SOME SPECTRAL PROPERTIES OF POSITIVE LINEAR OPERATORS

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It is well known (Perron [12], Frobenius [6, 7]) that if A is an $n \times n$ matrix over the real field with elements $\geqq 0$, the spectral radius[1] of A, $r(A)$, is a characteristic number, with at least one characteristic vector whose coordinates are $\geqq 0$. If A has positive elements throughout, then r is > 0, of algebraic and geometric multiplicity one, and exceeds all other elements of the spectrum in absolute value.[2] Generalizations of this theorem to integral equations were obtained by Jentzsch [9] and E. Hopf [8]. In an operator-theoretic setting, the result did not appear until 1948 when Krein and Rutman published their most comprehensive work [11]. Further results were obtained by Bonsall [2]-[4] and, in the framework of a general locally convex space, by the author [15, 17] For compact positive operators in an order-complete Banach lattice, see Ando [1].

While the key to many results generalizing the Perron-Frobenius theorem is compactness in one form or another, a good many spectral properties of positive linear operators are independent of it. Such properties were established by Bonsall (e.c., cf. Prop. 1 below), the author [17], and recently Putnam [13] who considers, however, only the rather special case of a bounded matrix with non-negative elements in l_2. The present paper establishes new and more general results on the (spectral) character of the spectral radius r of a positive operator T, valid in arbitrary ordered Banach spaces.[3] Section 2 collects some theorems for which no hypothesis or r is made; leaning heavily on topological properties of the positive cone K, they apply to any positive operator. Throughout § 3, r is assumed to be a pole of the resolvent of T. The stress is here on the notion of quasi-interior map; together with the assumption on r, this concept yields strong results earlier obtained by Krein and Rutman [11] for strongly positive operators[17] which are compact and defined on a space whose positive cone K has interior points. This is interesting since in many concrete examples of partially ordered (B)-spaces, K has empty interior [16, p. 130]. The paper concludes with two problems.

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[1] For the terminology adopted, see § 1.

[2] A short proof in [14]. Cf. also [5].

[3] With only minor modifications, the results of the present paper carry over to bounded positive endomorphisms of a partially ordered, quasi-complete locally convex space.

1. Auxiliary material. A (real or complex) Banach space E is *partially ordered* if an order relation[4], denoted $x \leqq y$ and invariant under addition and multiplication by positive scalars, is defined on E. It is well known that such an order structure is completely determined by the set $\{x: x \geqq 0\}$ of positive elements which will be called the *positive cone* K. Unless otherwise stated, we shall always suppose that K is closed in E and *proper*, i.e., such that $K \cap -K = \{0\}$[5]. K is *generating* if $E = K - K$, *normal* if $\| x + y \| \geqq \| y \|$ for all $x, y \in K$ and some real norm $x \to \| x \|$ generating the topology of E. K is a *B-cone* (BZ-Kegel in [**16**]) if for some fundamental system of bounded sets $\boldsymbol{B}$, the closed convex symmetric hulls of the sets $B \cap K, B \in \boldsymbol{B}$, form again a fundamental system of bounded subsets of E.[6] We say K is spanned by a set C if $K = \bigcup_{\lambda \geqq 0} \lambda C$. If E' is the topological dual of E, $K' \subset E'$ is the set of those linear forms which are $\geqq 0$ on K (resp. if E is complex, whose real parts are $\geqq 0$ on K). K' is called the cone *conjugate* to K. An $f \in E'$ is *positive* (resp. *strictly positive*) with respect to a given partial ordering of E if $\operatorname{Re} f(x) \geqq 0$ for $x \in K$ (resp. if $\operatorname{Re} f(x) > 0$ for $0 \neq x \in K$). If E is a real Banach space, F its complexification in the usual sense, and K is a normal cone (resp. a B-cone) in E, then $K + iK$ is a normal cone (resp. a B-cone) in F [**17**, p. 264].

Let E denote a real or complex Banach space, partially ordered by a proper closed cone K.

LEMMA 1. *If K is normal, then $E' = K' - K'$. If K is a normal B-cone, then so is K' for the strong topology on E'.*

The first part is proved (for real spaces) in [**10**]. For the second part, see [**3**, p. 146], and [**17**, p. 262/3] in the complex case. (It follows from a simple category argument that in a Banach space, every generating cone is a B-cone.)

An *order interval* in E is a set $[x, y] = \{z: x \leqq z \leqq y\}$. We note that if K is normal, every order interval is bounded.

DEFINITION. *A point x is quasi-interior to K if the order interval $[0, x]$ is a total subset of E.*

It is clear that every interior point of K is quasi-interior, and that every quasi-interior point of K is a non-support point of K in the sense of V. L. Klee. If K has non-empty interior, the three notions coincide; this is the case, in particular, if E is finite dimensional and K is total (hence K, resp. $K + iK$ if E is complex, is generating) in E.

[4] i.e., a binary relation which is reflexive and transitive. We assume always that $E \neq \{0\}$.

[5] K is proper if and only if the order relation is anti-symmetric.

[6] $S \subset E$ is symmetric if $x \in S$ implies $-x \in S$. In the (present) case of a normed space, K is a B-cone if and only if there exists an $m > 0$ such that every x in the unit ball U of E is of the form $x = \lim_{n \to \infty} (u_n - v_n)$ with $u_n, v_n \in K \cap mU$.

LEMMA 2. *Let P be a continuous projection in E such that $PK \subset K$. If $x \in PK$ is quasi-interior to K, it is quasi-interior to PK in PE.*

It is readily observed that $[0, x] \cap PE = P[0, x]$ under the conditions stated; since the linear hull of $[0, x]$ is dense in E, it follows that the linear hull of $P[0, x]$ is dense in PE.

A bounded endomorphism T of E is a *positive operator* if the positive cone K is invariant under T, i.e., if $TK \subset K$. The *spectral radius* r of T is the maximum modulus of the points in its spectrum[7] $\sigma(T)$. The complement of $\sigma(T)$ in the complex plane is denoted by $\rho(T)$, and the resolvent $(\lambda - T)^{-1}$, locally holomorphic in $\rho(T)$, by R_λ. The point spectrum of T is the set of all its *characteristic numbers*, i.e., the set of those λ for which $\lambda - T$ fails to be (1,1). For a characteristic number λ, $d(\lambda)$ denotes the (linear) dimension of the kernel of $\lambda - T$ (the *characteristic space*); an $x \neq 0$ in this kernel is called a *characteristic vector* (of T for λ). It is well known that every pole of the resolvent is a characteristic number of T.

If T is a positive operator, then so is its adjoint T' with respect to the conjugate cone K', which is a proper cone in E' if and only if K is total in E.

DEFINITION. *A positive operator T is quasi-interior if there exists $\lambda > r$* (r the spectral radius of T) *such that $TR_\lambda x$ is quasi-interior to K for every x, $0 \neq x \in K$.*[8]

This condition on T is not stronger than requiring that for each $x, 0 \neq x \in K$, the union of order intervals $\bigcup_{n=1}^{\infty} [0, T^n x]$ be total in E. (It is clear that K is a total cone in E if the set of quasi-interior positive operators on E is not empty.)

LEMMA 3. *If K is a normal B-cone or, more generally, if K and K'* (K' in the strong dual E') *are normal cones, then the set $\mathfrak{K}$ of all positive operators is a normal cone in the Banach space $\mathfrak{L}(E)$ of bounded endomorphisms of E.*

It is known [**17**, p. 269] that the assertion holds if K is a normal B-cone in E. If K and K' are both normal, then K' is a normal B-cone for the strong topology on E' (this follows from Lemma 1 and the subsequent remark); therefore by Lemma 1, the cone K'' conjugate to K' in the Banach space E'', bidual of E, is a normal B-cone. Thus the cone $\mathfrak{K}''$ of positive operators on E'' (with respect to K'') is normal

[7] If E is a real space, the terms spectrum, resolvent etc. will be understood with respect to the extension of T to the complexification of E, which may be considered as ordered with positive cone K or $K + iK$.

[8] E.g., if $E = l_2$, K the cone of all vectors with non-negative coordinates, a bounded matrix $A = (a_{i,k})$ with non-negative elements is quasi-interior if and only if for each pair (i, k) of indices, there exists $n = n(i, k)$ such that $(A^n)_{i,k} > 0$. Cf. [**13**].

in $\mathfrak{L}(E'')$ and this implies that $\mathfrak{K}$ is normal in $\mathfrak{L}(E)$ because the norm-preserving natural imbedding of $\mathfrak{L}(E)$ into $\mathfrak{L}(E'')$ maps $\mathfrak{K}$ into $\mathfrak{K}''$.

2. Some properties of the spectral radius.

Throughout this section, E denotes a (real or complex) partially ordered Banach space with positive cone K; E' is the (topological) dual of E, equipped with the strong topology unless otherwise stated. T is a positive operator on E with spectral radius r.

The first part of the following proposition is due to Bonsall [3, p. 148] but the proof given here, which also yields the second assertion, is entirely different from that in [3].

PROPOSITION 1. *Let K and K' be normal cones in E* resp. E'. *For each positive operator T, r is in the spectrum of T. If r is a pole of the resolvent R_λ of order k, every other pole of R_λ on $|\lambda| = r$ is of an order $\leqq k$.*

Proof. It follows from Lemma 3 that the cone $\mathfrak{K}$ of positive operators is normal in $\mathfrak{L}(E)$ with respect to the uniform topology. It is shown in [18] that if $z \to f(z)$ is an analytic function with values in a Banach space, holomorphic at 0, such that its expansion at 0, $\sum_{n=0}^{\infty} a_n z^n$, has radius of convergence 1 and the set of coefficients $\{a_n\}$ is contained in a normal cone, then $z = 1$ is singular for f and if it is a pole of order k, there is no pole of f on $|z| = 1$ of order $> k$. The proposition follows immediately by letting $f(z) = R(r/z)$ if $r > 0$ ($R_\lambda = R(\lambda)$ the resolvent of T). If $r = 0$, the result is trivial.

PROPOSITION 2. *R_λ is a positive operator for each (real) $\lambda > r$; if R_λ is positive for some $\lambda \in \rho(T)$, then λ is real and > 0.*[9] *If K, K' are normal* (hence, if K is a normal B-cone), *then $\lambda > r$ is a necessary and sufficient condition in order that R_λ be positive.*

Proof. From the expansion of R_λ at ∞, it is easily seen that the condition $\lambda > r$ is sufficient. Now assume that for some $\lambda \in \rho(T)$, R_λ is a positive operator. Select an $x_0 \in K$, $x_0 \neq 0$, and define recursively $x_n = R_\lambda x_{n-1} (n \in N)$.[10] Each x_n satisfies the equation

$$(*) \qquad \lambda x_n = T x_n + x_{n-1} .$$

We have $x_n \in K (n \in N)$ and since $x_n = 0$ for some n would imply $x_0 = 0$, $x_n \neq 0$ for all n. From (*) it follows that $\lambda x_1 \in K$, and by induction it is established that $\lambda^n x_n \in K$, $\lambda^{n-1} x_n \in K$ for all $n \in N$. Also,

[9] For this statement, we have to assume that $K \neq \{0\}$.

[10] N stands for the set of positive integers.

$$\lambda^n x_n \geqq \lambda^{n-1} x_{n-1} \geqq x_0 \qquad (n \in N) .$$

Thus $\lambda \neq 0$ and without loss of generality, we may assume that $|\lambda| = 1$. (For if R_λ is positive at $\lambda \neq 0$, then the resolvent of $|\lambda^{-1}| T$ is positive at $\lambda |\lambda^{-1}|$.) Let $\lambda = e^{i\varphi}$, $0 \leqq \varphi < 2\pi$, and suppose that $\varphi > 0$. It is clear that $n\varphi \neq \pi (n \in N)$ or K would not be a proper cone. Hence there is an $n_0 \in N$ such that the triangle in the complex plane with vertices 1, $e^{i(n_0 - 1)\varphi}$, $e^{in_0\varphi}$ contains 0 in its interior. Consider the 2-dimensional real subspace L of E (resp. of $E + iE$)[7] containing x_{n_0} and ix_{n_0}. $K \cap L$ (resp. $(K + iK) \cap L$) is a proper convex cone of vertex 0 in L containing the points x_{n_0}, $\lambda^{n_0 - 1} x_{n_0}$, $\lambda^{n_0} x_{n_0}$. Hence this cone contains 0 as an interior point in L which is contradictory. Thus $\varphi = 0$, and $\lambda > 0$.

Let K and K' be normal in E resp. E'; then the cone $\mathfrak{K}$ of positive operators is normal in $\mathfrak{L}(E)$ by Lemma 3. If we had $R_\lambda \in \mathfrak{K}$ for some λ, $0 < \lambda < r$, from the resolvent equation

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$$

it would follow that $R_\mu \leqq R_\lambda$ (with respect to the order relation on $\mathfrak{L}(E)$ whose positive cone is $\mathfrak{K}$) for all $\mu > \lambda$, for which $R_\mu \geqq 0$ therefore, in particular, for all $\mu > r$. This would imply $\| R_\mu \| \leqq \| R_\lambda \|$ for all $\mu > r$ and some real norm $A \to \| A \|$ generating the topology of bounded convergence on $\mathfrak{L}(E)$. This is impossible since $r \in \sigma(T)$ by Prop. 1 and consequently, $\| R_\mu \| \to \infty$ as $\mu \downarrow r$. The proof is finished.

PROPOSITION 3. *If there exists y, $0 \neq y \in K$, such that $T^p y \geqq \delta y$ for some $p \in N$ and $\delta > 0$, then $r \geqq \delta^{1/p}$.*

Proof. Since K is closed and $\neq E$, a routine argument shows that there exists a continuous linear form $h \in E'$ such that the real part $f(x) = \operatorname{Re} h(x)$ is $\geqq 0$ on K and $f(y) > 0$. For $\lambda > r$, we have

$$f(R_\lambda y) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} f(T^n y) \geqq \sum_{k=1}^{\infty} \frac{1}{\lambda^{kp+1}} f(T^{kp} y) \geqq f(y) \sum_{k=1}^{\infty} \frac{\delta^k}{\lambda^{kp+1}} = f(y) \cdot \frac{\delta}{\lambda(\lambda^p - \delta)}$$

because $T^p y \geqq \delta y$ implies $T^{kp} y \geqq \delta^k y (k \in N)$. It follows that $f(R_\lambda y)$ is unbounded as $\lambda^p \downarrow \delta$. Consequently $r \geqq \delta^{1/p}$.

THEOREM 1. *Let K be spanned by a convex set not containing 0 and compact for some locally convex topology (on E) for which T is continuous on K*[11]. *There exists a non-negative characteristic number*

[11] i.e., for which the restriction of T to K is continuous.

of T with (at least one) *characteristic vector in K. If in addition K is a normal cone generating E, then r is such a number.*[12]

Proof. Let C be the convex set and $\mathfrak{T}$ the locally convex topology in question. There exists a $\mathfrak{T}$-closed real hyperplane $H = \{x: f(x) = 1\}$ separating C strictly from 0. It is clear that $f(x) > 0$ for $0 \neq x \in K$. K is closed for $\mathfrak{T}$: Let $\boldsymbol{F}$ be a filter on K converging to $x_0 \in E$ for $\mathfrak{T}$; since f is continuous, there exists $F \in \boldsymbol{F}$ such that $\sup\{f(x): x \in F\} \leqq 1 + f(x_0)$, therefore $F \subset (1 + f(x_0))C_1$, where C_1 is the convex hull of $\{0\}$ and C. Since C_1 is compact[13], x_0 which is in the closure of F, is in K. Because $H \cap K$ is a closed subset of C_1, $H \cap K$ is compact; so $f(x_n) \to 0$ implies $x_n \to 0$ and thus $Tx_n \to 0$ for any sequence $\{x_n\} \subset K$, (all statements in this sentence referring to $\mathfrak{T}$).

Consider the real subspace $\hat{E} = K - K$ of E, equipped with the norm

$$z \to \|z\| = \inf\{f(x) + f(y): z = x - y;\, x, y \in K\} .$$

$\hat{E}$ is a Banach space. Given an arbitrary Cauchy sequence in $\hat{E}$, there exists a subsequence $\{z_k\}$ such that $\|z_{k+1} - z_k\| < 1/2^k$. By definition of the norm in $\hat{E}$, there exist two sequences $\{x_k\}$, $\{y_k\}$ in K with $z_{k+1} - z_k = x_k - y_k (k \in N)$ and $\|x_k\| + \|y_k\| \leqq 1/2^k$. Since C_1 is compact for $\mathfrak{T}$, the sequence

$$\left\{\sum_{\nu=1}^{n} x_\nu : n \in N\right\}\left(\text{resp.}\left\{\sum_{\nu=1}^{n} y_\nu : n \in N\right\}\right)$$

has a limit point x (resp. y) in K, and it is now easy to see that $\{z_k\}$ (and hence the given sequence) converges to $x - y$, in $\hat{E}$. It is readily verified that the restriction $\hat{T}$ of T to $\hat{E}$ is a continuous endomorphism. Moreover, K is a normal closed cone in $\hat{E}$, and it is a B-cone since it is generating (cf. the remark following Lemma 1). If $\hat{r}$ is the spectral radius of $\hat{T}$, we have $\hat{r} \in \sigma(\hat{T})$ by Prop. 1. Thus, since $\hat{R}_\lambda x$ is non-decreasing for each $x \in K$ if $\lambda \downarrow \hat{r}$, we have $\|\hat{R}_\lambda y\| \to \infty$ for some $y \in K$ as $\lambda \downarrow \hat{r}$. Let $\lambda_n \downarrow \hat{r}$ and set $x_n = \hat{R}(\lambda_n)y / \|\hat{R}(\lambda_n)y\|$. Then $\lambda_n x_n - \hat{T}x_n \to 0$ in $\hat{E}$ and also $(\hat{r} - \hat{T})x_n \to 0$ because of $\|x_n\| = 1$. By Proposition 2, $x_n \in K$; and, since $1 = \|x_n\| = f(x_n)$, it follows that $x_n \in H \cap K (n \in N)$. Now $H \cap K$ is compact for $\mathfrak{T}$ and as $\hat{r} - \hat{T}$ is continuous for $\mathfrak{T}$ on K, it follows that $(\hat{r} - \hat{T})x = 0$ for some $x \in H \cap K$. The proof of the first part is finished.

[12] The assumption that K be closed in E is not needed in Th. 1 and the corollary; the first assertion of Th. 1 is also independent of E being a Banach space and of T being bounded.

[13] In any linear topological space, the convex hull of a finite number of convex compact sets is compact. A locally convex topology is assumed to be Hausdorff by definition.

If K is a normal generating cone in E, then $r \in \sigma(T)$ by Prop. 1. It is clear that $\hat{r} \leqq r$. On the other hand, $\hat{r} < r$ would imply that $r - T$ is an algebraical automorphism of E, which is impossible.

REMARK. Using the notation of the preceding proof, the number $\hat{r}$ (which was shown to be in the point spectrum of T) may be characterized as follows:

(a) *$\hat{r}$ is the greatest real number α such that $\alpha - T$ is not an algebraical automorphism of the real subspace $K - K$ of E.*

(b) *$\hat{r}$ is the smallest real number α such that R_λ is positive for $\lambda > \alpha$, $\lambda \in \rho(T)$.*

(c) *If g is a real $\mathfrak{T}$-continuous linear form on E with $0 \notin g(C)$, then*

$$\hat{r} = \lim_{n \to \infty} \{\sup | g(T^n x) | : x \in C\}^{1/n} .$$

As an application of Th. 1, we list a proposition which is equivalent to the combination of [2, Th. 1] and [4, Th. C].

COROLLARY. *If K has non-empty interior, there exists a non-negative number in $\sigma(T)$ which is a characteristic number of T' with* (at least one) *characteristic vector in K'. If in addition K is normal, then r is such a number.*

Proof. If x_0 is interior to K, the real hyperplane $H = \{x' \in E' : \mathrm{Re}\langle x', x_0\rangle = 1\}$ intersects K' is a set compact for the weak* topology on E'. For the linear forms in this intersection are uniformly bounded on the order interval $[0, x_0]$ (which has interior points), hence equicontinuous. Obviously $H \cap K'$ spans K', and T' is continuous for the weak* topology. The assertion concerning T follows from $\sigma(T) = \sigma(T')$. Finally, if in addition K is normal, K' is a normal (B)-cone in E' spanning E' by Lemma 1 which completes the proof.

REMARK. If K is normal with non-empty interior $\mathring{K}$, then for each $x_0 \in \mathring{K}$, the norm $A \to \|A\|_{x_0} = \sup\{\|Ax\| : x \in [0, x_0]\}$ generates the topology of bounded convergence on $\mathfrak{L}(E)$. For a positive operator and a norm on E which is monotone on K, $\|T\|_{x_0} = \|Tx_0\|$. Thus:

If K is normal with $\mathring{K} \neq \phi$ (and T positive), then

$$r = \lim_{n \to \infty} \| T^n x_0 \|^{1/n}$$

for every $x_0 \in \mathring{K}$.

3. Operators for which r is a pole of R_λ. As in § 2, E denotes a (real or complex) partially ordered Banach space; but we shall assume that T is a positive operator for which the spectral radius r is a pole of the resolvent R_λ. The positive cone K is assumed proper and closed.

PROPOSITION 4. *The leading coefficient in the principal part of* R_λ *at* $\lambda = r$ *is a positive operator. Hence, if* K *is total in* E, *there exists* (at least) *one characteristic vector of* T *for* r *in* K, *and of* T' *for* r *in* K'.

Proof. Since the leading coefficient in the principal part of R_λ is the limit[14] (r being a pole of order k) of $(\lambda - r)^k R_\lambda$ as $\lambda \downarrow r$, the first assertion follows from the facts that R_λ is positive for $\lambda > r$ and that K is closed in E. Further, if K is a closed proper cone total in E, then K' is a closed proper cone weak* total in E'. The remainder is clear.

THEOREM 2. *Let* T *be quasi-interior. Then*:

1°. $r > 0$ *and* r *is a simple pole of* R_λ.

2°. *Every characteristic vector pertaining to* r, *of* T *in* K (resp. *of* T' *in* K') *is quasi-interior to* K (resp. *a strictly positive linear form*).

3°. *Each of these conditions implies that* $d(r) = 1$:

(a) K *has non-empty interior*

(b) $d(r)$ *is finite*

(c) E *is a Banach lattice.*[15]

Proof. The assumption $r = 0$ implies, by Prop. 4, that $Tx = 0$ for some x, $0 \neq x \in K$. (Since T is a quasi-interior map, K has quasi-interior points and is therefore total in E.) But then $TR_\lambda x = 0$ for every $\lambda \in \rho(T)$ which contradicts the definition of a quasi-interior map. Hence $r > 0$.

Let x_0, $0 \neq x_0 \in K$, be a characteristic vector of T for r. By definition, there exists $\lambda > r$ such that $TR_\lambda x_0$ is quasi-interior to K. From

$$TR_\lambda x_0 = \sum_1^\infty \frac{1}{\lambda^n} T^n x_0 = x_0 \sum_1^\infty \left(\frac{r}{\lambda}\right)^n$$

it follows that x_0 is quasi-interior to K. Similarly, if f is a characteristic vector of T' in K' for r, we have $r^n f(x) = f(T^n x) (n \in N)$ for $x \in E$, hence with $f_1(x) = \operatorname{Re} f(x)$

$$f_1(x) \sum_1^\infty \left(\frac{r}{\lambda}\right)^n = \sum_1^\infty \frac{1}{\lambda^n} f_1(T^n x) = f_1(TR_\lambda x) > 0$$

[14] For the topology of bounded convergence.

[15] In the sense of G. Birkhoff (Lattice Theory, New York 1948). A Banach lattice is by definition a real space; for our purposes, it is sufficient to assume that the underlying real space of E is a Banach lattice.

for every $0 \neq x \in K$, for f_1 must be > 0 at every quasi-interior point of K.

We show that r is a simple pole of R_λ. Let k be the order of r; if A is the leading coefficient in the principal part of R_λ at $\lambda = r$, we have $A = P(T - r)^{k-1}$ where

$$P = \frac{1}{2\pi i}\int_C R_\lambda d\lambda$$

(C a positively oriented circle enclosing r, and having no other elements of $\sigma(T)$ in its interior or on its boundary), is the continuous projection of E onto the subspace pertaining to the spectral set $\{r\}$. K being total in E, we have $Av \neq 0$ for some $v \in K$ and Av is quasi-interior to K by 2°. Let $f \in K'$ be a characteristic vector of T' for r (Prop. 4), then $P'f = f$ (P' the adjoint of P) and

$$f_1(Av) = f_1[(T - r)^{k-1}v] = [(T' - r)^{k-1}f]_1(v) > 0$$

which implies $k = 1$. Therefore, r is a simple pole.

We show now that 3°. holds. Since r is a simple pole of R_λ, P is a positive operator by Prop. 4. If $x_0 \in K$ is a characteristic vector of T for r, x_0 is quasi-interior to K by 2°. Therefore, the cone PK can have no boundary points $\neq 0$ which are not quasi-interior to PK in PE by Lemma 2. If a) K has interior points, then so has PK in PE; thus we must have $d(r) = 1$. If b) $d(r)$ is finite, i.e., if P is of finite rank, then every quasi-interior point of PK is actually interior to PK in PE and the conclusion is the same.

There remains to show that 3°. c) is sufficient for $d(r) = 1$. Let x_0 be any characteristic vector of T for r. We have $rx_0 = Tx_0$ and consequently $r\,|x_0| \leqq T\,|x_0|$, $|x_0|$ denoting the absolute of x_0 in the lattice-theoretic sense. If in the latter relation equality does not hold, we obtain

$$rf_1(|x_0|) < f_1(T\,|x_0|) = rf_1(|x_0|)$$

for every characteristic vector $f \in K'$ of T' for r (f is then strictly positive by 2°). This is contradictory; hence, $r\,|x_0| = T\,|x_0|$ for every characteristic vector x_0, whether or not in K, of T for r. Now $x_0 = x_0^+ - x_0^-$ where the summands are disjoint. Since $|x_0| = x_0^+ + x_0^-$, x_0^+ and x_0^- are both in the characteristic space of T pertaining to r. Assume that for some x_0, both $x_0^+ \neq 0$ and $x_0^- \neq 0$. Since the order interval $[0, x_0^+]$ is disjoint from x_0^- and the lattice operations are continuous, x_0^+ cannot be quasi-interior to K which contradicts 2°.[16] Consequently, either

[16] It becomes clear from this that if E is a Banach lattice, the points quasi-interior to K are weak units of E in the sense of Birkhoff (l.c.).

$x_0^+ = 0$ or $x_0^- = 0$. This implies that for each characteristic vector of T in K (for r), either $x_0 \in K$ or $x_0 \in -K$; therefore $d(r) = 1$.

The theorem is proved.

If the assumptions that T be quasi-interior and r be a pole of R_λ are satisfied, r need not be the only element of $\sigma(T)$ on $|\lambda| = r$ even if E is finite dimensional. For let E be Euclidean 2-space in its natural order (i.e., K being the set of all vectors with non-negative coordinates). The positive operator on E represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is quasi-interior: for $\lambda = 2$, R_λ is the matrix $1/3 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The characteristic numbers of T are 1 and -1.

PROPOSITION 5. *Let T be such that for each x, $0 \neq x \in K$, there exists a positive integer $n = n(x)$ for which $T^n x$ is an interior point of K.*[17] *Then r is the only element in the point spectrum of T on $|\lambda| = r$.*

Proof. We note first that if T has the stated property and E is a real space, the extension of T to the complexification $E + iE$ has the same property provided $E + iE$ is considered as partially ordered with positive cone $K + iK$. Hence we assume E as complex.

By Theorem 2 (since T is obviously quasi-interior) there exists x_0 interior to K with $rx_0 = Tx_0$. Because of $r > 0$, we may assume that $r = 1$. Suppose that for some φ, $0 < \varphi < 2\pi$, $e^{i\varphi}$ is in the point spectrum of T and $e^{i\varphi}x = Tx (x \neq 0)$. Consider the 3-dimensional real subspace E_3 of E that contains x_0, x, ix; obviously E_3 is invariant under T. x_0, which is an interior point of K, is interior to $K_3 = K \cap E_3$ in E_3. Identifying E_3 (which we may for our purpose) as Euclidean 3-space with coordinate axes x_0, x, ix, the restriction of T to E_3 is a rotation through φ about x_0. Let $w \neq 0$ be a point of K_3 which has maximum angular distance from x_0; then $T^n w$ must have the same property for every $n \in N$. This implies that no $T^n w (n \in N)$ is interior to K_3 and a contradiction is established.

4. Problems. Let E be a partially ordered Banach space with positive cone K, T a positive operator on E with spectral radius r. Under what general conditions, if any, are these implications true:

a. If r is an isolated singularity of R_λ, every singularity of R_λ on $|\lambda| = r$ is isolated.

b. If r is a pole of R_λ, R_λ has no singularities on $|\lambda| = r$ other than poles.[18]

[17] Operators T with this property are called *strongly positive* in **[11]**.

[18] E.g., are a. and b. true if K is a normal B-cone in E?

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