t)$ from T. In this paper we shall use Michal's results to obtain approximations to the solution of the Volterra integral equation with error bounds.

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Consider the integral equation

$$y(x) + \int_a^x K(x, t)y(t)dt = f(x) \tag{2}$$

where $K(x, t)$ is in T and $f(x)$ is in I. It is known that the exact solution to (2) is given by

$$y(x) = f(x) + \int_a^x S(x, t)f(t)dt \tag{3}$$

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[1] The symbols $S[K \mid x, t]$ and $y[K \mid x]$ were used to indicate the functional dependence of $S(x, t)$ and $y(x)$ on $K(x, t)$.

where the resolvent kernel $S(x, t)$ is in T. Let $K_0(x, t)$ from T be another kernel such that $S_0(x, t)$, the resolvent of $K_0(x, t)$, is known and that $\|h(x, t)\| = \|K(x, t) - K_0(x, t)\|$ is small in the sense of (1). Then by (3) the solution to (2) with kernel $K_0(x, t)$ is

$$(4) \qquad y_0(x) = f(x) + \int_a^x S_0(x, t)f(t)dt \ .$$

Now treat $y(x)$ as a function of the kernel $K(x, t)$. The first Fréchet differential $dy(x)$ of $y(x)$ with increment $h(x, t)$ (applied to $K(x, t)$) is

$$dy(x) = -\int_a^x \left[h(x, t) + \int_t^x S(x, z)h(z, t)dz\right]y(t)dt$$

[2, p. 253]. In particular, the Fréchet differential of $y(x)$ evaluated at $K_0(x, t)$ with increment $h(x, t) = K(x, t) - K_0(x, t)$ will be

$$(5) \qquad dy_0(x) = -\int_a^x \left[h(x, t) + \int_t^x S_0(x, z)h(z, t)dz\right]y_0(t)dt \ .$$

Furthermore, by Theorem 2 of [2] the differential system

$$\begin{cases} dy_0(x) = -\int_a^x \left[h(x, t) + \int_t^x S_0(x, z)h(z, t)dz\right]y_0(t)dt \\ y_0(x) = f(x) \qquad (K_0(x, t) = 0) \end{cases}$$

has a unique solution which is given by (4). Thus a first order approximation to the solution $y(x)$ of (2) will be

$$y_0(x) + dy_0(x) \ .$$

The exact solution to (2) is given by the Taylor expansion [3; 1. p. 112]

$$(6) \qquad y(x) = y_0(x) + \sum_{j=1}^{\infty} (j!)^{-1}d^j y_0(x)$$

where, in terms of composition powers[2],

$$(7) \qquad d^j y_0(x) = (-1)^j j!\,[h + S_0 h]^j * y_0 \ .$$

Thus knowledge of the higher order differentials will allow closer approximations to $y(x)$.

We now take up the problem of establishing error bounds for any order of approximation to $y(x)$ from (6). If A_j $(j = 1, 2, \cdots, n)$ is in T and g is in I, and

[2] $VW = \int_t^x V(x, z)W(z, t)dz$, $W^2 = \int_t^x W(x, z)W(z, t)dz$, $W^n = \int_t^x W(x, z)W^{n-1}(z, t)dz$, and $W^n * g = \int_a^x W^n(x, t)g(t)dt$

$$A = A_1A_2 \cdots A_n = \int_t^x \int_t^{z_1} \cdots \int_t^{z_{n-2}} A_1(x, z_1)A_2(z_1, z_2) \cdots A_n(z_{n-1}, t)dz_{n-1} \cdots dz_1 ,$$

it is seen that

$$\| A \| \leq \frac{|b-a|^{n-1}}{(n-1)!} \prod_{j=1}^{n} \| A_j \| \tag{8}$$

and

$$\| A * g \| \leq \frac{\| g \| \, |b-a|^n}{n!} \prod_{j=1}^{n} \| A_j \| . \tag{9}$$

Let $P_{n-i,i}[h(S_0h)]$ denote the sum of terms obtained from the composition $h^{n-i}(S_0h)^i$ by a permutation on the n places occupied by

$$\underbrace{hh \cdots h}_{n-i}\underbrace{(S_0h)(S_0h) \cdots (S_0h)}_{i} = h^{n-i}(S_0h)^i .$$

For example, by setting

$$P_{2,1}[h(S_0h)] = h^2(S_0h) + h(S_0h)h + (S_0h)h^2$$

and

$$P_{1,2}[h(S_0h)] = h(S_0h)^2 + (S_0h)h(S_0h) + (S_0h)^2h$$

we can write with brevity

$$[h + S_0h]^3 = h_3 + P_{2,1}[h(S_0h)] + P_{1,2}[h(S_0h)] + (S_0h)^3 .$$

Now let

$$c = \| h(x, t) \|, \ m = \| y_0(x) \|, \ B = \| S_0(x, t) \|, \text{ and } u = | b - a | .$$

Then from (7), (8), (9), and the mechanics of composition we obtain

$$\begin{aligned} \| (n!)^{-1}d^ny_0(x) \| &= \| (-1)^n[h + S_0h]^n * y_0 \| \\ &= \| h^n * y_0 + P_{n-1,1}[h(S_0h)] * y_0 + \cdots + P_{1,n-1}[h(S_0h)] * y_0 + (S_0h)^n * y_0 \| \\ &\leq \| h^n * y_0 \| + \| P_{n-1,1}[h(S_0h)] * y_0 \| + \cdots + \| (S_0h)^n * y_0 \| \\ &\leq \frac{mc^nu^n}{n!} + \binom{n}{1} \frac{mc^nu^{n+1}B}{(n+1)!} + \cdots \binom{n}{n} \frac{mc^nu^{2n}B^n}{(2n)!} \\ &\leq mc^nu^n \sum_{j=0}^{n} \binom{n}{j} \frac{(uB)^j}{(n+j)!} \\ &\leq \frac{m[cu(1+uB)]^n}{n!} . \end{aligned} \tag{10}$$

Thus transposing the desired nth order approximation to $y(x)$ from the right side of (6) to the left side and applying (10) we get

$$\begin{aligned}\left\|y(x) - y_0(x) - \sum_{j=1}^{n} \frac{1}{j!} d^j y_0(x)\right\| &= \left\| \sum_{j=n+1}^{\infty} \frac{1}{j!} d^j y_0(x)\right\| \\ &\leq \sum_{j=n+1}^{\infty} m(j!)^{-1}\theta^j \qquad (11) \\ &\leq m\left[e^\theta - \sum_{j=0}^{n} (j!)^{-1}\theta^j\right]\end{aligned}$$

where $\theta = cu[1 + uB]$, For small values of θ we readily discern the asymptotic relation

$$\left\|y(x) - y_0(x) - \sum_{j=1}^{n-1} \frac{1}{j!} d^j y_0(x)\right\| = 0(\theta^n) . \qquad (12)$$

A simple numerical example will be given next.

Consider the Volterra equation

$$y(x) + \frac{1}{3}\int_0^x xt[3 + x^3 - t^3]y(t)dt = x \exp [1/3x^3] \qquad (13)$$

where $K(x, t) = 1/3\, xt[3 + x^3 - t^3]$ is in T, $f(x) = x \exp [1/3x^3]$ is in I and $a = 0, b = 1$. Take $K_0(x, t) = xt \exp [1/3(x^3 - t^3)]$. The resolvent kernel for $K_0(x, t)$ is $S_0(x, t) = -xt$. By (4) the solution to (13) with kernel $K_0(x, t)$ is

$$y_0(x) = x \exp [1/3x^3] + \int_0^x - xt^2 \exp [1/3t^3]dt = x . \qquad (14)$$

By virtue of (5), the Fréchet differential of $y(x)$ evaluated at $K_0(x, t)$ with increment

$$h(x, t) = K(x, t) - K_0(x, t) = \frac{1}{3} xt[3 + x^3 - t^3] - xt \exp [1/3(x^3 - t^3)]$$

is

$$\begin{aligned}dy_0(x) = &- \int_0^x \left\{\frac{1}{3} xt(3 + x^3 - t^3 - 3) \exp [1/3(x^3 - t^3)]\right. \\ &\left. + \int_t^x - xz\left(\frac{1}{3} zt(3 + z^3 - t^3 - 3) \exp [1/3(z^3 - t^3)]\right)dz\right\}tdt \qquad (15) \\ = &\frac{x^{10}}{162} .\end{aligned}$$

Thus a first order approximation to $y(x)$ will be

$$y(x) \approx x + \frac{x^{10}}{162} . \tag{16}$$

It is easily established that

$$\| h(x, t) \| < 0.04, \| S_0(x, t) \| = 1, \| y_0(x) \| = 1 .$$

Hence, with $\theta = 0.08$, it follows from (11) that

$$\| y(x) - y_0(x) - dy_0(x) \| < 0.0033 . \tag{17}$$

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