AREA AND NORMALITY

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1. Introduction. The simplest non-Riemannian a-dimensional area (concisely: a-area) is a translation invariant positive continuous measure (or area) defined on the a-dimensional linear subspaces, called a-flats, of an n-dimensional affine space A_n $(1 \leq a \leq n)$. Such areas have been studied by Wagner [15] and they are the subject of the present investigation which is in part related to Wagner's, but has no connection with the differential geometry of general area metrics persued principally in Japan by Kawaguchi, Iwamoto and others.

The simplest case, $a = 1$, is well known. In that case a segment with endpoints x, y has a translation invariant length $d(x, y)$. If the sphere $d(z, x) = 1$ (z fixed) has at x_0 a supporting $(n-1)$-flat (hyperplane) H_0 then H_0 is transversal to the 1-flat (line) L_0 through z and x_0, and L_0 is normal to H_0.

Therefore the existence of an $(n-1)$-flat transversal to a given line is equivalent to the convexity of the sphere $d(z, x) = 1$; which, in turn, is equivalent to the triangle inequality for $d(a, b)$, in other words, to the space being Minkowskian (normed linear).

If L_0 is normal to H_0 at x_0 then it is normal to every line L through x_0 in H_0 in the two-flat spanned by L_0 and L. A well-known theorem of Blaschke [2] states that for $n \geq 3$ normality between lines is symmetric only in euclidean space. However, as shown by Radon [13], this is not the case for $n = 2$.

Here we treat the analogous problems for arbitrary a, and then study the special case of Minkowski area.

We cannot give more than this vague hint without some definitions. Let $(x^1, \cdots, x^n)$ be affine coordinates of a point x in A^n with origin $z = (0, \cdots, 0)$. The *a-box* $[x_0, x_1, \cdots, x_a]$ consists of all points of the form $(1 - \theta_i)x_0 + \sum_{1=1}^{a} \theta_i x_i$ where $0 \leq \theta_i, \leq 1$; and hence is a (possibly degenerate) parallelepiped.

An *a-area* assigns to every Borel[1] set M in an a-flat a measure $\alpha(M)$ which is invariant under the translations of A^n, and continuous; that is, $\alpha([x_0, \cdots, x_a])$ depends continuously on $x_0, \cdots, x_a$. The invariance under translation applied to sets in the same a-flat A yields at once that the measure in A is determined up to a factor depending on A. If we introduce an auxiliary euclidean metric

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[1]All sets considered will be Borel sets.

$$e(x, y) = \left[\sum_{i,k=1}^{n} g_{ik}(x^i - y^i)(x^k - y^k)\right]^{1/2}$$

where the form $\sum g_{ik}x^i x^k$ is positive definite, then the a-dimensional Lebesgue measure, $|M|_a^e$, in A which results from this euclidean metric is invariant under translations so that

$$(1) \qquad \alpha(M) = f(A)|M|_a^e, \quad f(A) > 0 .^2$$

Translation invariance implies that $f(A) = f(A')$ if A and A' are parallel a-flats, and the continuity of α implies continuity of $f(A)$. Because of the invariance under translation we may also write.

$$\alpha([x_0, \cdots, x_a]) = F(x_1 - x_0, \cdots, x_a - x_0)$$

where the function $F(x_1, \cdots, x_a)$ satisfies some simple conditions $F_1, \cdots F_4$ listed at the end of § 2.

We call the area α convex if

$$(2) \qquad F(x_1' + x_1'', x_2, \cdots, x_a) \leq F(x_1', x_2, \cdots, x_a) + F(x_1'', x_2, \cdots, x_a)$$

and *strictly convex* if the strict inequality holds for independent x_1', x_1'', $x_2, \cdots, x_a$.

If an a-flat A and a b-flat B intersect in a d-flat D, where $0 \leq d < \min(a, b)$, then they span a q-flat Q with $q = a + b - d$. We call B *totally transversal to* A, or A *totally normal to* B (at D in Q, where ambiguities are possible) if $\alpha(M) \leq \alpha(M')$ for a projection[3] M parallel to B on A of any set M' which lies in an a-flat A' through D in Q. For $d = 0$, $b = n - a$ this is Caratheodory's concept of transversality[4]. If A is totally normal to B at D, $d > b + 1$, then A is totally normal to every b'-flat, $d < b' < b$ through D in B. We call A *normal to* B *at* D and B *transversal to* A, if A is totally normal to every $(d + 1)$-flat in B through D. For $d = 0$, $b = n - a$ this is Wagner's concept of transversality. Only for $d = \min(a, b) - 1$ does normality of A to B at D imply total normality. This is the only case with $d > 0$ which was studied previously in the literature, namely in [7] for Minkowski area.

We call α *totally convex* if an $(n - a)$-flat totally transversal to a given a-flat at a point exists. For totally convex α the a-flats minimize area in the sense that the α-area of the union of all but one face of a closed a-dimensional polyhedron is not less than the area of that face.

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[2] Therefore the case $a = n$ is uninteresting as long as only areas for one definite A^n are considered. Hence we assume $1 \leq a \leq n - 1$ except in the last three sections.

[3] This concept needs clarification when $d > 0$. The precise form is found in § 2.

[4] Caratheodory treats more general a-dimensional variational problems. His ideas on transversality are easiest understood by consulting volume 1 of his Gesammelte Mathematische Schriften, München 1954; see in particular p. 364 and paper XX pp. 404–426.

However, the a-flats may minimize α-area for α which are not totally covex. On the other hand for $1 < a < n - 1$ the a-flats need not minimize area when α is merely convex. They will minimize a-area if α is *extendably convex* which means the following; α assigns an area $\phi(\mathfrak{a})$ to every simple a-vector, $\mathfrak{a}$, in the space V_a^n of all a-vectors, if $\phi(\mathfrak{a})$ can be extended to a convex function in all of V_a^n then α is extendably convex. The difference between extendable and total convexity has a very palpable interpretation in V_a^n.

If $F^2(x_1, \cdots, x_a)$ is a quadratic form in each set of variables $x_i^1, \cdots, x_i^n; i = 1, \cdots, a$; then we call $\alpha(M)$ *quadratic*. If $\alpha(M)$ is euclidean, that is if $\alpha(M) = |M|_a^e$ for a suitable choice of $e(x, y)$, then it is quadratic, but a quadratic area is not necessarily euclidean when $1 < a < n - 1$. The quadratic areas enter naturally as follows.

Let $0 \leq d < a \leq b < n$ and let a convex a-area α and a convex b-area β be defined in A^n. If normality (with respect to α) of an a-flat A to a b-flat B at a d-flat D is equivalent to normality (with respect to β) of B to A at D then both areas are quadratic unless $a + b = n, d = 0$. Whether the latter cases are really exceptional is not known except for $a = 1, b = n - 1$ (see below). If, in particular, $a = b$ and $\alpha \equiv \beta$, then equivalence of normality means that normality of two a-flats at a d-flat is a symmetric relation. Hence symmetry of normality implies—except for $a = n/2$, $d = 0$—that the area is quadratic. It will be euclidean only in special cases, for instance when $a < n/2$ and $d = 0$ or $a > n/2$ and $d = 2a - n$. For $a = b = 1, n > 2$ this becomes the above mentioned result of Blaschke [2].

All the results on symmetry and equivalence of normality also hold for total normality.

The a-dimensional Minkowski area (or measure), $2 \leq a \leq n$, in an n-dimensional Minkowski space with distance $F(x - y)$ is the area of the above type for which an a-dimensional unit ball in any a-flat A, that is the set $\{x \mid F(x - x_0) \leq 1; x, x_0 \in A\}$, has the euclidean volume $\pi^{a/2}/\Gamma(a/2 + 1)$. It is shown in [7] that these areas are convex and are strictly convex or differentiable if $F(x) = 1$ is strictly convex or differentiable.

We do not know whether Minkowski area is totally or extendably convex for $1 < a < n - 1$.

If the a-dimensional area $1 \leq a \leq n - 1$ of a Minkowski space is quadratic then the space is euclidean. Hence if normality of an a-flat A to a b-flat B at a d-flat D with respect to the a-area of one Minkowski space is equivalent to normality of B to A at D with respect to the b-area of another, then both Minkowski spaces are euclidean, unless $a + b = n, d = 0$. However only the case $a = 1, b = n - 1, d = 0$ is really known to be exceptional when the two spaces are different. When they are identical then already this case leads for $n > 2$ to an unsolved problem on convex bodies [10, Problem 5].

There are many interesting and difficult problems involving two areas in a Minkowski space of which we settle only a few. In the last section we obtain from the method and result of [8] a result of a different nature. If $b > a$ and $f_b(B), f_a(A)$ are the functions of (1) for a-and b-dimensional area of the same Minkowski space, we give an estimate from above for $f_b(B)$ in terms of $f_a(A)$ with $A \subset B$.

2. Normality. Our first objects are the relations between the various concepts of normality arising from different choices of d and b. In all that follows let $0 \leq d < \min(a, b)$; $q = a + b - d \leq n$. Moreover, A, B, D, Q with or without subscripts denote a-, b-, d-, q-flats respectively with $D \subset B \subset Q, A \subset Q, A \cap B = D$.

Choose in B a c-flat $C, c = b - d$, which intersects D in exactly one point and hence intersects A in this point only. The association of the points of A and A_0 which lie in the same c-flat parallel to C is a projection of A_0 on A, which depends on the choice of C. The restriction of this mapping to a subset M_0 of A_0 gives the projection of M_0 on a set M in A.

If C' is a second c-flat in B which intersects D in a point, and B^* is any b-flat in Q parallel to B, then the projection of $B^* \cap A_0$ on A with the use of C' is the product of the projection of $B^* \cap A_0$ on A with the use of C and of a translation parallel to D (which depends continuously on B^*).

This and (1) imply.

(2.1) LEMMA. *If M_0 is a set in A_0 and M, M' are its projections on A with the use of C and C' respectively, then $\alpha(M) = \alpha(M')$.*

Thus the arbitrariness of C does not influence the measures of the projections. Moreover, if $0 < \alpha(M_0) < \infty$, then $\alpha(M)/\alpha(M_0)$ is according to (1) independent of the choice of M_0 in A_0.

We now define: A is totally *normal to B at D in Q*, or *B totally transversal to A at D in Q*, if for a fixed $M_0 \subset A_0$ with $0 < \alpha(M_0) < \infty$ and a fixed C the area $\alpha(M)$ of the projection of M_0 on A is minimal.

The preceding discussion shows that this definition is independent of the choice of A_0, M_0 and C; and hence depends only on D, B and Q.

The existence of an A normal to B at D in Q follows from two observations.

(i) The function $f(A)$ is continuous and has the same value for parallel A. Hence $f(A)$ attains its positive minimum f_1 and its finite maximum f_2 on the compact set of a-flats through z, so that

$$f_1|M|_a^e \leq \alpha(M) \leq f_2|M|_a^e .$$

(ii) $|M|_a^e \to \infty$ and hence $\alpha(M) \to \infty$ when A approaches a position for which $A \cap B$ is greater than D.

As previously observed, a B totally transversal to a given A at D in Q will in general fail to exist.

We now consider some properties of normality. In many of the following statements "totally" appears in parentheses, because they remain valid for the weaker concept of normality defined in the Introduction.

(2.2) *If A is* (totally) *normal toB at D in Q and the b'-flat B' lies in Q and contains B but does not contain A, then A is* (totally) *normal to B' at $D' = B' \cap A$ in Q.*

This is nearly obvious. A $(b-d)$-flat C in B which intersects D in exactly one point also intersects A and hence D' in this point only. Therefore the same C can be used for projection in both cases of normality.

(2.3) *If A is* (totally) *normal to B at D, $d<b'<b$, then A is* (totally) *normal to any b'-flat B' through D in B.*

Take a $(b'-d)$-flat C' in B' that intersects D in a point and choose a $(b-d)$-flat C in B which contains C' and intersects D in this point only. For any A' through D in the space spanned by A and B' the projection of A' on A parallel to B and B' respectively coincide if we use C and C'.

Proposition (2.3) implies in particular that A is totally normal to every $(d+1)$-flat through D in B. We shall see in § 5 that the converse is in general not true. It does hold in an important special case.

(2.4) THEOREM. *If $A \cap B = D$, $a = d+1$, $b - d \geq 2$ and A is normal to every $(d+1)$-flat in B through D, then A is totally normal to B at D.*

For an indirect proof, assume that A is not totally normal to B and let $\overline{A} \neq A$ be totally normal to B at D in the space Q spanned by A and B. A suitable b-flat B' in Q parallel to B intersects A and $\overline{A}$ in two distinct d-flats D' and $\overline{D}'$ parallel to D. These lie therefore in a $(d+1)$-flat $D_+ \subset B'$. In D^+ take a line L which intersects D' in a point.

Consider a set M in A with $0 < \alpha(M) < \infty$. Since A is normal to D^+, the projection $\overline{M}$ of M on $\overline{A}$ parallel to L satisfies $\alpha(\overline{M}) \geq \alpha(M)$.

On the other hand, let C' be a $(b-d)$-flat in B' which contains L and intersects D' in $L \cap D'$ only. Projection of M on $\bar{A}$ parallel to B with the use of C' again yields the set $\bar{M}$. Since $\bar{A}$ is totally normal to B and A is not, we would have $\alpha(M) > \alpha(\bar{M})$, a contradiction.

Defining normality of A to B at D as in the Introduction we conclude from (2.4) that normality and total normality coincide for $d = \min(a, b) - 1$. Obviously (2.3) remains valid for normalily instead of total normality. To prove (2.2) in this case we observe that a $(d'+1)$-flat E through D' in B' intersects B in a $(d+1)$-flat $F \supset D$. For $b' - b = d' - d$ and $E \cup B$ spans B' so that

$$\dim E \cap B + b' = \dim E + \dim B = d' + 1 + b = b' + d + 1 .$$

By hypothesis A is totally normal to F at D, by (2.2) it is also totally normal to E at D' and hence normal to B'.

Moreover (2.2) and (2.3) also show that the case $b = n - a, q = n$ is decisive in the following sense.

(2.5) *If an $(n-a)$-flat (totally) transversal to A exists, then for given $D \subset A \subset Q$, $q = a + b - d$, a b-flat (totally) transversal to A at D in Q exists.*

By hypothesis there is an $(n-a)$-flat N transversal to A through a point $p \in D$. By (2.2) A is normal to the $(n - a + d)$-flat B' spanned by D and N. This settles the case $q = n$. If $q < n$ then according to (2.3) A is normal to the b-flat $B = Q \cap B'$.

For later purposes we note the following consequence of (2.4) and (2.5).

(2.6) LEMMA. *A b-flat B transversal to A at D in Q for any given $D \subset A \subset Q$ will exist if and only if*

(i) *For $p \in A$, every $(a+1)$-flat through A contains a line transversal to A at p.*

(ii) *The set formed by the totality of all transversals to A at p in the different $(a+1)$-flats though A contains an $(n-a)$-flat N.*

The flat N is then transversal to A.

Also for later application we notice as a consequence of the continuity of $f(A)$ the following.

(2.7) LEMMA. *If $A_\nu \to A$, $D_\nu \to D$, $B_\nu \to B$ and A_ν is (totally) normal to B_ν at D_ν then A is (totally) normal to B at D.*

We follow these considerations up analytically using Barthel [1]. The invariance of $\alpha(M)$ under translation implies that the area of the box $[x_0, x_1, \cdots, x_a]$ has the form $F(x_1 - x_0, \cdots, x_a - x_0)$ and

$$(2.8) \qquad \begin{aligned} F(x_1, \cdots, x_a) &= F(x_1^1, \cdots, x_1^n, \cdots, x_a^1, \cdots, x_a^n) \\ &= f(A_x)\,|[z, x_1, \cdots, x_a]|_a^e , \end{aligned}$$

where A_x is the flat spanned by $x_1, \cdots, x_a$, if $x_1, \cdots, x_a$ are linearly independent and $F(x_1, \cdots, x_a) = 0$ otherwise. Thus $F(x_1, \cdots, x_a)$ has the following properties.

F_1: *$F(x_1, \cdots, x_a)$ is continuous in the $a \cdot n$ variables and symmetric in $x_1, \cdots, x_a$.*
F_2: *$F(x_1, \cdots, x_a) > 0$ if $x_1 \wedge \cdots \wedge x_a \neq 0$.*
F_3: *$F(\lambda x_1, x_2, \cdots, x_a) = |\lambda|\ F(x_1, \cdots, x_a)$.*
F_4: *$F(x_1 + \lambda x_j, x_2, \cdots, x_a) = F(x_1, \cdots, x_a)$ for $j > 1$.*

Conversely, if a function $F(x_1, \cdots, x_a)$ has the properties $F_1, \cdots, F_4$ then a well known argument (see e.g. [**14**, pp. 118, 124]) shows that $F(x_1, \cdots, x_a)$ has the form (2.8) with continuous $f(A_x)$ and vanishes for $x_1 \wedge \cdots \wedge x_a = 0$. Hence it defines an area function.

We now take definite independent vectors $u_1, \cdots, u_a$ and assume that $F(x_1, \cdots, x_a)$ possesses a differential as function of $x_1^1, \cdots, x_1^n$ at $x_i = u_i$. Then F_3 and F_4 yield for small $\lambda > 0$ and $j = 1, \cdots, a$

$$(2.9) \qquad \begin{aligned} \delta_j^1 \lambda F(u_1, \cdots, u_a) &= F(u_1 + \lambda u_j, u_2, \cdots, u_a) - F(u_1, \cdots, u_a) \\ &= \sum_i \frac{\partial F(u_1, \cdots, u_a)}{\partial x_1^i} \lambda u_j^i + o(\lambda) . \end{aligned}$$

For $\lambda \to 0$, using the symmetry of $F(x_1, \cdots, x_a)$ we obtain the following.

(2.10) *If $F(x_1, \cdots, x_a)$ possesses a differential as function of $x_k^1, \cdots, x_k^n$ at $u_1, \cdots, u_a$; $u_1 \wedge \cdots \wedge u_a \neq 0$, then*

$$(2.11) \qquad \sum_i \frac{\partial F(u_1, \cdots, u_a)}{\partial x_k^i} u_j^i = \delta_j^k F(u_1, \cdots, u_a) .$$

Let A be normal to B at D in Q, $z \in D$. Choose a non-degenerate q-box $[z, y_1, \cdots, y_b, u_{d+1}, \cdots, u_a]$ such that $y_1, \cdots, y_d$ lie in D; $y_{d+1}, \cdots, y_b$ in B and $u_{d+1}, \cdots, u_a$ in A. For any $\lambda_{d+1}, \cdots, \lambda_b$ the box $[z, y_1, \cdots, y_d, u_{d+1}, \cdots, u_a]$ originates from the box

$$\left[z, y_1, \cdots, y_d, u_{d+1} + \sum_{i=d+1}^{b} \lambda_i y_i, u_{d+2}, \cdots, u_a\right]$$

by projection parallel to B. If F possesses a differential at $y_1, \cdots, y_d, u_{d+1}, \cdots, u_a$ as function of $x_{d+1}^1, \cdots, x_{d+1}^n$, then normality of A to B implies that $F(y_1, \cdots, y_d, u_{d+1} + \sum \lambda_i y_i, u_{d+2}, \cdots, u_a)$ has a minimum for $\lambda_i = 0$. Hence

$$\sum \frac{\partial F(y_1, \cdots, y_d, u_{d+1}, \cdots, u_a)}{\partial x^i_{d+1}} y^i_j = 0 \qquad j = d+1, \cdots, b\ .$$

Thus we have found the following.

(2.12) *If the a-flat through z spanned by* $y_1, \cdots, y_d, u_{d+1}, \cdots, u_a$ *is normal to the b-flat B through z spanned by* $y_1, \cdots, y_b$ *and F is differentiable at* $y_1, \cdots, y_d, u_{d+1}, \cdots, u_a$ *as function of* $x^1_k, \cdots, x^n_k$ *for* $k = d+1, \cdots, a$ *then*

$$\sum_i \frac{\partial F(y_1, \cdots, y_d, u_{d+1}, \cdots, u_a)}{\partial x^i_k} y^i_j = 0\ , \qquad \begin{array}{l} k = d+1, \cdots, a \\ j = d+1, \cdots b\ . \end{array} \tag{2.13}$$

We conclude from (2.11) that the matrix $\partial F(y, u)/\partial x^i_k$ has rank $a - d$. Therefore, if D, A, Q are given there can be at most one b-flat transversal to A at D in Q. For brevity we say that $F(x_1, \cdots, x_a)$ is *individually differentiable* at $u_1, \cdots, u_a$ if it possesses a differential at $u_1, \cdots, u_a$ with respect to each of the sets of variables $x^1_k, \cdots, x^n_k$; $k = 1, \cdots, a$.

With property (ii) of (2.6) in mind we state explicitly the following consequence of our discussion.

(2.14) LEMMA. *If* $F(x_1, \cdots, x_a)$ *is individually differentiable at* $u_1, \cdots, u_a$ *with* $u_1 \wedge \cdots \wedge u_a \neq 0$, *and if in each* $(a+1)$*-flat containing the a-flat* A_u *spanned by* $u_1, \cdots, u_a$ *there exists a transversal to* A_u; *then this transversal is unique and the y corresponding to the different* $(a+1)$*-flats through* A_u *form the* $(n-a)$*-flat*

$$\sum_i \frac{\partial F(u_1, \cdots, u_a)}{\partial x^i_k} y^i = 0 \qquad k = 1, \cdots, a\ .$$

3. Convexity.

Convexity, strict convexity and differentiability for the area α were determined in terms of the function $F(x_1, x_2, \cdots, x_a)$ in the introduction as follows.

(3.1) DEFINITION. Writing $F(y, x) = F(y, x_2, \cdots, x_a)$ we say that α is *convex, strictly convex,* or *differentiable* according as the curve $F(\lambda_1 y_1 + \lambda_2 y_2, x) = 1$ has those properties in the plane spanned by y_1, y_2 for any linearly independent $y_1, y_2, x_2, \cdots, x_a$.

Thus for convex α we have

$$F(y_1 + y_2, x) \leq F(y_1, x) + F(y_2, x) \text{ for } y_1 \wedge y_2 \wedge \cdots \wedge x_a \neq 0$$

with strict inequality for strict convexity. If we do not exclude linear dependence of y_1, y_2, then setting $y_1 = \mu y_2$ we have

$$F(y_1+y_2,x)=|1+\mu|F(y_1,x)\begin{cases} = F(y_1,x)+F(y_2,x) \text{ if } \mu \geq 0 \\ < F(y_1,x)+F(y_2,x) \text{ if } \mu < 0 \,. \end{cases}$$

Thus we find the following.

(3.2) LEMMA. *The area function* α *is convex if and only if*

$$F(y_1+y_2, x_2, \cdots, x_a) \leq F(y_1, x_2, \cdots, x_a) + F(y_2, x_2, \cdots, x_a)$$

for $y_i \wedge x_2 \wedge \cdots \wedge x_a \neq 0$; *and is strictly convex if and only if equality implies* $y_1 = \mu y_2$, $\mu > 0$.

Let α be convex and $u_1 \wedge \cdots \wedge u_a \neq 0$. The function F has a differential with respect to $x_1^1, \cdots, x_1^n$ at $u_1, \cdots, u_a$ if and only if the curve

$$F(\lambda u_1 + \mu v, u_2, \cdots, u_a) = 1$$

is differentiable at $\lambda = 1, \mu = 0$ for all v with $v \wedge u_1 \wedge \cdots \wedge u_a \neq 0$. We have thus proved the following.

(3.3) *A convex area function* α *is differentiable if and only if the corresponding function* $F(x_1, \cdots, x_a)$ *is individually differentiable for* $x_1 \wedge \cdots \wedge x_a \neq 0$.

The differentiability properties of convex functions imply that for every convex α the corresponding F has strong differentiability properties, of which we need only the following.

(3.4) LEMMA. *If* α *is convex and* $u_1, \wedge \cdots \wedge u_a \neq 0$, *then there exist sequences* $\{u_{i\nu}\}$ *such that* $u_{i\nu} \to u_i$ $(i = 1, \cdots, a)$ *and such that* $F(x_1, \cdots, x_a)$ *is individually differentiable at* $u_{i\nu}, \cdots, u_{a\nu}$.

Reformulation of these properties in terms of the function $f(A)$ will prove useful. Since $f(A)$ is defined relative to a definite euclidean metric $e(x, y)$ we may use euclidean concepts. In particular we will speak of "perpendicularity" when we mean normality with respect to $e(x, y)$.

Consider a plane P perpendicular at z to the $(a-1)$-flat L_{a-1} and choose in L_{a-1} an $(a-1)$-box $[z, x_2, \cdots, x_a]$ with euclidean $(a-1)$-volume 1. On each ray R in P with origin z choose y_R such that $F(y_R, x_2, \cdots, x_a) = 1$. The euclidean a-volume of this box is $e(z, y_R)$. Hence, if A_R is the a-flat containing R and L_{a-1} then

$$F(y_R, x_2, \cdots, x_a) = f(A_R)e(z, y_R) = 1 \,.$$

If the t-flat L, $2 \leq t \leq n-a+1$ is perpendicular to L_{a-1} at z we denote by $S(L_{a-1}, L_t)$ the locus in L_t obtained by taking the point y_R

with $e(z, y_R) = f^{-1}(A_R)$ on a variable ray R in L_t with origin z. Then we can express our result as follows.

(3.5) LEMMA. *The area function α is convex, strictly convex, convex and differentiable if for any L_2, L_{a-1} and only if for all L_t, L_{a-1}, $2 \leq t \leq n - a + 1$ the surface $S(L_{a-1}, L_t)$ is convex, strictly convex, convex and differentiable.*

Following the arguments of [7] we now settle the case $d = \min(a, b) - 1$. The emphasis is not only on the result, but also on the *method* of constructing normal and transversal flats which the proof provides.

(3.6) THEOREM. *Let $d = \min(a,b) - 1, q = a + b - d \leq n$. For given d-, a-, q-flats $D \subset A \subset Q$, there exists a b-flat B transversal to A at D in Q if and only if the area function α is convex. B is unique when α is differentiable. The normal to B at D in Q is unique for all given $D \subset B \subset Q$ if and only if α is strictly convex.*

Proof. There are two cases.

CASE I: $a = d + 1, b = q - 1$. If $z \in D \subset Q$ are given we take the $(q - d)$-flat L_{q-d} perpendicular to D in Q at z and construct the surface $S = S(D, L_{q-d})$ of (3.5). An a-flat A through D in Q intersects L_{q-d} in two rays, each containing a point of S. Let y_A be one of these points. We claim that *B is transversal to A at D in Q if and only if it is spanned by D and a $(q - d - 1)$-flat through z parallel to a supporting flat H of S in L_{q-d} at y_A.*

The additional remarks on strict convexity and differentiability are then obvious. For if $H \cap S$ contains more points than y_A then the normal A to B at D in Q is not unique, and if S has two different supporting flats at y_A then B is not unique.

To prove our assertion we take A_1 perpendicular to B through D in Q, and in A_1 we take a set M_1 with $0 < \alpha(M_1) < \infty$. If we use $C = L_{q-d} \cap B$ to define projection parallel to B, then we have for the projection M of M_1 on any A

$$\alpha(M) = |M|_a^e f(A) = |M_1|_a^e |\sec(y_A z y_{A_1})| f(A) . \tag{3.7}$$

Therefore B is transversal to A if and only if $|\cos(y_A z y_{A_1})| f^{-1}(A)$ is maximal; or if and only if S has a supporting plane at y_A which is perpendicular to the ray from z through y_{A_1}, in other words is parallel to B.

The construction is easily freed from the intervening metric $e(x, y)$. Let $1 \leq a = d + 1 < q \leq n$ and let $z \in D \subset Q$ be given. Take a non-degenerate d-box $[z, x_1, \cdots, x_d]$ in D and a $(q-d)$-flat L_{q-d} in Q which intersects D at z only. In L_{q-d} construct the locus

$$S = \{x \mid F(x, x_1, \cdots, x_d) = 1\} .$$

Then the a-flat spanned by $x, x_1, \cdots, x_d$ with $x \in S$ is normal to the b-flat B in Q through D if and only if $B \cap L_{q-d}$ is parallel to a supporting $(q-d-1)$-flat of S at x.

CASE II: $b = d+1, a = q-1$. As in Case I take L_{q-d} perpendicular to D at z in Q. Instead of using S we now take the line perpendicular to a variable a-flat A through D in Q. The two points y_A with $e(z, y_A) = f^{-1}(A)$ generate a locus T. When α is convex, strictly convex, convex and differentiable then T has the corresponding property.

This time we claim that B is transversal to A at D in Q if and only if it is spanned by D and the perpendicular to a supporting $(q-d-1)$-flat of T in L_{q-d} at y_A. We define A_1 and M_1 as in Case I and use the line C perpendicular to A_1 at z for projection parallel to B. Then the projection M of M_1 on any A again satisfies (3.7) and $f^{-1}(A)|\cos(y_A z y_{A_1})|$ is maximal if and only if y_A lies on a supporting flat of T which is perpendicular to C. Since C is perpendicular to A_1 it lies in B. The additioned remarks follow as in Case I.

The definition of T cannot be entirely freed from extraneous concepts, but their role can be reduced.

If T is convex, let T' be the polar reciprocal in L_{q-d} of T with respect to the metric $e(x, y)$ (see [5, p. 28]). If T is strictly convex (differentiable) then T' is differentiable (strictly convex). In terms of T' we can interpret the normality relation in a manner similar to that of Case I; only the roles of normality and transversality are interchanged.

If $x \in T'$ then the $(d+1)$-flat spanned by x and D is transversal to the a-flat A through D in Q if and only if A is spanned by D and a $(q-d-1)$-flat parallel to a supporting flat of T' at x.

In the most interesting case, $d = 0$, the surface T' has a very interesting meaning. In $(Q = L_{q-d})$ take any $(q = a+1)$-measure invariant under translation. The only arbitrariness is then the unit of measure. Then T' is a *solution of the isoperimetric problem* to minimize the α-area among all closed convex hyper-surfaces in Q which bound a set of given $(a+1)$-measure. For details see [6]. Of course T' remains a solution even if we change the unit of $(a+1)$-measure.

Assume that α is convex and consider an a-flat A_u through z spanned by $u_1, \cdots, u_a$ and such that F is individually differentiable at $u_1, \cdots, u_a$. Then (3.6) (more particularly Case II) guarantees that in every $(a+1)$-flat containing A_u there exists a transversal to A_u at z. We conclude from (2.14) that the transversals at z to A_u in the different

$(a+1)$-flats form an $(n-a)$-flat N_A and from Theorem (2.6) that this N_A is transversal to A.

If F is not individually differentiable at $u_1, \cdots, u_a$ then we can find sequences $\{u_{i\nu}\}$ with $u_{i\nu} \to u_i$ $(i = 1, \cdots, a)$ such that F is individually differentiable at $u_{i\nu}, \cdots, u_{a\nu}$. Hence if A_ν contains $z, u_{i\nu}, \cdots, u_{a\nu}$ then there exists an $(n-a)$-flat N_ν transversal to A_ν at z. By the continuity of the area function every limit $(n-a)$-flat of a subsequence of N_ν is transversal to A. Thus if α is convex there exists an $(n-a)$-flat transversal to A. Using (2.14) and (2.5) we have proved

(3.8) THEOREM. *If the area function α is convex then, given an a-flat A, a d-flat $D \subset A$ and a q-flat $Q \supset A$ with $0 \leq d < a < q \leq n$; there exists a b-flat, $b = q - a + d$, transversal to A at D in Q, which is unique when α is differentiable.* (*Wagner* [15], *for* $d = 0$).

The conditions in (3.8) are also necessary, but we conclude from (2.5) and (3.6) that we need consider only fixed d and q.

(3.9) THEOREM. *With the notation of* (3.8); *if for fixed d, q and all A, D, Q a b-flat transversal to A at D in Q exists* (*and is unique*) *then α is convex* (*and differentiable*).

A normal to B at D in Q is in general not unique even for strictly and extendably convex α (as we shall see in (5.14)) when $d < \min(a, b) - 1$. For in that case normality is not equivalent to total normality. However, because total normals exist and are normal we have

(3.10) *If the a-flat A normal to B at D in Q is unique, then A is totally normal to B.*

Even the total normal is not necessarily unique for strictly and extendably convex α, see (5.14).

4. Area minimizing a-flats. Total and extendable convexity. The area $\alpha(\Delta)$ of an a-dimensional polyhedron Δ is defined as the sum of the a-areas of its a-faces. In the following we reserve Δ for the union of all a-faces but one, Δ_0, of an a-dimensional polyhedron in A^n which is abstractly a closed orientable a-dimensional manifold but may have self interersections in A^n. By A_Δ we denote the a-flat containing the face Δ_0 and hence the boundary of Δ.

We say that the a-flat A (strictly) minimizes α-area in the q-flat $Q \supset A$, $q > a$, if $\alpha(\Delta) \geq \alpha(\Delta_0)$ $(\alpha(\Delta) > \alpha(\Delta_0))$ for all choices of $\Delta \neq \Delta_0$ in Q for which $A_\Delta = A$. If this is true for all a-flats A in Q we say that the a-flats (strictly) minimize area in Q.

The case $a = 1$ is familiar; with the help of (3.6) we may formulate these results as follows.

The line L minimizes α-length in the q-flat Q if and only if a $(q-1)$-flat B transversal to L in Q at a point z exists. The line L strictly minimizes length in Q if and only if L is the only line normal to B at z.

The lines (strictly) minimize α-length in A^n if and only if α is (strictly) convex.

A few of these facts extend to the general case.

(4.1) *The a-flat A minimizes α-area in Q if a $(q-a)$-flat B totally transversal to A at a point z exists. Let B exist. Then A strictly minimizes α-area when A is the only a-flat totally normal to B at z or when a is strictly convex.*

Project $\varDelta$ on $A_\varDelta$ parallel to B. For topological reasons this projection covers $\varDelta_0$. Let σ be an a-dimensional face of $\varDelta$ which lies in the a-flat A and let σ_0 be its projection on $A_\varDelta$. If $\dim(B \cap A) > 0$ then obviously $0 = \alpha(\sigma_0) < \alpha(\sigma)$. If $\dim(B \cap A) = 0$ then the transversality of B to $A_\varDelta$ implies $\alpha(\sigma_0) \leq \alpha(\sigma)$. This proves $\alpha(\varDelta) \geq \alpha(\varDelta_0)$.

If α is strictly convex and $\varDelta \neq \varDelta_0$ then $\alpha(\varDelta) > \alpha(\varDelta_0)$ is obvious when $\dim(B \cap A) > 0$ for some A containing an a-face of $\varDelta$. Assume therefore $\dim(B \cap A) = 0$ for all such A. There is at least one pair of a-faces σ^1, σ^2 of $\varDelta$ which have a common $(a-1)$-face and at least one of which is not parallel to $A_\varDelta$. If A^i is the a-flat containing σ^i then not both A^1, A^2 can be normal to B. For, if A_i is the a-flat parallel to A^i through $A_\varDelta \cap B$ then $\dim(A_1 \cap A_2) = a - 1$ and hence $A_1 \cup A_2$ spans an $(a+1)$-flat Q which intersects B in a line L through $A_\varDelta \cap B$. Since α is strictly convex at least one of the two a-flats, say A_1, is by (2.3) and (3.6) not normal to B. Hence A' is not normal to B and $\alpha(\sigma') > \alpha(\sigma_0')$. Hence $\alpha(\varDelta) > \alpha(\varDelta_0)$.

If A is the only total normal to B at z then at least one a-face σ' of $\varDelta$ is not totally normal to B and again $\alpha(\sigma') > \alpha(\sigma_0')$.

The case $q=a+1$ is completely known essentially through Minkowski (Theorie der konvexen Körper, § 27, Ges. Abh. 2, Leipzing 1911, 131–229). His terminology is so different that we give the argument here.

For each $(a+1)$-flat Q through z we construct the surface T_Q, analogous to T in the discussion of Case II in the proof of (3.6), as the locus T_Q of the points y_A with $e(z, y_A) = f^{-1}(A)$ on the perpendiculars to the a-flats A through z in Q.

(4.2) *The a-flat A minimizes α-area in the $(a+1)$-flat Q if and only if a line transversal to A in Q exists.*

A strictly minimizes area in Q if and only if a line transversal to in Q exists and y_A is not an interior point of an a-flat region on T_Q.

The sufficiency of the first part of (4.2) follows from (4.1) and the fact that a line transversal to A is totally transversal to A. We next prove the necessity statements in both parts of (4.2).

We choose rectangular coordinates such that Q is the flat $x^{a+2} = \cdots = x^n = 0$ and define, as usual,

$$H(0) = 0 \text{ for } x = 0, H(x) = |x| f(A_x) \text{ for } x \neq 0 ,$$

where A_x is the α-flat through z in Q with normal x and $|x| = (\sum x_i^2)^{1/2}$, so that T_Q has the equation $H(x) = 1$. The function $H(x)$ is convex with α.

If no transversal to A exists then, according to Case II in (3.6), T_Q does not possess a supporting α-flat at y_A; so that y_A is an interior point of the convex closure of T_Q. Hence independent points $x_1, \cdots, x_{a+1}$ on T_Q exist such that

$$(4.3) \qquad H(y_A) > \sum_{i=1}^{a+1} \lambda_i H(x_j) , \qquad y_B = \sum_{i=1}^{a+1} \lambda_i x_i , \qquad \lambda_i > 0 .$$

If y_A is an interior point of an α-flat set on T_Q then independent $x_1, \cdots, x_{a+1}$ on T_Q exist with

$$(4.4) \qquad H(y_A) = \sum_{i=1}^{a+1} \lambda_i H(x_i) , \qquad y_A = \sum_{i=1}^{a+1} \lambda_i x_i , \qquad \lambda_i > 0 .$$

Setting $\xi = y_A / |y_A|$, $\xi_i = x_i / |x_i|$ we have $-|y_A| \xi + \sum \lambda_i |x_i| \xi_i = 0$.

Therefore (see Bonnesen-Fenchel [5, p. 118]),[5] an $(\alpha + 1)$-simplex in Q exists whose faces have exterior normals, $-\xi, \xi_1, \cdots, \xi_a$ and area $|x|, \lambda_1 |x_1|, \cdots, \lambda_a |x|$. The total area of the faces with normals $\xi_1, \cdots, \xi_a$ is

$$\sum \lambda_i |x_i| f(A_{x_i}) = \sum \lambda_i H(x_i)$$

and $|x| f(A_x) = H(x)$ is the area of the face with normal $-\xi$.

The relations (4.3), (4.4) prove the necessity statements in (4.2).

To establish sufficiency in the second part of (4.2) we resume the notation used in the last part of the proof of (4.1). We assume that Δ lies in Q and replace B by a line L transversal to $A = A_\Delta$.

For $\alpha(\Delta) = \alpha(\Delta_0)$ it is necessary that the mapping of Δ on Δ_0 by projection parallel to L be one-to-one and that all α-flats carrying α-faces of Δ be normal to L.

Now there are two supporting flats A', A'' of T_Q perpendicular to L. On the other hand the construction of the transversal in the discussion of Case II in (3.6) shows that at the points y_A which corresponds to an A normal to L the surface T_Q has supporting planes perpendicular to

[5] The proof there is involved but becomes very simple in the present case where the number of faces is $a + 1$.

L. Therefore A' and A'' each contain one of the two points y_A and one of the two points y_A for each A which carries an a-face of Δ.

Since projection of Δ on Δ_0 is one-to-one and $\Delta \neq \Delta_0$ it follows that among the points y_A, y_A in A' there are $a+1$ which do not lie in an $(a-1)$-flat. These points span an a-simplex which lies on T_Q.

(4.5) COROLLARY. *The a-flats minimize α-area in all $(a+1)$-flats if and only if α is convex. They strictly minimize area if and only if in addition the surface T_Q contains no a-flat piece for any $(a+1)$-flat Q.*

Our results are not as complete for $q>a+1$, $a\neq 1$. Consider the vector space V_a^n of all contravariant a-vectors $\mathfrak{A}$ in A^n. A simple a-vector $\mathfrak{A} \neq 0$ determines an oriented a-flat in A^n through the origin. For the α-area determined by $\mathfrak{A}$ we obtain a function $\Phi(\mathfrak{A})$ defined on all simple $\mathfrak{A}$-vectors whose relation to F is given by

$$a!\,\Phi(x_1 \wedge \cdots \wedge x_a) = F(x_1, \cdots, x_a)\,.$$

Obviously Φ satisfies the conditions

$$\Phi_1 \qquad \Phi(\mathfrak{A}) > 0 \quad \text{for} \quad \mathfrak{A} \neq 0$$

$$\Phi_2 \qquad \Phi(\lambda\mathfrak{A}) = |\lambda|\,\Phi(\mathfrak{A}) \quad \text{for all real } \lambda\,.$$

All a-vectors are simple only when $a=1$ and $a=n-1$. (If we exclude the trivial cases $a=0, n$). We shall prove at the end of this section that for $1<a<n-1$ and convex α it is in general impossible to extend $\Phi(\mathfrak{A})$ to a convex function defined for all a-vectors. An obviously necessary condition for extendability is

$$\Phi(\mathfrak{A}) \leq \sum_{i=1}^{r} \Phi(\mathfrak{A}_i) \quad \text{for simple } \mathfrak{A}, \mathfrak{A}_i, \text{ with } \quad \mathfrak{A} = \sum_{i=1}^{r} \mathfrak{A}_i\,. \tag{4.6}$$

Condition (4.6) is also sufficient. The simple a-vectors form a basis of V_a^n. Hence if $\mathfrak{A}$ is any a-vector then simple a-vectors $\mathfrak{A}_i$ exist so that

$$\mathfrak{A} = \sum_{i=1}^{r} \mathfrak{A}_i\,, \tag{4.7}$$

since any scalar multiple of a simple vector is simple. We can now extend $\Phi(\mathfrak{A})$ to all of V_a^n by defining

$$\Phi(\mathfrak{A}) = \inf \sum_{i=1}^{r} \Phi(\mathfrak{A}_i)$$

where the $\{\mathfrak{A}_i\}$ traverse all sets of simple vectors whose sum is $\mathfrak{A}$. Because of (4.6) $\Phi(\mathfrak{A})$ is not changed by this definition for simple $\mathfrak{A}$, and the extended function obviously is convex and satisfies Φ_1 and Φ_2.

We call α *extendably convex* if it satisfies (4.6). As before consider a polyhedron $\Delta \cup \Delta_0$. Orient it and let $\mathfrak{A}_1, \cdots, \mathfrak{A}_r$ be the simple a-vectors corresponding to the a-faces in Δ. Let $\mathfrak{A}_0$ correspond to Δ_0. Then

$$\sum_{i=0}^{r} \mathfrak{A}_i = 0 \quad \text{or} \quad \mathfrak{A} = -\mathfrak{A}_0 = \sum_{i=1}^{r} \mathfrak{A}_i$$

so that $\alpha(\Delta) \geq \alpha(\Delta_0)$ is equivalent to condition (4.6). In general the relation $\mathfrak{A} = \sum_{i=1}^{r} \mathfrak{A}_i$ for simple $\mathfrak{A}, \mathfrak{A}_i$ does not imply that $-\mathfrak{A}, \mathfrak{A}_1, \cdots, \mathfrak{A}_r$ correspond to the faces of a closed polyhedron. For example, the a-flats corresponding to $\mathfrak{A}, \mathfrak{A}_i$ through the origin z may intersect at z alone. However it is not unlikely that the validity of (4.6) for $\mathfrak{A}, \mathfrak{A}_i$ deriving from polyhedra implies its general validity. We have not been able to prove this. Thus we can only state:

(4.7) *If α is extendably convex then the a-flats minimize area.*

We call α *totally convex* if an $(n-a)$-flat totally transversal to a given a-flat at a point exists. If the condition in (4.7) is necessary then (4.1) shows that total convexity entails extendable convexity. We shall prove this directly, obtaining at the same time a very interesting geometric interpretation for the two types of convexity. The arguments are closely related to those of Wagner [15].

Denote by W_a the affine space associated with the vector space V_a^n, so that we may speak of hyperplanes etc. which do not pass through 0. The simple vectors in V_a^n form the *Grassmann cone* and the equation $\Phi(\mathfrak{A}) = 1$ defines on that cone the indicatrix I of the area α.

Extendable convexity of α means that I lies on the boundary of its convex closure in W_a; that is, that I possesses at every point a supporting hyperplane in W_a.

In order to interpret total convexity we provide A^n with the euclidean metric $g_{ik} = \delta_{ik}$. This metric induces a scalar product $\mathfrak{A} \cdot \mathfrak{B}$ for the simple a-vectors in A^n whose geomentric meaning, apart from sign, is the product of the (euclidean) area of one vector and the area of the orthogonal projection of the other on the a-flat of the first.

This scalar product for the vectors on the Grassmann cone can be extended to an inner product in V_a^n and hence induces a euclidean metric in W_a. To the projection of an a-flat A_1 on an a-flat A parallel to the $(n-a)$-flat B perpendicular to the a-flat B^* at a point there corresponds in V_a^n the projection of the line A_1 on the line A parallel to the hyperplane H_B perpendicular to the line B^*.

Assume now $\mathfrak{A} \in I$ and that I possesses at $\mathfrak{A}$ a simple supporting hyperplane H_B; that is a hyperplane H_B perpendicular to a line B^* on the Grassmann cone. If $\mathfrak{A}_1$ is a simple vector lying on H_B

(that is, interpreted in A^n, if $|\mathfrak{A}| = |\mathfrak{A}_1'|$ for the projection $\mathfrak{A}_1'$ of $\mathfrak{A}_1$ on the a-flat of $\mathfrak{A}$ parallel to the $(n-a)$-flat B which is perpendicular to B^*), then $\Phi(\mathfrak{A}_1) \geq \Phi(\mathfrak{A})$ since H_B is a supporting plane of I. Therefore B is totally transversal to A.

Conversely, if B is totally transversal to A at a point, then any simple $\mathfrak{A}_1$ whose projection parallel to H_B is $\mathfrak{A}$ satisfies $\Phi(\mathfrak{A}_1) \leq \Phi(\mathfrak{A}) = 1$, so that H_B is a supporting hyperplane of I. This could, of course, be formulated without the use of an auxiliary metric:

(4.10) *The area α is totally convex if and only if the indicatrix I posseses at every point $\mathfrak{A} = (a^\lambda)$ a simple supporting hyperplane $\sum a^\lambda b_\lambda = 1$, where $\mathfrak{B} = (b_\lambda)$ satisfies the conditions of a simple vector.*

If I is differentiable at $\mathfrak{A}$, so that the $a(n-a)$-flat, T, tangent to I at $\mathfrak{A}$ exists, then any supporting hyperplane of I at $\mathfrak{A}$ must pass through T. Through a given $a(n-a)$-flat there is exactly one simple hyperplane (see [15]). Since extendable convexity means only the existence of some supporting hyperplane of I at a given point we deduce from (4.10):

(4.11) *Total convexity implies extendable convexity but not conversely.*

That the converse is not valid does not follow from the preceding arguments, but in (5.13) we give an example of an extendably but not totally convex area.

We now show that convexity of α does not imply extendable convexity (Wagner [15] states this fact for $\min(a, n-a) > 2$; but, as it seems to us, he only proves that a certain definite extension of convex area is in general not convex). For this purpose we prove a lemma which seems to be of some independent interest.

(4.12) LEMMA. *Let S_a be a simple closed $(a-1)$-surface in an a-flat A so that at every point of S_a there is both an interior and an exterior supporting $(a-1)$-sphere of radius c in A. Let $z \in A$ be in the interior of S_a so that at the line zx from z to any $x \in S_a$ makes an angle no less than $\alpha > 0$ with the tangent $(a-1)$-flat of S_a at x.*

Then for every $\varepsilon > 0$ there exists a hypersurface $S \supset S_a$ such that every L_2 through z which contains a line that makes an angle greater than ε with A intersects S in convex curve.

Proof. For sufficiently small $\delta > 0$ the interior parallel surface S_a', which is the locus in the interior of S_a of points whose distance from S_a is δ, obviously satisfies the hypotheses of the lemma provided the constants c and α are replaced by suitable constants c' and α'. Let T_a' be the a-body bounded by S_a'.

Let S be the locus of points whose distance from T'_a is δ. Clearly $S_a \subset S$. Every $L_2 \ni z$ intersects S in a curve C. Assume that C is not convex; then there is an $x \in C$ at which C does not have a line of support in L_2 and therefore S does not have a plane of support at x. Thus the point x' nearest to x on T'_a must lie on S'_a and the line zx makes an angle less than $\tan^{-1}[d(x, x')/d(z, x')] \leq \tan^{-1}(\delta/d)$ with A, where d is the distance from z to S_a.

Now let L be the tangent line to C at x. Since L intersects the interior of C, the cylinder L_δ, which is the locus of points whose distance from L is δ, must intersect the interior of T'_a. Since the quadric $Q_\delta = L_\delta \cap A$ is tangent to S'_a at x' it follows that the minimal curvature of Q_δ at x' is less than $1/c'$. Let L' be the tangent line to Q_δ at x' in the direction of minimal curvature then the tangent of the angle between L and L' is less than $\sqrt{\delta/c'}$.

Thus for sufficiently small δ the two lines L and zx make arbitrarily small angles with the lines L' and zx' in A. Since the last named lines make an angle with each other which exceeds α' it follows that every line in L_2 makes an arbitrarily small angle with A.

Now, for example, in the space V_2^4 of 2-vectors in A^4 we can find a three-plane generated by simple vectors which contains no two-plane of simple vectors. Such a three-plans is L_3 generated by $e_1 \wedge e_2$, $e_3 \wedge e_4$ and $(e_1 + e_3) \wedge (e_2 + e_4)$. The simple vectors which it contains are all of the form $\lambda(e_1 + \mu e_3) \wedge (e_3 + \mu e_4)$. We can now define the area function F so that the indicatrix I does not lie on the boundary of its convex hull in L_3, for instance by $F(e_1, e_2) = F(e_3, e_4) = F(e_1 + e_3, e_2 + e_4) = 1$ and $F(e_1 + 2e_3, e_2 + e_2 + 2e_4) > 6$ in violation of (4.6); but so that $I \cap L_3$ satisfies all the conditions of Lemma (4.12) where z is the zero element of V_2^4. By Lemma (4.12) we can now extend I in such a way that its intersection with every two-plane of simple vectors is convex, in other words, so that F is convex. However, since I does not lie on the boundary of its convex hull, the area is not extendably convex.

5. Equivalence of normality. Example. Quadratic area. The normality relations determine the area up to a constant factor in the following sense.

(5.1) THEOREM. *Let α and α' be two a-dimensional convex area functions, $a + b - d \leq n$ and $d \leq \min(a, b) - 1$. For any d-flat D and any b-flat B through D let A be normal to B at D with respect to α' whenever this is the case with respect to α. Then $\alpha'(M)$ and $\alpha(M)$ differ only by a constant factor.*

The same holds for total normality if there exists a b-flat totally transversal with respect to α for any given a-flat at any given d-flat *in any given* $(a + b - d)$*-flat* (*in particular, when* α *is totally convex*).

Proof. Let $a = d + 1$. With the notation of Case I in (3.6) we construct the surfaces S, S' belonging to α and α' respectively. The hypothesis of (5.1) means in terms of S, S': If H and H' are parallel supporting $(q - d - 1)$-flats of S and S' then a line through z containing a point x of $S \cap H$ also contains a point of $S' \cap H'$. It folllows that S and S' are homothetic, and this conclusion remains valid when this condition on the line zx is assumed only for those $x \in S$ at which S is differentiable, that is H is unique.

This weakening of the hypothesis amounts to requiring that A be normal to B at D with respect to α' only when B is the unique transversal to A at D in $A \oplus B$ with respect to α.

The fact that S and S' are homothetic means that $\alpha'(M)/\alpha(M)$ is constant for all M lying in a-flats through a fixed $(a - 1)$-flat in an $(a + b - d)$-flat. This yields the general answer, because two arbitrary a-flats A', A'' can be joined by a finite number of a-flats $A_1 = A', A_2, \cdots, A_r = A''$ such that $\dim A_i \cap A_{i+1} = a - 1$ for $i = 1, \cdots, r - 1$.

Application of the result just obtained to the pencils determined by A_i and A_{i+1} proves the theorem.

The case $d < a - 1$ is reduced to $d = a - 1$ as follows. Let B^+, $\dim B^+ = b + a - d - 1$ be the unique transversal to A at an $(a - 1)$-flat D^+ in $A \oplus B^+$. In D^+ chose a d-flat D and an $(a - d - 1)$-flat E such that $D^+ = D \oplus E$. Then $D^+ = D \oplus E$ where B is a b-flat and $A \oplus B = A \oplus B^+$ because $E \subset A$.

For normality we know, and for total normality we assume, that a b-flat B' totally transversal to A at D in $A \oplus B^+$ exists. By (2.2) $B' \oplus E$ is transversal to A at D^+ in $A \oplus B^+$ and $B' \oplus E = B^+$ because B^+ is unique. By hypothesis B' is transversal to A at D with respect to α', and again by (2.2) B^+ is transversal to A at D^+ with respect to α'.

This means that the hypothesis of the theorem is satisfied for $d = a - 1$ and $b = a + b - d - 1$, so that the assertion follows from the first part of the proof.

Let $0 \leq d < a < n$. For a given a-area α we say that normality at d-flats is *symmetric*, if normality of an a-flat A to an a-flat A' at a d-flat D implies that A' is normal to A at D.

If $0 \leq d < a < n$ and an a-area α and a b-area β are given, we say that α-normality and β-normality at d-flats are *equivalent*, if normality of an a-flat A to a b-flat B at a d-flat D with respect to α implies that B is normal to A at D with respect to β and conversely, normality of B to A at D implies that A is normal to B at D.

This formulation admits the possibility that $a = b$. If at the same time $\alpha = \beta$ then equivalence means symmetry. If $a = b$ but $\alpha \neq \beta$ then equivalence means that normality in one norm is equivalent to transversality in the other.

Symmetry and equivalence of total normality are defined in the same way by replacing everywhere normality and transversality by total normality or transversality.

In the next section we discuss the implications of symmetry or equivalence of normality. Here we give some examples where these phenomena occur and the area is not euclidean.

(5.2) *For $d = 0, a = 1, n = 2$ symmetry of normality does not imply that the length,* i.e. *the corresponding two-dimensional Minkowski metric, is euclidean.* All these metrics have been determined by Radon [13], (see also [9, p. 104]).

(5.3) *For any $(n-1)$-dimensional convex area function β there is a convex one-dimensional area,* i.e. *a Minkowski metric $F(x-y)$, such that normality of a hyperplane to a line for β is equivalent to normality of the line to the hyperplane for $F(x-y)$.*

To see this we construct the surface T' of Case II of (3.6) for β and $d = 0$. That is, on the perpendicular to a variable hyperplane $B \ni z$ at z we take the two points y_B with $e(y_B, z) = f^{-1}(B)$. These points y_B traverse a convex hypersurface T and T' is the polar reciprocal of T. As Minkowski metric $F(x-y)$ we take the metric with T' as unit sphere $F(x) = 1$. Then the discussion under (3.6) shows that the hyperplanes normal (for β) to a line zw at $w \in T'$ are the supporting planes of T' at w and these are exactly the planes transversal to zw at w for $F(x-y)$.

The a-area $\alpha, 1 \leq a \leq n-1$ is euclidean if $\alpha(M) = |M|_a^e$ for a suitable choice of $e(x, y)$. With the summation convention $\sum_k g_{ik}x^k = g_{ik}x^k$ this means for F that

$$(5.4) \qquad F^2(x_1, \cdots, x_a) = \sum_{i_1 < \cdots < i_a} \begin{vmatrix} x_1^{i_1} & \cdots & x_1^{i_a} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x_a^{i_1} & \cdots & x_a^{i_a} \end{vmatrix} \begin{vmatrix} g_{i_1k}x_1^k & \cdots & g_{i_ak}x_1^k \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ g_{i_1k}x_a^k & \cdots & g_{i_ak}x_a^k \end{vmatrix} .$$

We shall call α *quadratic* if F^2 is a quadratic form in each set of variables $x_i^1, \cdots, x_i^n$ $(i = 1, \cdots, a)$. A quadratic F^2 is a quadratic form in the Plücker coordinates.

$$P^{i_1 \cdots i_a} = \begin{vmatrix} x_1^{i_1} & \cdots & x_1^{i_a} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x_a^{i_1} & \cdots & x_a^{i_a} \end{vmatrix}, \qquad 1 \leq i_1 < \cdots < i_a \leq n ,$$

of the a-flat through z spanned by $x_1, \cdots, x_a$, since $F(x_1, \cdots, x_a) = f(A)|[z, x_1, \cdots, x_a]|_a^e$ where the terms on the right depend only on the Plücker coordinates.

If F is quadratic then for any L_{a-1} and L_2 perpendicular to L_{a-1} at z the curve $S(L_{a-1}, L_2)$ of (3.5) is an ellipse and conversely. If Q is any $(a+1)$-flat through z we construct in Q the surface T of Case II of (3.6) for $D = z$. The section of T with any plane $L_2 \ni z$ is obtained from $S(_{a-1}, L_2)$, where L_{a-1} is perpendicular at z to L_2 in Q, by a rotation through $\pi/2$. Hence T is an ellipsoid. This implies that the area restricted to Q is euclidean Thus we have the following.

(5.5) THEOREM. *An a-area is quadratic if and only if it is euclidean in every $(a+1)$-flat; that is to say, if and only if normality of a-flats at $(a-1)$-flats in $(a+1)$-flats is symmetric.*

We now wish to determine under what conditions a quadratic area is euclidean.

(5.6) *A quadratic a-area is euclidean if $a = 1$ or $a = n-1$, and in general is not euclidean if $1 < a < n-1$.*

The first part of the statement is obvious since a quadratic length is euclidean by definition and a quadratic $(n-1)$-area is euclidean in n-space by (5.5).

A simple counting argument convinces us of the truth of the second part since a euclidean quadratic area is determined by the metric (g_{ij}) so that the manifold of euclidean quadratic areas is $n(n+1)/2$-dimensional, while the manifold of Plücker coordinates is of dimension $1 + a(n-a)$; or, in other words, there are $\binom{n}{a} - a(n-a) - 1$ independent (quadratic) identities satisfied by the $P^{i_1 \cdots i_a}$ (see e.g. [2]). The distinct quadratic form in the Plücker coordinates therefore have dimension

$$\frac{1}{2}\binom{n}{a}\left[\binom{n}{a} + 1\right] - \binom{n}{a} + a(n-a) + 1$$

which exceeds $\binom{n+1}{2}$ whenever $1 < a < n-1$.

If, for example, we restrict our attention to a-areas for which

$$F^2(x_1, \cdots, x_a) = \sum_{i_1 < \cdots < i_a} A_{i_1 \cdots i_a} (P^{i_1 \cdots i_a})^2$$

then no two different forms can be identical. Thus the dimension of this set is $\binom{n}{a}$ while the dimension of each equivalence class is no greater than $\binom{n+1}{2}$.

In particular, the Cartesian form

$$F^2(x_1, \cdots, x_n) = \sum (P^{i_1 \cdots i_a})^2 \tag{5.7}$$

is preserved only under orthogonal transformations. For, if we assume the existence of a matrix $g_{ij} \neq \delta_{ij}$ which preserves the form (5.7) then, for a suitable choice of Cartesian coordinates, we have $g_{ij} = A_i \delta_{ij}$ and (5.7) becomes

$$F^2(x_1, \cdots, x_n) = \sum A_{i_1} \cdots A_{i_a} (P^{i_1 \cdots i_a})^2 \tag{5.8}$$

Now, if (5.8) is Cartesian in one Cartesian coordinate system then it is Cartesian in all. Thus $A_{i_1} \cdots A_{i_a} = 1$ for all $i_1 < \cdots < i_a$. Since $n > a$ this implies $A_i = 1$ and $g_{ij} = \delta_{ij}$.

Since every euclidean area can be brought to Cartesian form we have also proved the following (which also follows from Theorem 9.1).

(5.9) *If two metrics g_{ij} and g'_{ij} give rise to identical a-areas, $a < n$, then $g_{ij} = g'_{ij}$.*

We can now determine the relations which suffice to make a quadratic a-area euclidean:

(5.10) THEOREM. *A quadratic a-area is euclidean if it is euclidean in every $(a + 2)$-flat.*

Proof. We proceed by induction. Assuming the area is euclidean is every m-flat, $m \geq a + 2$, we wish to prove it euclidean in every $(m + 1)$-flat. Let the $(m + 1)$-flat L_{m+1} have the equations $x^{m+2} = \cdots = x^n = 0$. Since the area is euclidean in every sub-flat $x^i = 0$ $(i = 1, \cdots, m + 1)$, there exists a matrix $g^{(i)}_{pq}$ $(1 \leq p, q \leq m + 1; p, q \neq i)$ so that the area function has the form (5.4) in this sub-flat. By (5.9) we have $g^{(i)}_{pq} = g^{(j)}_{pq}$ if $p, q \neq i, j$ since that is the unique metric in the common sub-flat $x^i = x^j = 0$. Thus there exists a matrix $g_{pq} = g^{(i)}_{pq}$ $(i \neq p, q)$ that defines a euclidean a-area in L_{m+1} which coincides with the given a-area in every coordinate sub-flat.

Without loss of generality we may assume the coordinates in L_{m+1} chosen so that $g_{pq} = \delta_{pq}$. Then on L_{m+1} we have

$$F^2(x_1, \cdots, x_a) = \sum (P^{i_1 \cdots i_a})^2 + R \tag{5.11}$$

where R involves the products of distinct Plücker coordinates so that every index $1, \cdots, m + 1$ appears in every product (if $m + 1 > 2a$ then there are no such terms and the proof is complete).

Consider the sub-flat $x^{m+1} = \lambda x^m$ of L_{m+1} and introduce the coordinates $y^i = x^i$ $(i = 1, \cdots, m - 1)$, $y^m = (1 + \lambda^2)^{-1} x^m$. In terms of these coordinates (5.11) becomes

$$(5.12) \qquad F^2(x_1, \cdots, x_a) = \sum (P^{i_1 \cdots i_a})^2 + R'$$

where R' involves only products in which there appears every index $1, \cdots, m$. Now (5.12) is euclidean by hypothesis and the matrix g'_{ij} which represents it in the form (5.4) reduces to the identity matrix in every coordinate sub-flat. Hence $g'_{ij} = \delta_{ij}$ and $R' \equiv 0$. This means $R = 0$ in every sub-flat $x^{m+1} = \lambda x^m$, that is, $R \equiv 0$ so that (5.11) is euclidean.

The simplest case of an a-area with $1 < a < n - 1$, namely quadratic 2-area in A^4, already povides examples to show that:

(5.13) *For* $1 < a < n - 1$ *an extendably convex a-area need not be totally convex.*

For, denote the euclidean area in A^4 which belongs to $g_{ik} = \delta_{ik}$ by $E(x_1, x_2)$ and put $e_i = (\delta_{i1}, \cdots, \delta_{i4})$. For any $\varepsilon > 0$

$$F^2(x_1, x_2) = \varepsilon E^2(x_1, x_2) + \left(\begin{vmatrix} x_1^1 & x_1^2 \\ x_2^1 & x_2^2 \end{vmatrix} + \begin{vmatrix} x_1^3 & x_1^4 \\ x_2^3 & x_2^4 \end{vmatrix} \right)^2$$

defines a quadratic 2-area which obviously is extendably convex. The (x^1, x^2)-plane P^{12} is normal to every line in the (x^3, x^4)-plane P^{34}, because for arbitrary λ, μ, ρ we have

$$\begin{aligned} F^2(e_1 + \lambda e_3 + \mu e_4, e_2 + \rho\lambda e_3 + \rho\mu e_4) &\geq \varepsilon E^2(e_1, e_2) + \left(\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} \lambda & \mu \\ \rho\lambda & \rho\mu \end{vmatrix} \right)^2 \\ &= \varepsilon + 1 = F^2(e_1, e_2) \, . \end{aligned}$$

Thus P^{12} is normal to P^{34}. However, for small ε, P^{12} is not totally normal to P^{34}, since then

$$F^2(e_1 + e_3 + e_4/2, e_2 + e_3 - e_4/2) = \varepsilon E^2(e_1 + e_3 + e_4/2, e_2 + e_3 - e_4/2) < 1 + \varepsilon .$$

According to (3.10) the plane normal to P^{34} at z cannot be unique. Actually there is a one-parameter family of planes totally normal to P^{34} at z. To see this we observe that

$$\begin{aligned} &F^2(e_1 + \lambda e_3 + \mu_4, e_2 + \rho e_3 + \sigma e_4) \\ &\qquad = \varepsilon(1 + \lambda^2 + \mu^2 + \rho^2 + \sigma^2 + (\lambda\sigma - \mu\rho)^2) + (1 + \lambda\sigma - \mu\rho)^2. \end{aligned}$$

For a given ε with $0 < \varepsilon < 1$ this expression attains the minimal value $4\varepsilon/(1+\varepsilon)$ for $\lambda = -\sigma = \delta \cos\theta$, $\mu = \rho = \delta \sin\theta$ where $\delta = (1-\varepsilon)^{1/2}(1+\varepsilon)^{-1/2}$ and θ is arbitrary. Hence

(5.14) *If* $1<a<n-1$ *then extendable strict convexity of an a-area does not imply that the a-flat totally normal to an* $(n - a)$*-flat at a point is unique. More generally, the a-flat totally normal to a b-flat at a d-flat is not necessarily unique when* $d < \min(a, b) - 1$.

6. Equivalence of normality. Implications. Equivalence of normality for two convex areas implies for most combinations of the dimensions a, b, d that both areas are quadratic.

(6.1) THEOREM. *Let $0 \leq d < a \leq b < n$ $a+b-d \leq n$, but not $a+b=n$ and $d=0$. If a convex a-area α and a convex b-area β have the property that (total) α-normality and (total) β-normality at d-flats are equivalent, then both α and β are quadratic.*

(6.2) COROLLARY. *If for a convex a-area (total) normality at d-flats is symmetric then the area is quadratic unless $n = 2a$ and $d = 0$.*

We know from (5.3) that $a = 1, b = n - 1$ is actually exceptional but no examples are known for $a > 1$. (See note at end of paper).

The following proof is arranged so that only the existence of normals and not of transversals is used. Since the total normals exist, the proofs remain valid when normality is replaced everywhere by total normality.

Since normals and total normals do exist for non-convex areas, it is possible that (6.1) also holds without the assumption that α and β be convex. However the present proof uses convexity.

The hypothesis on the dimensions means that either (1) $a + b < n + d$ or (2) $a + b = n + d$ and $d > 0$. We consider the two cases separately.

In case (1) we show first (denoting an i-flat by L_i):

(A) Given[6] $L_{a-1} \subset L_{a+1} \subset L_{a+2}$ there exists an $L_a \subset L_{a+2}$ with $L_a \cap L_{a+1} = L_{a-1}$ such that the a-flats through L_{a-1} in L_{a+1} are normal to L_a.

(A′) The same as (A) with b replacing a.[6]

The proofs are entirely analogous with a slight simplification for (A) which we shall point out.

To prove (A) take $L_{n-b+a} \supset L_{a+1}$ with $L_{n-b+a} \cap L_{a+2} = L_{a+1}$, then take B normal to L_{n-b+a} at $D \subset L_{a-1}$. If $d + 1 < a$ choose the $(a - 1 - d)$-flat C such that $D \oplus C = L_{a-1}$. Since $B \oplus L_{n-b-a} = A^n$ we can find L_a with $L_{a-1} \subset L_a \subset B \oplus C$ and $L_a \oplus L_{a+1} = L_{a+2}$. (Here we can take $L_a \subset B$, but in the proof of (A′) there would exist no $L_b \subset A$ for $b > a$, whereas $L_b \subset A \oplus C$ exists because C is a $(b - 1 - d)$-flat and hence dim $A \oplus C = b - 1 - d + a \geq b$.) B is normal, hence by hypothesis transversal at D to any a-flat A' in L_{n-b+a} through D. If $L_{n-b+a} \supset A' \supset L_{a-1}$ then A' is normal to $B \oplus C$ at L_{a-1} and hence is normal to L_a at L_{a-1}.

We now show that (A) implies that α is quadratic. Let $z \in L_a \subset L_{a+2}$ and take L_3 through a perpendicular to L_{a-1} in L_{a+2}. Construct the surface $S = S(L_3, L_{a-1})$ of (3.5). It follows from the discussion of (3.6) Case I that for two lines G, H through z in L_3 the a-flat $G \oplus L_{a-1}$ is normal to $H \oplus L_{a-1}$ if and only if H is parallel to a supporting line of S at one of the two points $G \cap S$.

[6] $a + 2 \leq b + 2 \leq n$ since $b < n + d - a \leq n - 1$.

Now it follows from (A): Given L_2 through z in L_3 there exists in L_3 a $G \ni z$ such that for $z \in H \subset L_2$ the a-flat $H \oplus L_{a-1}$ is normal to $G \oplus L_{a-1}$. In terms of S this means that every intersection of S with a plane through z lies in some circumscribed cylinder of S.

A well known theorem of Blaschke [3] (see also [4, p. 157]) states that a closed convex surface S' in A^3 is an ellipsoid if every cylinder touches S' in a plane curve. Blaschke assumes that S' is differentiable but not that S' has a center. The differentiability hypothesis is very easily removed (see e.g. [9, p. 93]).

Under the hypothesis that S' has a center z the hypothesis may be relaxed in two ways.

(B_1) *S' is an ellipsoid when every plane section of S' through z lies on a circumscribed cylinder.*

(B_2) *S' is an ellipsoid when every circumscribed cylinder contains a plane section of S' through z.*

(B_1) is proved by a trivial modification of the proof of Blaschke's theorem and is also well known from the theory of Banach spaces.

The proof of (B_2) requires a less obvious but far from difficult modification of Blaschke's proof. (B_1) and (A) show that S is an ellipsoid. It follows that $S(L_{a-1}, L_{n-a+1})$ is also an ellipsoid (compare for example [9, p. 91]).

In the same way we deduce from (A′) and (B_1) that the surface $S(L_{b-1}, L_3)$ constructed with the area β is an ellipsoid so that β is also quadratic.

We now turn to the case $a + b = n + d$, $d > 0$ and prove:

(C) *Given $z \in L_{a-2} \subset L_{a-1} \subset L_{a+1}$ there is an a-flat A in L_{a+1} with $A \cap L_{a-1} = L_{a-2}$ such that the a-flats A_G in L_{a+1} through L_{a-1} are normal to A. The same holds with b replacing a.*

Take B normal to L_{a-1} at an $L_{a-1} \subset L_{a-2}$. Such a B exists because $a - 1 + b = n + d - 1$, moreover $L_{a-1} \oplus B = A^n$.

For any line G through z in B the a-flat $A_G = L_{a-1} \oplus G$ is transversal to B at $D_G = L_{a-1} \oplus G$. Hence A_G is, by hypothesis, normal to B at D_G. If $a > d + 1$ choose an $(a - d - 1)$-flat C through z in L_{a-1} such that $L_{a-1} \oplus C = L_{a-2} \subset L_{a-1}$. Then A_G is normal to $B \oplus C$ at $L^G_{a-1} = D_G \oplus C$. Let $z \in L_2 \subset B$, $L_2 \cap L_{a-1} = z$, $L_2 \oplus L_{a-1} = L_{a-1}$. This L_2 exists because $L_{a-1} \cap B = L_{a-1}$ and $L_{a-1} \oplus B = A^n$. Then $A = L_2 \oplus L_{a-2} \subset B \oplus C$ and for $z \in G \subset L_2$ the a-flat A_G is normal to A at L^G_{a-1}.

We now construct a surface T as in Case II of (3.6). On the line perpendicular to a given a-flat A' through z in L_{a+1} we take $y_{A'}$ with $e(z, y_{A'}) = f^{-1}(A')$. The points $y_{A'}$ traverse T.

Also, for a given L_{a-2} with $z \in L_{a-2} \subset L_{a+1}$ we take the L_3

perpendicular to L_{a-2} through z ($L_3 = L_{a+1}$ if $a = 2$). If $A' \supset L_{a-2}$ then the perpendicular to A' at z lies in L_3. The perpendiculars to the $A' \supset L_{a-2}$ therefore intersect T in a surface T_0 and it suffices to prove that T_0, or its polar reciprocal T_0' in L_3, is an ellipsoid.

According to the discussion of Case II the a-flat spanned by $x \in T_0'$ and L_{a-2} is transversal to the a-flat $A' \supset L_{a-2}$ if and only if A' is spanned by L_{a-2} and a plane L_2 through z parallel to a supporting plane of T_0' at x. Then A' is normal also to every a-flat in L_{a+1} through L_{a-2} and x.

Statement (C) means in terms of T_0', that given a line H through z ($H \oplus L_{a-2}$ is the L_{a-1} in the hypothesis of (C)) the cylinder parallel to H circumscribed to T_0' touches T_0' in a set containing a section of T_0' by a plane L_2 through z ($H \oplus L_{a-2}$ is the L_a in the assertion of (C)). It now follows from (B_2) that T_0' is an ellipsoid.

The proof that β is quadratic for $a + b = n + d, d > 0$ is again entirely analogous.

The Corollary (6.2) can be improved in special cases as follows:

(6.3) THEOREM. *If $a < n/2$ and $d = 0$ or $a > n/2$ and $d = 2a - n$ and for a (totally) convex a-area α (total) normality at d-flats is symmetric, then α is euclidean.*

The area function is differentiable because, according to (6.2), it is quadratic (in other respects the present proof is independent of (6.2)).

Let $a < n/2, A \ni z$ and let B_A be the $(n - a)$-flat transversal to A at z. Then each a-flat $A' \ni z$ in B_A is transversal to A. Hence by hypothesis A' is normal to A so that $B_{A'} \supset A$. Thus $A' \supset B_A$ implies $B_{A'} \supset A$. The mapping $A \to B_A$ can therefore be extended to a correlation φ on itself of the bundle consisting of all i-flats $(1 \leq i \leq n - 1)$ through z (see [2, pp. 51–53]). Moreover φ is a polarity because $A\varphi^2 = A$, and if $L_1 \ni z$ then $L_1\varphi$ does not contain L_1. Thus φ coincides with the mapping which belongs to a suitable ellipsoid E with center z which associates $L_1 \ni z$ with its diametral hyperplane $L_1\varphi$. This nearly obvious fact may be seen as follows.

We extend A^n to a projective space P^n and the correlation φ to a correlation of P^n by first associating $z = (0, \cdots, 0,1)$ with the hyperplane at infinity $H = (0, \cdots, 0,1)$. With the intersection $L_1 \cap H = (x_1, \cdots, x_n, 0)$ we associate the hyperplane $L_1\varphi = (\xi_1, \cdots, \xi_n, 0)$. If T is the (symmetric) matrix of $(x_1, \cdots, x_n, 0) \to (\xi_1, \cdots, \xi_n, 0)$ then $\begin{pmatrix} T & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix of a polarity in P^n which defines the ellipsoid E with the above property.

This ellipsoid taken as unit sphere defines a euclidean metric in A^n and also a euclidean a-area. By construction normality of a-flats at z for this area coincides with α-normality of a-flats at z. According to

(5.1) this shows that the two areas differ only by a constant factor so that α is also euclidean.

The case $a > n/2, d = 2a - n$ is very similar. If $A \ni z$ we take the $(n-a)$-flat B_A transversal to A at z. This time if $A' \supset B_A$ then A' is transversal to A at $A' \cap A$ where $\dim A' \cap A = 2a - n$. By hypothesis A' is normal to A so that $B_{A'} \subset A$. Since $A' \supset B_A$ implies $B_{A'} \subset A$, the mapping $A \to B_A$ can again be extended to a correlation of the bundle of all i-flats $(1 \leq i \leq n-1)$ through z on itself. From here on the proof proceeds exactly as in the first case.

7. Minkowski area. We now apply our results to the special cases from which the general theory originated.

Consider a symmetric Minkowski metric (or a 1-dimension convex area) $F(x)$ in A^n. We denote its unit ball $F(x) \leq 1$ by U and let $U(A)$ denote the intersection of U with the a-flat A through z. For any a-flat $\bar{A}$ parallel to A the intersection $(F(x-\bar{z}) \leq 1) \cap \bar{A}, \bar{z} \in \bar{A}$ originates from $U(A)$ by translation and is a unit ball in $\bar{A}$ for the metric induced by $F(x)$ in $\bar{A}$. Following [7] we define an a-dimensional area $1 \leq a \leq n$ in A^n by stipulating that the measure of $U(A)$ have the euclidean volume

$$\pi_a = \pi^{a/2}/\Gamma\left(\frac{a}{2}+1\right),$$

(in particular $\pi_1 = 2, \pi_2 = \pi$) so that for a definite euclidean metric e we have

$$f_a(\bar{A}) = f_a(A) = \pi_a/|U(A)|_a^e .$$

The functions corresponding to our previous $\alpha(M)$ and $F(x_1, \cdots, x_a)$ will be denoted by $|M|_a$ and $F_a(x_1, \cdots, x_a)$ so that

$$|M|_a = f_a(A)|M|_a^e ,$$

$$F_a(x_1, \cdots, x_a) = f_a(A)|[z, x_1, \cdots, x_a]|_a^e$$

and $F_1(x) = F(x)$. Since we admitted $a = n$ we also have an n-dimensional measure

$$|M|_n = f_n|M|_n^e = \pi_n|M|_n^e/|U|_n^e .$$

For $a < n$ let $L_{a-1} \ni z$ be an $(a-1)$-flat and L_2 the plane perpendicular to L_{a-1} at z. On a variable ray R with origin z in L_2 take the point y_R with

$$e(z, y_R) = f_a^{-1}(A_R) = |U(A_R)|_a^e \pi_a^{-1} , \qquad A_R = L_{a-1} \oplus R .$$

That is the curve $S(L_{a-1}, L_2)$ for f_a as constructed in (3.5). It is a fundamental and non-trivial fact (see [7, p. 164]) that $S(L_{a-1}, L_2)$ for f_a is always convex and is strictly convex or differentiable when the unit sphere $F(x) = 1$ of the space is strictly convex or differentiable respectively. Thus we have the following.

(7.1) THEOREM. *The Minkowski areas* $|M|_a, (1 \leq a \leq n-1)$ *are all convex. They are strictly convex or differentiable if the unit sphere* $F(x) = 1$ *is strictly convex or differentiable.*

The question whether Minkowski areas are totally convex for $1 < a < n-1$ is equivalent to a difficult problem on convex bodies. Even extendable convexity is not known (see Problem 10 in [10]).

We mention the following further property of Minkowski area which is important for differential geometric investigations and was proved by Barthel [1].

(7.2) *If* $F(x)$ *is of class* C^r *for* $x \neq 0$ *then* $F_a(x_1, \cdots, x_a)$ *is of class* C^r *for* $x_1 \wedge \cdots \wedge x_a \neq 0$.

We also note:

(7.3) *If the a-area,* $1 \leq a \leq n-1$, *of a Minkowski space is quadratic then the space is euclidean.*

For, if $a > 1$ then we conclude from (6.3) that the area in any $(a+1)$-dimensional subspace is euclidean. It is easily seen and contained in Theorem (9.1) that therefore the metric in this subspace is euclidean. It is well known (see e.g. [9, (16.12) p. 91]) that then the metric of the whole space is euclidean. Therefore (6.1) and (6.2) yield the following.

(7.4) THEOREM. *Let* $0 \leq d < a \leq b < n$ *but not* $a + b = n$ *and* $d = 0$. *If* α, β *are Minkowski a-and b-areas respectively* (*not necessarily relative to the same Minkowski metric*) *and* α*-normality and* β*-normality at d-flats are equivalent then both Minkowski metrics are euclidean.*

If normality of a-flats at d-flats in a Minkowski space is symmetric then the space is euclidean unless $a = n/2, d = 0$.

We note in particular that for all $n > 2$ symmetry of normality of a-flats at $(a-1)$-flats suffices to make the Minkowski space euclidean. From (5.2) we know that the case $a = 1, b = n-1, d = 0$ is exceptional for two distinct Minkowski metrics. Whether this case is exceptional when α and β belong to the same Minkowski space amounts (unless $a = b = 1$) to an interesting open problem on convex bodies (see [10, Problem 5]).

Finally we see from Theorem (7.4) and the example at the end of § 6 that convex area functions—even quadratic area functions—are in general not Minkowskian. The problem of characterization of Minkowski areas among the convex area functions remains open, (see § 9).

The fact that area functions are now defined for all a leads to new concepts, in particular to a sine function. If $A \cap B = D \ni z$, where $0 \leq d \leq \min(a, b) - 1$ and $A \oplus B = Q$, take a non-degenerate q-box, $[z, y_1, \cdots, y_b, x_{d+1}, \cdots, x_a]$, such that $y_1, \cdots, y_d \in D; y_{d+1}, \cdots, y_b \in B - D$ and $x_{d+1}, \cdots, x_a \in A - D$. Now put

$$(7.5) \qquad \operatorname{sm}(A, B) = \frac{F_d(y_1, \cdots, y_d)F_q(y_1, \cdots, y_b, x_{d+1}, \cdots, x_a)}{F_a(y_1, \cdots, y_d, x_{d+1}, \cdots, x_a)F_b(y_1, \cdots, y_b)} ,$$

where $F_0 = 1$. The number $\operatorname{sm}(A, B)$ is called the *Minkowski sine* of the flats A, B because it depends only on the latter and not on the choice of the q-box. For example, if $d > 0$ then replacing $y_1, \cdots, y_d$ by other independent $\bar{y}_1, \cdots, \bar{y}_d \in D$ amounts to multiplying all four terms F_d, F_q, F_a, F_b in (7.5) by

$$|[z, \bar{y}_1, \cdots, \bar{y}_d]|_d^e / |[z, y_1, \cdots, y_d]|_d^e .$$

If D does not contain z, but $\bar{z} \in D$ then the vectors y_i, x_j in (7.5) must be replaced by $y_i - \bar{z}, x_j - \bar{z}$.

The sine function is *not* the function of a number, "the angle between A and B". Even in the euclidean case this angle is defined *only* for $d = \min(a, b)$-1. Hence the restriction to this case in [7] and [1]. The sine function for the euclidean metric will be denoted by se. Then obviously, with $f(L_0) = 1$, we have

$$(7.6) \qquad \operatorname{sm}(A, B) = \operatorname{se}(A, B)f_d(D)f_q(0)f_a^{-1}(A)f_b^{-1}(B) .$$

For any λ_j^k, $k = d + 1, \cdots, b$; $j = d + 1, \cdots, a$ put

$$y_k(\lambda) = \sum_{j=d+1}^{b} \lambda_j^k y_j .$$

Then the boxes of the form $[z, y_1, \cdots, y_d, x_{d+1} + y_{d+1}(\lambda), \cdots, x_a + y_a(\lambda)]$ have $[z, y_1, \cdots, y_d, x_{d+1}, \cdots, x_a]$ as projection in Q parallel to B on A. Since

$$F_a(y_1, \cdots, y_b, x_{d+1} + y_{d+1}(\lambda), \cdots, x_a + y_a(\lambda))$$

does not depend on the λ_j^k, the a-flat A is totally normal to B at D in Q if and only if

$$\operatorname{sm}(A, B) \geq \operatorname{sm}(A^*, B) \text{ for } A^* \cap B = D, A^* \subset Q .$$

We denote this maximal value of $\mathrm{sm}(A^*, B)$ for given B, D, Q by $\alpha(B, D, Q)$. If $q = n$ then $Q = A^n$ is unique and we write simply $\alpha(B, D)$.

(7.7) *If*

$$\mathrm{sm}(A_1, B_1) = \max_A \alpha(A, D, Q),$$

then

$$\mathrm{sm}(A_1, B_1) = \max_B \alpha(B, D, Q),$$

and conversely, hence A_2 is totally normal to B_2 and B_2 is totally normal to A_2.

Proof. If A is normal to B then

(7.8) $$\alpha(B, D, Q) = \mathrm{sm}(A, B) \leq \alpha(A, D, Q)$$

hence

(7.9) $$\max_B \alpha(B, D, Q) \leq \max_A \alpha(A, D, Q)$$

Similarly, if B' is totally normal to A' then

(7.10) $$\alpha(A', D, Q) = \mathrm{sm}(A', B') \leq \alpha(B', D, Q)\,.$$

Whence together with (7.9) we have

(7.11) $$\max_B \alpha(B, D, Q) = \max_A \alpha(A, D, Q)\,.$$

If

$$\mathrm{sm}(A_1, B_1) = \max_A \alpha(A, D, Q) = \alpha(A_1, D, Q)$$

then B_1 is totally normal to A_1. Hence (7.11) and (7.10) imply

$$\mathrm{sm}(A_1, B_1) = \alpha(B_1, D, Q) = \max(B, D, Q)$$

so that A_1 is totally normal to B_1.

(7.12) *If for given A (B) in Q through D there exists a b-flat (a-flat) totally transversal to B (A) at D in Q (which is always the case for* $\min(a, b) = d + 1$) *and*

$$\mathrm{sm}(A_2, B_2) = \min_A \alpha(B, D, Q)$$

then

$$\mathrm{sm}(A_2, B_2) = \min_B \alpha(B, D, Q)$$

and A_2, B_2 are normal to each other.

For, if A is totally transversal to a given B, then

$$\alpha(A_2, D, Q) = \operatorname{sm}(A, B) \leq \alpha(B, D, Q) .$$

Hence

$$\min_A \alpha(B, D, Q) \leq \min_B \alpha(B, D, Q) .$$

The proof is analogous to that of (7.7).

As a consequence of (7.11) and (7.12) we have the following.

(7.13) COROLLARY. *If the function $\alpha(A, D, Q)$ is constant for fixed D, Q then $\alpha(B, D, Q)$ is constant and conversely. Moreover the constants have the same value. If $\alpha(A, D, Q)$ or $\alpha(B, D, Q)$ is constant then total normality of A to B and total normality of B to A are equivalent.*

The equivalence of total normality follows from the fact that for any A totally normal to B we have

$$\operatorname{sm}(A, B) = \max_A \alpha(A, D, Q) .$$

The equivalence of normality implies that B (A) totally transversal to A (B) at D in Q exist. Therefore both (7.11) and (7.12) apply.

Whether the converse of the second statement in (7.13) always holds is not known. However the proof of (3.6) yields the following special case.

(7.14) *If $d = \min(a, b) - 1$ and normality of A to B at D in Q is equivalent to normality of B to A, then $\alpha(A, D, Q)$ and $\alpha(B, D, Q)$ are constant.*

Proof. If $z \in D$ we take as in Case I of (3.6) the $(q-d)$-flat L_{q-d} perpendicular to D at z and construct, if $a \leq b$ say, the surface S by taking on each ray R in L_{q-d} with origin z the point y_R with $e(z, y_R) = f_a^{-1}(A_R)$ where $A_R = D \oplus R$.

For the b-area we construct T as in Case II by taking on the perpendicular in Q to a b-flat B through D in Q the two points y'_R with $e(z, y'_R) = f_b^{-1}(B)$, and denote by T' the polar reciprocal of T in L_{q-d} with respect to the metric $e(x, y)$.

If w_R is the point $R \cap T'$ then the supporting $(q-d-1)$-flat of T' at w_R spans together with the d-flat parallel to D through w_R a b-flat B normal to A_R. The reciprocity of T and T' implies that B has distance $f_b(B)$ from z. Hence by (7.6) we have[7]

[7] Because $d = \min(a, b) - 1$ the function se is the ordinary sine of the angle between A_R and B in the metric $e(x, y)$.

$$\alpha(A_R, D, Q) = \mathrm{sm}\,(A_R, B) = \mathrm{se}\,(A_R, B) f_d(D) f_q(Q) f_a^{-1}(A_R) f_b^{-1}(B) .$$

But

$$f_b(B) = \mathrm{se}\,(A_R, B) e(z, w_R),\, f_a(A_R) = e^{-1}(z, y_R) ,$$

so that we have the following nice *interpretation for* $\alpha(A_R, D, Q)$:

$$\alpha(A_R, D, Q) = f_d^{-1}(D) f_q^{-1}(Q) e(z, y_R)/e(z, w_R) .$$

If normality of A to B at D in Q is equivalent to that of B to A then S and T' are homothetic. Hence $e(z, y_R)/e(z, w_R)$ is constant, which proves (7.14).

8. The range of the sine functions. Problems regarding the ranges of $\alpha(B, D, Q)$ are important for Minkowskian geometry and are geometrically very attractive, but unfortunately often quite difficult—only in the simplest case $n = 2$ hence $a = b = 1, d = 0$ do we have complete answers owing to Petty [12] who found the following.

For any line L_1 *in* A^2 *through* z *we put* $\alpha(L_1, z, A^2) = \alpha(L)$ *and denote by* C_F *the unit circle* $F(x) = 1$. *Then*

$$\min_{L_1, F} \alpha(L_1) = \pi/4, \quad \max_{L_1, F} \alpha(L_1) = \pi/2 ,$$

and $\alpha(L_1) = \pi/4$ *or* $\alpha(L_1) = \pi/2$ *imply that* C_F *is a parallelogram and* L_1 *a suitable line (different in the two cases).*

Also

$$\max_F \min_{L_1} \alpha(L_1) = \pi/3 ,$$

where the maximum is attained only when C_F *is a hexagon which is regular for a suitable* $e(x, y)$.

Finally

$$\min_F \max_{L_1} \alpha(L_1) = 1 ,$$

where the minimum is attained only when C_F *is an ellipse, that is when the metric is euclidean.*

By (7.13) and (7.14) we have $\alpha(L_1) = k_F$, that is $\alpha(L_1)$ is independent of L_1, if and only if normality of lines in the plane is symmetric. This means that C_F is one of the curves discovered by Radon [13] which we encountered already several times implicitly and which we shall call Radon curves. Their construction is also found in Petty [12] and in [9, p. 104]. Since the regular hexagon is a Radon curve we find $1 \leq k_F \leq \pi/3$ with $k_F = 1$ only for the euclidean metric and $k_F = \pi/3$ only when C_F is a regular hexagon.

Under the hypothesis of (7.14), if $a = b$ and hence $d = a - 1$ then S and T' are Radon curves and we can derive the range of $\alpha(L_a, L_{a-1}, L_{a+1})$ (when constant) from Petty's results. Otherwise the ranges for $\alpha(A, D, Q)$ with D, Q fixed are not known. For variable D, Q we deduce from (7.13) and (7.14) the following.

(8.1) THEOREM. *If* $0 \leq d < a \leq b < n$ *but not* $a + b = n$ *and* $d = 0$ *then* $\alpha(L_a, L_d, L_{a+b-d})$ *is independent of* L_a, L_d, L_{a+b-d} *only in the euclidean geometry (where all* α*-functions are equal to* 1).

Beyond this result only very few facts on the ranges of the sine functions are known for $n > 2$, which we shall now discuss.

$$\min_{F,L_1} \alpha(L_1, L_0) = \min_{F,L_{n-1}} \alpha(L_{n-1}, L_0) = \pi_n/2\pi_{n-1} \tag{8.2}$$

$$\max_{F,L_1} \alpha(L_1, L_0) = \max_{F,L_{n-1}} \alpha(L_{n-1}, L_0) = n\pi_n/2\pi_{n-1} \,. \tag{8.3}$$

In the first of these relations equality is obtained only when the unit sphere S, that is $F(x) = 1$, is a *cylinder* and in the second only when S is a *double cone.*

The proof is very simple. The equality of the first two members in (8.2) or (8.3) follows from (7.12) and (7.7). Let H be a hyperplane through z and L_1 normal to H at z. If p, p' are the points $L_1 \cap S$ and $U_H = U \cap H$ then the hyperplanes parallel to H through p and p' are supporting planes of U. Moreover U_H has maximal $(n-1)$-dimensional volume among all sections U by hyperplanes parallel to H. Therefore

$$\pi_n = |U|_n \leq F(p - p')|U_H|_{n-1} \operatorname{sm}(L_1, H) = 2\pi_{n-1}\alpha(H, z)$$

with equality only for cylinders.

On the other hand U contains the double cone formed by the cones with apexes p, p' and bases U_H so that

$$\pi_n \geq n^{-1} 2\pi_{n-1}\alpha(H, z)$$

with equality only for double cones.

These relations successively provide bounds for all $\alpha(L_a, L_d)$, but these bounds are not sharp. We exemplify the procedure with $\alpha(L_{n-2}, L_0)$. If L_{n-2} is normal to L_2 at z then we consider in L_2 lines L_1' and L_1 through z such that L_1' is normal to L_1. Since L_{n-2} is normal to L_1 and L_1' we have, with $L_{n-1} = L_{n-2} \oplus L_1$,

$$\operatorname{sm}(L_{n-2}, L_2) \operatorname{sm}(L_1', L_1) = \operatorname{sm}(L_{n-2}, L_1) \operatorname{sm}(L_1', L_{n-1})$$

or

$$\begin{aligned}\operatorname{sm}(L_{n-2}, L_2) &= \alpha(L_1, z, L_{n-1}) \operatorname{sm}(L_1', L_{n-1})\alpha^{-1}(L_1, z, L_2) \\ &\leq \frac{n-1}{2}\,\frac{\pi_{n-1}}{\pi_{n-2}}\,\frac{n}{2}\,\frac{\pi_n}{\pi_{n-1}}\,\frac{4}{\pi} = \frac{n(n-1)}{\pi}\,\frac{\pi_n}{\pi_{n-2}} \,.\end{aligned}$$

It is easily seen that with a proper choice of L_1, L'_1 in L_2 the line L'_1 is normal to L_{n-1}. Hence

$$\operatorname{sm}(L_{n-2}, L_2) \geq \frac{1}{2}\frac{\pi_{n-1}}{\pi_{n-2}}\frac{1}{2}\frac{\pi_n}{\pi_{n-1}}\frac{2}{\pi} = \frac{1}{2\pi}\frac{\pi_n}{\pi_{n-2}},$$

so that

$$\frac{1}{2\pi}\frac{\pi_n}{\pi_{n-2}} \leq \alpha(L_{n-2}, L_0) \leq \frac{n(n-1)}{\pi}\frac{\pi_n}{\pi_{n-2}}.$$

The only exact bound other than (8.2) and (8.3) which has been determined is the following.

$$\max_{F, L_{n-1}} \alpha(L_{n-1}, L_{n-2}) = 2\pi_{n-2}\pi_n/\pi_{n-1}^2. \tag{8.4}$$

This equality holds only for a cylindrical unit sphere with $(n-2)$-dimensional generators and a parallelogram as 2-dimensional crosssection whose exact definition will emerge from the proof.

If an L_{n-2} is given we choose coordinates so that its equations are $x_{n-1} = x_n = 0$ and put $x_{n-1} = \rho\cos\varphi$, $x_n = \rho\sin\varphi$ so that $x_1, \cdots, x_{n-2}, \rho, \varphi$ are our coordinates. Set $U(L_{n-2}) = V$. For given x, φ with $x \in V$ let $(x, r(x, \varphi), \varphi)$ lie on the unit sphere S. Then, with $e^2(x, y) = \sum(x^i - y^i)^2$,

$$|U|_n^e = \frac{1}{2}\int_0^{2\pi}\int_V r^2(x, \varphi)dx\,d\varphi \geq \frac{1}{2|V|_{n-2}^e}\int_0^{2\pi}\left(\int_V r(x, \varphi)dx\right)^2 d\varphi$$

with equality only when $r(x, \varphi)$ is independent of x.

Now $\int_V r(x, \varphi_0)dx$ is the euclidean volume $A(\varphi_0)$ of the intersection of U with the half-hyperplane $\varphi = \varphi_0$. Hence if P_{φ_0} is the hyperplane containing $\varphi = \varphi_0$ we have

$$\begin{aligned}\operatorname{sm}(P_{\varphi_1}, P_{\varphi_2}) &= \frac{\sin|\varphi_1 - \varphi_2|2A(\varphi_1)2A(\varphi_2)}{|V|_{n-2}^e|U|_n^e}\cdot\frac{\pi_{n-2}\pi_n}{\pi_{n-1}^2}\\ &\leq \frac{4\sin|\varphi_1 - \varphi_2|A(\varphi_1)A(\varphi_2)}{(1/2)\int_0^{2\pi}A^2(\varphi)d\varphi}\cdot\frac{\pi_{n-2}\pi_n}{\pi_{n-1}^2}\end{aligned}$$

Considering the convex curve $\rho = A(\varphi)$ in $x_1 = \cdots = x_{n-2} = 0$ we see that the first factor on the right attains its maximum 2 when the curve is a parallelogram and φ_1, φ_2 fall in the diagonals. There will be equality in (8.4) if and only if in addition $r(x, \varphi)$ is independent of x. For $n = 3$ we have equality only for a parallelepiped.

The most important questions regarding the ranges of the sine functions concern

$$\min_F \max_{L_a} \alpha(L_a, L_d) = \min_F \max_{L_{n-a+d}} \alpha(L_{n-a+d}, L_d),$$

in particular whether, or for which a, d this number equals 1; and whether the value 1 characterizes euclidean geometry. The case $a = 1, d = 0$ is Problem 6 in [10].

9. Relations between the functions f_a. The Minkowski areas are derived from—and hence determined by—the Minkowski length. The question arises whether in a Minkowski geometry any of the areas $(1 < a < n)$ determine the remaining ones.

(9.1) THEOREM. *An a-dimensional area function $F_a(x_1, \cdots, x_a)$, $1 \leq a \leq n - 1$, is an a-dimensional Minkowski area for at most one Minkowski geometry. In other words, if $F_a(x_1, \cdots, x_a)$ is known then $F(x)$ and hence the remaining $F_b(x_1, \cdots, x_b)$ are determined.*

This follows from a theorem of P. Funk [11]:

Let S_e be the sphere $e(z, x) = 1$ in B and let $S(A)$ be its intersection with $A \ni z$. Let $g_i(x)$, $i = 1, 2$ be an even continuous function on S_e and denote by $S(A, g_i)$ the integral of $g_i(x)$ over $S(A)$ with respect to $(a - 1)$-dimensional area. If $S(A, g_1) = S(A, g_2)$ for each A with $z \in A \subset B$ then $g_1(x) \equiv g_2(x)$.

Induction reduces this statement to $a = b - 1$.

A proof for $b = 3$ is found in [5, p. 138]. A proof for general b is obtained by using expansion in terms of spherical harmonics. If $x \in S_e$ then $xF^{-1}(x)$ lies on $F(x) = 1$. Hence $|U(A)|_a^e = S(A, a^{-1}F^{-a}(x))$ so that by Funk's theorem this relation determines $F(x)$.

An explicit expression of $F(x)$ in terms of $f_a(A)$ can be found in [4, pp. 154, 155], and this yields, in principle, the value $f_b(B)$ for given B. Actually the expression thus obtained is much too involved to deduce pertinent information from it. There is however an inequality of a very simple form, although its proof is involved, which relates f_b and f_a and which we are now going to derive from the results of [8].

If $n \geq b > a > 1$, $B \ni z$ then

$$D(b, a)f_b^{-a}(B) \geq \int_{B \supset A \ni z} f_a^{-b}(A)dA \tag{9.2}$$

with equality only for the ellipsoid. In this formula dA is the kinematic density for a-flats in B, the quantity $D(b, a)$ is the measure of all a-flats through z in B and hence is a constant which depends only on a and b.

Since in (9.2) B acts as the whole space we may take $b = n$. The inequality is a special case of a relation between the functions

$$f_{i,a}(A) = \pi_a / |U_i(A)|_a^e, f_{i,n} = \pi_n / |U_i|_n^e \qquad i = 1, \cdots, a$$

belonging to different Minkowski metrics with unit spheres $U_1, \cdots, {}_aU$

with common center z:

$$D(n,a)\prod_{i=1}^{a} f_{i,n}^{-1} \geq \int_{A\ni z}\prod_{i=1}^{a} f_{i,a}^{-n/a}(A)dA\ , \tag{9.3}$$

with equality only when the U_i are homothetic ellipsoids, i.e. when the corresponding Minkowski metrics are proportional euclidean metrics.

The inequality (9.3) is in turn a consequence of a still more general inequality.

Let $M_1, \cdots, M_a$ be convex bodies in the n-dimensional euclidean space $E^n, n \geq 3, 2 \leq a \leq n-1$ then

$$\begin{aligned} |M_1|_n \cdots |M_a|_n &\geq \pi_n^a \pi_a^{-n} D^{-1}(n,a) \\ &\times \int_{A\ni z} |M_1 \cap A|_a^{n/a} \cdots |M_a \cap A|_a^{n/a} dA\ , \end{aligned} \tag{9.4}$$

with equality for $|M_i|_n > 0$ only when the M_i are homothetic ellipsoids with center z. The measure $|M|_i$ is of course, the i-dimensional Lebesgue measure in E^n.

We deduce (9.4) from the following relation for any closed bounded sets $M_1, \cdots, M_a$.

$$\begin{aligned} |M_1|_n \cdots |M_a|_n &= C^1(n,a) \\ &\times \int_{A\ni z}\int_{M_1\cap A}\cdots\int_{M_a\cap A} T^{n-a}(P_1, \cdots, P_a, z)dV_{P_1}^a \cdots dV_{P_a}^a dA \end{aligned} \tag{9.5}$$

where $T(P_1, \cdots, P_a, z)$ is the a-dimensional measure of the (possibly degenerate) simplex with vertices $P_1, \cdots, P_a, z$ and $dV_{P_i}^a$ is the area element of A at $P_i \in M_i \cap A$. The symbol $C^i(n,a)$ denotes a constant which depends only on n and a.

For $a = n-1$ and $a = n-2$ (9.5) is proved in [8, (2), (17)], hence we prove (9.5) by induction for decreasing a. Assume (9.5) to hold for some $a+1 \leq n-1$. As M_{a+1} we take the euclidean unit ball U with center z. Then if B denotes an $(a+1)$-flat we have

$$\begin{aligned} |M_1|_n \cdots |M_a|_n &= \pi_n^{-1} C^1(n, a+1) \\ &\times \int_{B\ni z}\int_{M_1\cap B}\cdots\int_{M_{a+1}\cap B} T^{n-a-1}(P_1, \cdots, P_{a+1}, z)dV_{P_1}^{a+1} \cdots dV_{P_{a+1}}^{a+1} dB\ . \end{aligned}$$

Now $M_{a+1} \cap B$ is an $(a+1)$ − dimensional unit ball $\overline{U}$, and if φ is the angle between the a-flats spanned by $P_1, \cdots, P_a$ and the line through z and P_{a+1}, then

$$T(P_1, \cdots, P_{a+1}, z) = (a+1)^{-1}e(z, P_{a+1})|\sin\varphi|T(P_1, \cdots, P_a, z)\ .$$

Since

$$\int_{\bar{U}} e^{n-a-1}(z, P_{a+1}) |\sin^{n-a-1}\varphi| d V_{P+1}^{a+1}$$

depends only on n and a we obtain, after carrying out the integration ever U,

$$|M_1|_n \cdots |M_a|_n = C^2(n, a)$$
$$\times \int_{B \ni z} \int_{M_1 \cap B} \cdots \int_{M_a \cap B} T^{n-a-1}(P_1, \cdots, P_a, z) d V_{P_1}^{a+1} \cdots d V_{P_{a+1}}^{a+1} dB .$$

For a variable a-flat A through z in B we have (see [8, (12)])

$$d V_{P_1}^{a+1} \cdots d V_{P_a}^{a+1} = a!\, T(P_1, \cdots, P_a, z) d V_{P_1}^{a} \cdots d V_{P_a}^{a} dA .$$

Integration first over all A through z in B, and then over all B through z can, according to the properties of kinematic measure, be interpreted as an integration over all A through z (except for a factor which depends only on a), and this proves (9.5).

Steiner's symmetrization leads from (9.5) to (9.4). Consider a fixed a-flat A through z and let $M_1, \cdots, M_a$ be convex bodies. It is shown in [8, pp. 8–10] that under simultaneous symmetrization of the sets $M_i \cap A$ in any $(a-1)$-flat C through z in A the integral

$$\int_{M_1 \cap A} \cdots \int_{M_a \cap A} T^{n-a}(P_1, \cdots, P_a, z) d V_{P_1}^{a} \cdots d V_{P_a}^{a}$$

decreases unless the centers of all chords of all $M_i \cap A$ perpendicular to C are coplanar with z. Hence the $M_1 \cap A$ are homothetic ellipsoids with center z if the last integral is to be minimized. The minimum is actually attained for such ellipsoids [8, pp. 10, 11] and the integral has then the value

$$C^3(n, a) |M_1 \cap A|_a^{n/a} \cdots |M_a \cap A|_a^{a/a} .$$

This proves (9.4).

We note two consequences of these results.

(9.6) *The ellipsoids with center z maximize* $\int_{A \in z} |M \cap A|_a^n dA$ *among all convex bodies with a given volume.*

Application of (9.4) to the case $M_2 = \cdots = M_a = U$ yields

$$(9.7) \qquad |M|_n \geq \pi_n \pi_a^{-n/a} D^{-1}(n, a) \int_{A \in z} |M \cap A|_a^{n/a} dA ,$$

with equality only for the sphere. Hence the sphere gives the maximum of $\min_A |M \cap A|_a^n |M|_n^{a-n}$ for given volume $|M| > 0$.

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Note: While this paper was in print it was shown in H. Busemann: *Areas in affine space* II (to appear in the Rend. Circ. Mat. Palermo) that the case $a+b=n, d=1$ in (6.1) is exceptional for all a and also that $a=n/2,\ d=0$ in (6.2) is always exceptional.