ON UNIQUENESS QUESTIONS FOR HYPERBOLIC DIFFERENTIAL EQUATIONS

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1. Statement of results. This note is concerned with the existence, uniqueness, and successive approximations for solutions of the initial value problem

$$z_{xy} = f(x, y, z, p, q),\ z(x, 0) = \sigma(x),\ z(0, y) = \tau(y)\ ,$$

where $\sigma(0) = \tau(0) = z_0$, on a rectangle $R: 0 \leqq x \leqq a,\ 0 \leqq y \leqq b$. By a solution is meant a continuous function having partial derivatives almost everywhere and satisfying the integral equation

$$(1) \qquad z(x, y) = \sigma(x) + \tau(y) - z_0 + \int_0^x\int_0^y f(s, t, z(s, t), z_x(s, t), z_y(s, t))\,dsdt.$$

Actually it will be clear from the conditions imposed on σ, τ and f that any solution of (1) is uniformly Lipschitz continuous. Let D be the five-dimensional set $D = \{(x, y, z, p, q,) : (x, y) \in R \text{ and } z, p, q \text{ arbitrary}\}$. Let $f(x, y, z, p, q)$ be defined and continuous on D, such that $|f(x, y, z, p, q,)| < N = \text{const.}$ for $(x, y, z, p, q) \in D$. Let $\sigma(x)$, $\tau(y)$ be defined and uniformly Lipschitz continuous on $0 \leqq x \leqq a$, $0 \leqq y \leqq b$, respectively (so that $|\sigma(x) - \sigma(\bar{x})| \leqq K|x - \bar{x}|$, $|\tau(\mathrm{y}) - \tau(\bar{y})| \leqq K|y - \bar{y}|$ for some constant K) and let $\sigma(0) = \tau(0) = z_0$. In addition, for $(x, y) \in R$ and arbitrary $z, p, q, \bar{z}, \bar{p}, \bar{q}$ assume that

$$(2) \qquad |f(x, y, z, p, q,) - f(x, y, \bar{z}, \bar{p}, \bar{q})| \leqq \varphi(x, y, |z - \bar{z}|, |p - \bar{p}|, |q - \bar{q}|)\ ,$$

where $\varphi(x, y, z, p, q)$ is a continuous, non-negative function defined for $(x, y) \in R$ and non-negative z, p, q, non-decreasing in each of the variables z, p, q, and with the property that for every (α, β), where $0 < \alpha \leqq a$, $0 < \beta \leqq b$, the only solution of

$$(3) \qquad z(x, y) = \int_0^x\int_0^y \varphi(s, t, z(s, t), z_x(s, t), z_y(s, t))\,dsdt$$

in the rectangle $R_{\alpha\beta}: 0 \leqq x \leqq \alpha,\ 0 \leqq y \leqq \beta$ is $z \equiv 0$.

THEOREM (*). *Under the above assumptions on σ, τ, f and φ,* (1) *possesses one and only one solution on R. This solution is the uniform limit of the successive approximations defined by*

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$$(4_0) \qquad z_0(x, y) = \sigma(x) + \tau(y) - z_0$$

and, for $n = 1, 2, 3, \cdots$, *by*

$$(4_n) \quad z_n(x, y) = z_0(x, y) + \int_0^x\int_0^y f(x, y, z_{n-1}(s, t), z_{n-1\,x}(s, t), z_{n-1\,y}(s, t))\,dsdt\,.$$

The existence assertion of (*) neither implies nor is implied by that in Hartman-Wintner [3] and its generalizations due to Conti, Szmydt, Ciliberto, Kisynski (for references, see [6] and [2]). The uniqueness assertion of (*) can be considered as a crude analogue of Kamke's uniqueness theorem (cf. [5], p. 139) in the theory of ordinary differential equations. Finally, the assertion concerning the convergence of successive approximations is an analogue of a result on ordinary differential equations (cf. Viswanatham [8] and references there to van Kampen, to Wintner and to Dieudonne, and Coddington and Levinson [1]).

A theorem similar to (*), in which f and φ do not depend on p, q is proved by Guglielmino [2]. The proof of (*) below will be a generalization of that of [2]. A uniqueness theorem for (1) involving a majorant function of the form $\varphi(z, p, q) = \varphi(|z| + |p| + |q|)$ is given in [6]. (After the completion of this manuscript, I learned[1] of a paper "On the existence theorem of Caratheodory for ordinary and hyperbolic differential equations" by W. Walter, written at about the same time, which contains a theorem in the direction of the uniqueness assertion of (*). Walter's assumptions, however, are somewhat different.)

REMARK. It will be clear from the proofs that (*) *remains valid if* f, z, p, q, σ, τ *are n-vectors* (say, with the norm $|z| = \sum_{k=1}^n |z^k|$ or $|z| = \max(|z^1|, \cdots, |z^n|)$ if $z = (z^1, \cdots, z^n)$). Of course φ will still be a function of 5 variables, (not of $(3n + 2)$ variables as f is).

A theorem suggested by Nagumo's uniqueness theorem (cf. [5], p. 97) for ordinary differential equations is the following:

THEOREM (**). *Let* $f(x, y, z, p, q)$ *be defined, continuous and bounded on* D, *and satisfy, for* $xy > 0$ *and arbitrary* $z, p, q, \bar{z}, \bar{p}, \bar{q}$,

$$(5) \quad |f(x, y, z, p, q,) - f(x, y, \bar{z}, \bar{p}, \bar{q})| \leqq c_1(x, y)|z - \bar{z}|/xy + c(x, y)|p - \bar{p}|/y + c_3(x, y)|q - \bar{q}|/x\,,$$

where $c_i(x, y)$, $i = 1, 2, 3$, *are non-negative, continuous functions such that*

$$c_1 + c_2 + c_3 \equiv 1\,.$$

Let $\sigma(x)$, $\tau(y)$ *be as in* (*), *and, in addition, let*

[1] Added in proof, 4 April 1960. Since this paper was accepted for publication, the following related articles have appeared: W. L. Walter, *Ueber die Differentialgleichung* $u_{xy}=f(x, y, u, u_x, u_y)$, I and II, Math. Zeit., **71** (1959), 308–324 and 436–453; my attention has also been called to the paper of J. B. Diaz and W. L. Walter, *On uniqueness theorems for ordinary differential equations and for partial differentiale equatitions of hyperbolic type*, to appear in Trans. A.M.S..

$$(6) \qquad \sigma_x(+0) = \lim_{x \to +0} \sigma_x(x), \ \tau_y(+0) = \lim_{y \to +0} \tau_y(y)$$

exist. Then (1) *has at most one solution* $z = z(x, y)$. *Furthermore, if*

$$(6*) \qquad c_1(0,0) > 0 \ ,$$

then a solution exist and is the uniform limit of the successive approximations (4).

In (6), x[or y] tends to $+0$ through the set of values on which σ_x [or τ_y] exists.

Nagumo's theorem follows from Kamke's (with $\varphi(x, y) = y/x$). However (**) does not follow from (*) because $\varphi(x, y, z, p, q)$ is assumed continuous on $x = 0$ and on $y = 0$.

REMARK 1. (**) is valid if f, z, p, q, σ, τ are n-vectors (say $z = (z^1, \cdots, z^n)$ and either $|z| = \sum_{k=1}^{n} |z^k|$ or $|z| = \max(|z^1|, \cdots, |z^n|)$).

REMARK 2. A modification of an example of Perron [7] in the theory of ordinary differential equations will show that (**) is false if $c_1 = \text{const.} > 1$, $c_2 \equiv c_3 \equiv 0$ (so that f does not depend on p, q). Also, a modification of an example of Haviland [4] shows that successive approximations need not converge if $c_1 = \text{const.} > 1$, $c_2 = c_3 \equiv 0$.

The proof of (*) will be given in §§ 2-4 below; that of (**) in §§ 5-6; finally, the proof of the last remark will be indicated in § 7.

The results above answer some questions suggested by Professor P. Hartman. I also wish the acknowledge helpful discussions with him.

2. Proof of (*). Preliminaries. In the proof of (*) below, there is no loss of generality in supposing that φ is bounded, say $0 \leqq \varphi(x, y z, p, q,) \leqq 2N$ on D. For otherwise φ can be replaced by $\bar{\varphi}$, where $\bar{\varphi}(x, y, z, p, q)$ equals $\varphi(x, y, z, p, q)$ or $2N$ according as $\varphi(x, y, z, p, q)$ does not or does exceed $2N$. It is clear that $\bar{\varphi}$ is continuous and non-decreasing in each of the variables z, p, q. Furthermore, the only solution $z(x, y)$ of

$$(3') \qquad z(x, y) = \int_0^x \int_0^y \bar{\varphi}(s, t, x(s, t), z_x(s, t,), z_y(s, t))\, ds dt$$

on any rectangle $R_{\alpha\beta} : 0 \leqq x \leqq \alpha(\leqq a)$, $0 \leqq y \leqq \beta(\leqq b)$ is $z \equiv 0$.

In order to see this, note that $\varphi(x, y, 0, 0, 0) \equiv 0$ because $z = 0$ is a solution of (3). Hence there exists an $\varepsilon > 0$ such that $0 \leqq \varphi(x, y, z, p, q) \leqq 2N$ if $|z|, |p|, |q| < \varepsilon$. Suppose that $z(x, y) \not\equiv 0$ is a solution of (3′) on $R_{\alpha\beta}$. Let $d, 0 \leqq d \leqq (\alpha^2 + \beta^2)^{\frac{1}{2}}$, be the largest value of r for which $z(x, y) \equiv 0$ in the intersection S_r of $x^2 + y^2 \leqq r^2$ and $R_{\alpha\beta}$. If U is any neighborhood of S_d (relative to $R_{\alpha\beta}$), there exists a rectangle $R_{\gamma\delta}$ in U on which $z \not\equiv 0$. Since $z \equiv 0$ on S_d, it is clear that if U is "sufficiently small", then, on U (hence on $R_{\gamma\delta}$), $|z| < \varepsilon$ and, almost everywhere, $|z_x| + |z_y| < \varepsilon$. But then $z \not\equiv 0$ is a solution of (3) on $R_{\gamma\delta}$. Since this is impossible, the only solution of (3′) on $R_{\alpha\beta}$ is $z \equiv 0$.

It will be convenient to have the following notation. R_1 denotes a subset (not always the same) of R of the from $E \times [0, b]$, where E is a (Lebesgue) measurable subset of $[0, a]$ with means $E = a$. Similary, R_2 is a subset (not always the same) of the form $[0, a] \times E$, where E is a measurable subset of $[0, b]$ and means $E = b$. Partial derivatives z_x, z_y of a function z will be denoted by p, q.

3. **Lemma for** (*). The proof of (*) will depend on the following lemma.

LEMMA 1. *Let $\alpha(x, y)$, $\beta(x, y)$ $\gamma(x, y)$ be non-negative, measurable functions defined on R, R_1, R_2, respectively, such that α is continuous, β is uniformly Lipschitz continuous with respect to y and γ is uniformly Lipschitz continuous with respect to x, In addition, let*

$$\alpha(x, y) \leqq \int_0^x \int_0^y \varphi(s, t, \alpha(s, t), \beta(s, t), \gamma(s,t))dsdt \ , \tag{7}$$

$$\beta(x, y) \leqq \int_0^y \varphi(s, t, \alpha(x, t), \beta(x, t), \gamma(x, t))dt \ , \tag{8}$$

$$\gamma(x, y) \leqq \int_0^x \varphi(s, y, \alpha(s, y), \beta(s, y), \gamma(s, y))\, ds \ , \tag{9}$$

where φ satisfies the conditions of (*) *and is bounded. Then $\alpha \equiv \beta \equiv \gamma \equiv 0$.*

Note that the Lipschitz continuity of β [or α] with respect to y [or x] is assumed to be uniform with respect to x and y.

The proof of the lemma below follows a suggestion made by R. Sacksteder. My original proof, which will be omitted, depended on two results. The first result is an existence theorem for

$$z(x, y) = \psi(x, y) + \int_0^x \int_0^y \varphi(s, t, z(s, t), p(s, t), q(s, t))dsdt \ , \tag{10}$$

where ψ is a non-negative, uniformly Lipschitz continuous function which is non-decreasing in x and in y. This existence theorem is proved by using the successive approximations $z_0 = \psi(x, y)$ and

$$z_n(x, y) = z_0(x, y) + \int_0^x \int_0^y \varphi(s, t, z_{n-1}, p_{n-1}, q_{n-1})dsdt \tag{11}$$

which satisfy

$$z_n \leqq z_{n+1}, \ p_n \leqq p_{n+1}, \ q_n \leqq q_{n+1} \ . \tag{12}$$

The second result is the fact that if ψ is replaced by another function $\bar{\psi}$ with similar properties and, almost everywhere,

$$\psi \leqq \bar{\psi}, \ \psi_x \leqq \bar{\psi}_x, \ \psi_y \leqq \bar{\psi}_y \ , \tag{13}$$

then the corresponding solution $\bar{z}$ satisfies

$$z \leqq \bar{z}, \ p \leqq \bar{p}, \ q \leqq \bar{q} \ . \tag{14}$$

Proof. Define sequences of successive approximations as follows: Let

$$z_0(x, y) = \alpha(x, y), \ u_0(x, y) = \beta(x, y), \ v_0(x, y) = \gamma(x, y) \tag{15}$$

and, for $n \geqq 1$,

$$z_n(x, y) = \int_0^x \int_0^y \varphi(s, t, z_{n-1}(s, t), u_{n-1}(s, t), v_{n-1}(s, t))\, ds dt\ , \tag{16}$$

$$u_n(x, y) = \int_0^y \varphi(x, t, z_{n-1}(x, t), u_{n-1}(x, t), v_{n-1}(x, t))\, dt, \tag{17}$$

$$v_n(x, y) = \int_0^x \varphi(s, y, z_{n-1}(s, y), u_{n-1}(s, y), v_{n-1}(s, y)) ds\ . \tag{18}$$

The functions z_n, u_n, v_n are defined on sets R, R_1, R_2, respectively, which can be taken independent of n. The inequalities (7), (8), (9) give the case $n = 0$ of

$$z_n \leqq z_{n+1}, \ u_n \leqq u_{n+1}, \ v_n \leqq v_{n+1}\ . \tag{19}$$

The cases $n > 0$ of these inequalities follow by induction by virtue of the monotony of φ.

The boundedness of φ implies the uniform boundedness of the functions z_n, u_n, v_n. Hence, as $n \to \infty$

$$z = \lim z_n, \ u = \lim u_n, \ v = \lim v_n\ , \tag{20}$$

exist on R, R_1, R_2, respectively. It is clear from (15) and (19), (20) that

$$0 \leqq \alpha \leqq z, \ 0 \leqq \beta \leqq u, \ 0 \leqq \gamma \leqq v\ . \tag{21}$$

Lebesgue's theorem on term-by-term integration under bounded convergence implies

$$z(x, y) = \int_0^x \int_0^y \varphi(s, t, z(s, t), u(s, t), v(s, t))\, ds dt\ , \tag{22}$$

$$u(x, y) = \int_0^y \varphi(x, t, z(x, t), u(x, t), v(x, t))\, dt\ , \tag{23}$$

$$v(x, z) = \int_0^x \varphi(s, y\, z(s, y), u(s, y), v(s, y))\, ds\ . \tag{24}$$

It is clear that $z_y = u$, $z_y = v$ almost everywhere. Thus the assumption on φ concerning (3) shows that $z \equiv u \equiv v \equiv 0$. Lemma 1 follows from (21).

4. Proof of (*). (i). Let $z(x, y)$ be a solution of (1). There exist functions $u(x, y)$, $v(x, y)$ defined on sets R_1, R_2, respectively, such that

$$z(x, y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y f(s, t, z(s, t), u(s, t), v(s, t)\, ds dt\ , \tag{25}$$

$$u(x, y) = \sigma_x(x) + \int_0^y f(x, t, z(x, t), u(x, t), v(x, t))\, dt\ , \tag{26}$$

$$(27)\qquad v(x, y) = \tau_y(y) + \int_0^x f(s, y, z(s, y), z_x(s, y), z_y(s, y))ds ,$$

and the relations $u = z_x$ and $v = z_y$ hold almost everywhere. In order to see this, note that almost everywhere on R,

$$z_x(x, y) = \sigma_x(x) + \int_0^y f(x, t, z(x, t), z_x(x, t), z_y(x, t))dt ,$$

$$z_y(x, y) = \sigma_y(y) + \int_0^x f(s, y, z(s, y), z_x(s, y), z_y(s, y))ds ,$$

The expressions on the right side of these equations are defined for (x, y) on sets R_1, R_2, respectively. Define $u(x, y), v(x, y)$ to be these expressions on R_1, R_2. In particular $z_x = u$ and $z_y = v$ almost everywhere. Hence (26), (27) hold on (possibly different) sets R_1, R_2. Clearly (25) is valid for all (x, y) on R.

(ii). *Uniqueness in* (*). Suppose that (1) possesses two solutions $z = z_1(x, y), z_2(x, y)$ on R. Let $u_1(x, y), v_1(x, y)$ and $u_2(x, y), v_2(x, y)$ be the functions associated with z_1, z_2 by (i). Let $\alpha = |z_1 - z_2|$, $\beta = |u_1 - u_2|$, $\gamma = |v_1 - v_2|$. If the relations (25) for $z = z_1, z_2$ are subtracted, it is seen that the inequality (2) for f implies (7). Similarly (26), (27) imply (8), (9) respectively.

The functions α, β, γ satisfy the assumptions of Lemma 1. Hence the uniqueness assertion in (*) follows from Lemma 1.

(iii). *Existence and successive approximations*. Let $z_0(x, y), z_1(x, y), \cdots$ be the successive approximations defined by (4). Corresponding to each $z_n(x, y)$, it is possible to introduce functions $u_n(x, y)$, $v_n(x, y)$ defined on sets R_1, R_2, respectively, and satisfying $u_0 = \sigma_x(x)$, $v_0 = \tau_y(y)$,

$$(28_n)\qquad z_n(x, y) = \sigma(x) + \tau(y) - z_0 + \int_0^x\int_0^y f(s, t, z_{n-1}(s, t), u_{n-1}(s, t), v_{n-1}(s, t))dsdt ,$$

$$(29_n)\qquad u_n(x, y) = \sigma_x(x) + \int_0^y f(x, t, z_{n-1}(x, t), u_{n-1}(x, t), v_{n-1}(x, t))dt ,$$

$$(30_n)\qquad v_n(x, y) = \tau_y(y) + \int_0^x f(s, y, z_{n-1}(x, t), u_{n-1}(s, y), v_{n-1}(x, t))ds .$$

The sets R_1, R_2 can be assumed to be independent of n.

Let $Z_{mn} = |z_m - z_n|$, $U_{mn} = |u_m - u_n|$, $V_{mn} = |v_m - v_n|$ and

$$(31)\qquad \alpha_k(x, y) = \operatorname*{l.u.b.}_{m,n\geqq k} Z_{mn},\quad \beta_k(x, y) = \operatorname*{l.u.b.}_{m,n\geqq k} U_{mn},\quad \gamma_k(x, y) = \operatorname*{l.u.b.}_{m,n\geqq k} V_{mn} .$$

It is clear that Z_{mn}, U_{mn}, V_{mn} are uniformly Lipschitz continuous with respect to $(x, y), x, y$, respectively, and that a corresponding statement holds for $\alpha_k, \beta_k, \gamma_k$.

By subtracting the relation (28_n) from (28_{n-1}) and using the inequality (2) for f, it is seen that

$$Z_{mm}(x, z) \leqq \int_0^x\int_0^y \varphi(s, t, Z_{m-1\ n-1}(s, t), U_{m-1\ n-1}(s, t), V_{m-1\ n-1}(s, t))dsdt .$$

Thus, if $m, n \geqq k$, the monotony of φ shows that

$$Z_{mn}(x, y) \leqq \int_0^x\int_0^y \varphi(s, t, \alpha_{k-1}(s, t), \beta_{k-1}(s, t), \gamma_{k-1}(s, t))dsdt .$$

Hence

$$\alpha_k(x, y) \leqq \int_0^x\int_0^y \varphi(s, t, \alpha_{k-1}(s, t), \beta_{k-1}(s, t), \gamma_{k-1}(s, t))dsdt .$$

Similarly

$$\beta_k(x, y) \leqq \int_0^y \varphi(x, t, \alpha_{k-1}(x, t), \beta_{k-1}(x, t), \gamma_{k-1}(x, t))dt ,$$

$$\gamma_k(x, y) \leqq \int_0^x \varphi(s, y, \alpha_{k-1}(s, y), \beta_{k-1}(s, y), \gamma_{k-1}(s, y))ds .$$

By (31), the sequences $\{\alpha_k(x, y)\}$, $\{\beta_k(x, y)\}$, $\{\gamma_k(x, y)\}$ are non-increasing (and non-negative). Let $\alpha(x, y), \beta(x, y), \gamma(x, y)$ denote the respective limits of these sequence, The Lipschitz continuity of $\alpha_k, \beta_k, \gamma_k$ is preserved under the limiting process. Lebesgue's theorem on term-by-term integration under bounded convergence gives the inequalities (7), (8), (9). Hence Lemma 1 shows that $\alpha \equiv 0, \beta \equiv 0, \gamma \equiv 0$ on R, R_1, R_2, respectively. This implies the existence of the functions $z = \lim z_n$, $u = \lim u_n$, $v = \lim v_n$ on R_1, R_2, as $n \to \infty$, satisfying (25), (26), (27). It is clear that the limit function $z(x, y)$ is a solution of (1).

Finally, the equicontinuity of the functions $z_n(x, y)$ (implied by their uniform Lipschitz continuity) shows that $z(x, z)$ is the *uniform* limit of the $z_n(x, y)$. This proves (*).

5. Lemma for (**). The proof of (**) will depend on the following lemma:

LEMMA 2. *Let $\alpha(x, y), \beta(x,y), \gamma(x, y)$ be non-negative, measurable functions defined on R, R_1, R_2, respectively, so that α is continuous, β is uniformly Lipschitz continuous with respect to y and γ is uniformly Lipschitz continuous with respect to x. Furthermore, assume that*

(32) $$\alpha(x, y)/xy \to 0 \ \text{as}\ 0 < xy \to 0$$

and that, uniformly with respect to x and y, respectively,

(33) $$\beta(x, y)/y \to 0 \ \text{as}\ y \to 0 \ \text{and}\ \gamma(x, y)/x \to 0 \ \text{as}\ x \to 0 .$$

Finally, suppose that

(34) $$\alpha(x, y) \leqq \int_0^x\int_0^y \{c_1(s, t)\alpha(s, t)/st + c_2(s, t)\beta(s, t)/t + c_3(s, t)\gamma(s, t)/s\}\, dsdt ,$$

$$\beta(x, y) \leqq \int_0^y \{c_1(x, t)\alpha(x, t)/xt + c_2(x, t)\beta(x, t)/t + c_3(x, t)\gamma(x, t)/x\}\,dt\,, \tag{35}$$

$$\gamma(x, y) \leqq \int_0^x \{c_1(s, y)\alpha(s, y)/sy + c_2(s, y)\beta(s, y)/y + c_3(s, y)\gamma(s, y)/s\}\,ds\,, \tag{36}$$

where c_1, c_2, c_3 *are as in the first part of* (**). *Then* $\alpha \equiv \beta \equiv \gamma \equiv 0$.

Proof. By (32), if $\alpha(x, y)/xy$ is defined as 0 when $xy = 0$, it becomes a continuous function on R. Hence, it assumes its maximum M_1 at some point $(x^1, y^1) \in R$. Let $M_2 = \text{l.u.b.}\ \beta(x, y)/y$ and $M_3 = \text{l.u.b.}\ \gamma(x, y)/x$ for $(x, y) \in R$.

Note that there exist numbers M_{jk}, where $j, k = 1, 2, 3$, satisfying

$$M_{jk} \geqq 0 \text{ and } \sum_{k=1}^{3} M_{jk} = 1 \qquad \text{for } j = 1, 2, 3\,, \tag{37}$$

and

$$M_j \leqq \sum_{k=1}^{3} M_{jk} M_k\,. \tag{38_j}$$

If $M_1 \neq 0$, then $M_1 = \alpha(x^1, y^1)/x^1y^1$ holds for some point (x^1, y^1) of R with $x^1y^1 > 0$. In this case, (38_1) follows from (34) with $(x, y) = (x^1, y^1)$ if

$$M_{1k} = (x^1y^1)^{-1} \int_0^{x^1}\int_0^{y^1} c_k(s, t)\,ds dt\,. \tag{39}$$

If $M_1 = 0$, let $M_{1k} = c_k(0, 0)$.

In order to obtain (38_2), let (x_j, y_j), where $j = 1, 2, \cdots$, be points of R such that $\lim (x_j, y_j) = (x^2, y^2)$ exists, $\lim \beta(x_j, y_j)/y_j = M_2$ and $\lim \beta(x_j, y) = \beta(y)$ exists uniformly for $0 \leqq y \leqq b$. Then (35) leads to (38_2) with

$$M_{2k} = (y^2)^{-1} \int_0^{y^2} c_k(x^2, t)\,dt \text{ or } M_{2k} = c_k(x^2, 0) \tag{40}$$

according as $y^2 > 0$ or $y^2 = 0$. A relation of the type (38_3) is obtained similarly.

Let $M_J = \max(M_1, M_2, M_3)$. Suppose, if possible, that $M_J > 0$. Assume, for the moment, that $M_J > M_j$ if $j \neq J$. Then, by (37) and (38_J), $M_{JJ} = 1$ and $M_{Jk} = 0$ for $k \neq J$. But the derivation of (38_J) can then be modified to obtain $M_J < M_J$. For example, if $J = 1$, then $c_1(s, t) \equiv 1$ and $c_2(s, t) = c_3(s, t) = 0$ in (34) when $(x, y) = (x^1, y^1)$, while $\alpha(s, t)/st$ is nearly zero for small st, so that one obtains $M_1 < M_1$. Or if $J = 2$, then $y^2 > 0$ and $c_1(x^2, t) = 1, c_2(x^2, t) = c_3(x^2, t) = 0$ for $0 \leqq t \leqq y^2$, while the relations

$$\beta(y) \leqq \int_0^y \beta(t)\,dt/t, \qquad \beta(y^2)/y^2 = M_2$$

give $M_2 < M_2$ since $\beta(t)/t$ is nearly 0 for small t by the uniformity of the first limit relation in (33).

Similar arguments show that if two or three of the numbers M_1, M_2, M_3 are equal to $M_J > 0$, one is led to a contradiction. Hence $M_J = 0$. This proves the lemma.

6. Proof of (**). (i). *Uniqueness* in (**). Let $z = z_1(x, y)$, $z_2(x, y)$ be two solutions of (1) on R. Let $u_1(x, y)$, $v_1(x, y)$ and $u_2(x, y)$, $v_2(x, y)$ be the functions associated with them as in the proof of (*). Let $\alpha = |z_1 - z_2|$, $\beta = |u_1 - u_2|$, $\gamma = |v_1 - v_2|$. It will be verified that, as x (or y) $\to 0$, then, except for sets of measure zero,

$$\alpha(x, y),\ \beta(x, y),\ \gamma(x, y) \to 0\ . \tag{41}$$

Consider the case $x \to 0$. The assertions (41) concerning α and γ are clear. In order to verify assertion (41) for the function β, it will first be shown that if $z = z(x, y)$ is any solution of (1) (say, $z = z_1$ or $z = z_2$) and if $u(x, y)\, v(x, y)$ are its associated functions, then

$$\lim u(x, y) = \rho(y), \text{ as } x \to 0, \text{ exists uniformly in } y\ . \tag{42}$$

To see this, let x_j, where $j = 1, 2, 3, \cdots$ be a sequence of x values such that $\lim x_j = 0$ and $\lim u(x_j, y) = \rho(y)$ exists uniformly as $j \to \infty$. Putting $x = x_j$ in (26) and letting $j \to \infty$, it is seen that

$$\rho(y) = \sigma_x(+0) + \int_0^y f(0, t, \tau(t), \rho(t), \tau_y(t))\,dt\ . \tag{43}$$

We note that $\rho(y)$ is continuous. Furthermore, $\rho(y)$ does not depend on the sequence $x_1, x_2, \cdots$. Suppose that another sequence leads to a different limit $\bar\rho(y) \not\equiv \rho(y)$. By substituting $\bar\rho$ for ρ in (43), and subtracting, we get

$$|\bar\rho(y) - \rho(y)| \leqq \int_0^y |f(0, t, \tau(t), \bar\rho(t), \tau_y(t)) - f(0, t, \tau(t), \rho(t), \tau_y(t))|\,dt\ . \tag{44}$$

Since $f, \rho, \bar\rho$ are continuous and $\rho(0) = \bar\rho(0) = \sigma_x(+0)$, the integrand of (44) can be made small by making y small. Hence

$$|\bar\rho(y) - \rho(y)|/y \to 0, \text{ as } y \to 0\ . \tag{45}$$

By relation (5),

$$|\bar\rho(y) - \rho(y)|/y \leqq y^{-1}\int_0^y c_2(0, t)|\bar\rho(t) - \rho(t)|\,dt/t\ ,$$

Using (45) as before, this leads to a contradiction. Hence $\bar\rho \equiv \rho$. Therefore every sequence, for which the limit in (42) exists, leads to the same limit. Hence (42) holds.

If $\lim u_1(x, y) = \rho_1(y)$ and $\lim u_2(x, y) = \rho_2(y)$, as $x \to 0$, we can repeat the above argument and obtain $\rho_1 \equiv \rho_2$. This completes the verification of (41).

We now verify assumptions (32) and (33) of Lemma 2. Consider, for example, the assertion

$$\beta(x, y)/y \to 0 \text{ as } y \to 0 . \tag{46}$$

By putting $u = u_1, u_2$ in (26) and subtracting we get

$$\begin{aligned}\beta(x, y) \leqq \int_0^y |f(x, t, z_1(x, t), u_1(x, t), v_1(x, t)) \\ - f(x, t, z_2(x, t), u_2(x, t), v_2(x, t))|dt .\end{aligned} \tag{47}$$

Now the integrand of (47) can be made small, by making y small, and using (41). This proves (46). The other limits in (32) and (33) are verified similarly. The other assumptions of Lemma 2 are quite straightforward. Therefore $\alpha \equiv \beta \equiv \gamma \equiv 0$. This proves "uniqueness".

(ii). *Existence and successive approximations in* (**). Let $z_0(x, y)$, $z_1(x, y), \cdots$, be the successive approximations defined by (4). Corresponding to $z_n(x, y)$ it is possible to introduce, as in the proof of (*), functions $u_n(x, y)$, $v_n(x, y)$ defined on sets R_1, R_2 (independent of n) and satisfying $u_0 = \sigma_x(x)$, $v_0 = \tau_y(y)$, (28_n), (29_n) and (30_n). Let Z_{mn}, U_{mn}, V_{mn} be defined as in the existence proof (*) above. It will be verified that, given ε, there exists a $\delta(\varepsilon)$ and an $N(\varepsilon)$, such that

$$Z_{mn}(x, y),\ U_{mn}(x, y),\ V_{mn}(x, y) < \varepsilon \tag{48}$$

for $x < \delta(\varepsilon)$ and for all $m, n > N(\varepsilon)$. A similar statement will be seen to hold when x is replaced by y. The assertion (48) concerning Z_{mn} and V_{mn} is clear. In order to verify (48) for the function U_{mn} it will first be shown that

$$\lim u_n(x, y) = h_n(y), \text{ as } x \to 0, \text{ exists uniformly in } y \text{ and } n . \tag{49}$$

It is easily verified, by induction, that $h_n(y)$ exists uniformly in y for fixed n, where

$$h_n(y) = \sigma_x(+0) + \int_0^y f(0, t, \tau(t), h_{n-1}(y), \tau_y(t))\,dt . \tag{50_n}$$

To see the uniformity in n, define

$$\begin{aligned}\bar{z}_n(x, y) = z_n(x, y) - \sigma(x) - \tau(y) + z_0;\ \bar{u}_n(y, y) = u_n(y, y) - \sigma_x(y); \\ \bar{v}_n(x, y) = v_n(x, y) - \tau_y(y) ;\end{aligned} \tag{51_n}$$

$$g(x, y, z, p, q) = f(x, y, z + \sigma(x) + \tau(y) - z_0, p + \sigma_x(x), q + \tau_y(y)) . \tag{52}$$

For $\bar{u}_n$ we define $\bar{h}_n$ corresponding to h. Clearly g satisfies a condition analogous to (5), $\bar{u}_0(x, y) = \bar{h}_0(y) \equiv 0$, and

$$\bar{u}_n(x, y) = \int_0^y g(x, t, \bar{z}_{n-1}(x, t), \bar{u}_{n-1}(x, t), \bar{v}_{n-1}(x, t))\,dt,\ n \geqq 1 \tag{53_n}$$

$$\bar{h}_n(y) = \int_0^y g(0, t, 0, \bar{h}_{n-1}(t), 0)\,dt,\ n \geqq 1 . \tag{54_n}$$

To prove (49) it suffices to verify that

(55) $\lim \bar{u}_n(x, y) = \bar{h}_n(y)$, as $x \to 0$, exists uniformly in y and n.

By subtracting (54_n) from (53_n), it is seen that

$$|\bar{u}_n(x, y) - \bar{h}_n(y)| \leqq \int_0^y \{|g_1 - g_2| + |g_2 - g_3|\} dt \tag{56}$$

where $g_1 = g(x, t, \bar{z}_{n-1}(x, t), \bar{u}_{n-1}(x, t), \bar{v}_{n-1}(x, t))$, $g_2 = g(0, t, 0, \bar{u}_{n-1}(x, t), 0)$ and $g_3 = g(0, t, 0, \bar{h}_{n-1}(t), 0)$. We note that, given $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ such that $|g_1 - g_2| < \varepsilon$ if $x < \delta$ for all y and n. Hence, noting (5),

$$|\bar{u}_n(x, y) - \bar{h}_n(z)| \leqq \int_0^y \{\varepsilon + t^{-1} c_2(0, t)|\bar{u}_{n-1}(x, t) - \bar{h}_{n-1}(t)|\} dt . \tag{57_n}$$

By continuity, because of (6*), $c_2(0, t) < 1$ for small $t > 0$. Hence there exists a number θ, $0 < \theta < 1$, such that

$$\int_0^y c_2(0, t)\, dt \leqq \theta y \text{ for } 0 < y \leqq b .$$

A simple induction shows that

$$|\bar{u}_n(x, y) - \bar{h}_n(y)| \leqq (1 - \theta^n)\, \varepsilon\, y/(1 - \theta) \leqq b\, \varepsilon\, /(1 - \theta) . \tag{58}$$

This proves (55). Hence (49) is established.

Next we note that $h_n(y)$, $n = 0, 1, 2, \cdots$, are the successive approximations for the initial value problem

$$dw/dt = F(t, w),\ w(0) = \sigma_x(+0) , \tag{59}$$

where $F(t, w) = f(0, t, \tau(t), w, \tau_y(t))$ is bounded, measurable and continuous in w (for almost all fixed t). By (5),

$$|F(t, w) - F(t, \bar{w})| \leqq |w - \bar{w}|/t . \tag{60}$$

Note that the existence of $\tau_y(+0)$ implies that $F(t, w) \to F(0, w) = f(0, 0, \tau(0), w, \tau_y(+0))$ as $t \to +0$. The proof of the main theorem in [8] shows that these successive approximations converge uniformly, (60) being Nagumo's uniqueness condition (cf. [5], p. 97). Hence

(61) $\lim h_n(y) = h(y)$, exists uniformly in y as $n \to \infty$.

Now (61) and (49) together give (48) for $U_{mn}(x, y)$. Hence (48) is established.

By an argument similar to that used in verifying (46) it is seen that, given $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that

$$\begin{aligned} &(xy)^{-1} Z_{mn}(x, y) < \varepsilon \text{ for } xy < \delta(\varepsilon) \text{ and for } m, n > N(\varepsilon) \\ &x^{-1} U_{mn}(x, y) < \varepsilon \text{ for } x < \delta(\varepsilon) \text{ and for } m, n > N(\varepsilon) \\ &y^{-1} V_{mn}(x, y) < \varepsilon \text{ for } y < \delta(\varepsilon) \text{ and for } m, n > N(\varepsilon) . \end{aligned} \tag{52}$$

Now defining $\alpha_k, \beta_k, \gamma_k$ as in (31), we note that we can substitute them for Z_{mn}, U_{mn}, V_{mn}, respectively, in (62) changing $m, n > N(\varepsilon)$ to $k > N(\varepsilon)$. Proceeding as in the analogous section of the proof of theorem $(*)$, we conclude that α, β, γ, satisfy (34), (35) and (36), also (32) and (33). Therefore, by Lemma 2, the successive approximations converge uniformly to a solution of (1).

7. Counter-examples. (a). Let $a = b = 1, 1 + \varepsilon = \delta^2, \varepsilon > 0, \delta > 1$. Let $f(x, y, z, p, q)$ be independent of p, q and defined by

$$f(x,y,z,p,q) = \begin{cases} 0 & \text{if } (x, y) \in R, z \leqq 0\ , \\ (1+\varepsilon)z/xy & \text{if } (x, y) \in R, 0 < z < (xy)^\delta\ , \\ (1+\varepsilon)(xy)^{\delta-1} & \text{if } (x, y) \in R, (xy)^\delta \leqq z\ . \end{cases}$$

Then $f(x, y, z, p, q)$ is continuous and satifies (5) for $c_1(x, y) = 1 + \varepsilon$, (and $c_2 = c_3 \equiv 0$). Let $\sigma(x) = \tau(y) \equiv 0$. Then (1) has an infinity of solutions, namely, $z = c(xy)^\delta$, where $0 < c < 1$.

(b). Let $a = b = 1$, $R^0 = \{(x, y) : 0 < x,\ y \leqq 1\}$, $1 + \varepsilon = \delta^2$, $\varepsilon > 0$, $\delta > 0$ and

$$f(x,y,z,p,q) = \begin{cases} 0 & \text{if } x = 0, y = 0\ , \\ (xy)^{\delta-1} & \text{if } (x, y) \in R^0, z < 0\ , \\ (xy)^{\delta-1} - (1+\varepsilon)z/xy & \text{if } (x, y) \in R^0, 0 \leqq z \leqq (xy)^0\ , \\ -\varepsilon(xy)^{\delta-1} & \text{if } (x, y) \in R^0, (xy)^\delta < z\ . \end{cases}$$

Then $f(x, y, z, p, q)$ satisfies the same relation (5) as in example (a). However, in (4), $z_{2n} = 0$, $z_{2n+1} = (xy)^\delta/\delta^2$, so that successive approximations (4) do not converge.

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