THE ABSOLUTE CONTINUITY OF TOEPLITZ'S MATRICES

Marvin Rosenblum

1. Introduction. Suppose W is a real $L^2(-\pi, \pi)$ function that is bounded below but not equivalent to a constant function. The *Toeplitz matrix* associated with W is $T_0 = [w_{j-k}]$, $j, k = 0, 1, 2, \cdots$, where

$$(1.1) \qquad w_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\phi)e^{-in\phi} \, d\phi, \; n = 0, \pm 1, \pm 2, \cdots$$

The hermitian matrix T_0 gives rise to a semi-bounded transformation T_1 on complex sequential Hilbert space l_2, and thus the Friedrichs extension T of T_1 is a self-adjoint operator. $T = T(W(\phi))$ is the *Toeplitz operator* associated with W.

In [5], [6] Hartman and Wintner show that the case in which W is not semi-bounded (which we prudently avoid here) presents special difficulty. However for semi-bounded W they prove that

(i) the spectrum of T fills the interval

[ess inf W, ess sup W],

and

(ii) T has no point spectrum.

Thus the spectral measure ([4], p. 58) $E(\cdot)$ of T is such that $\langle E(\cdot)F, F \rangle$ is a nonatomic Borel measure for each $F \in l^2$. If $\langle E(\cdot)F, F \rangle$ is AC (absolutely continuous with respect to Lebesgue measure) for each $F \in l^2$, then we say that T is AC.

Our investigation continues work of C. R. Putnam [11]. He proves that T is AC in each of the following cases:

(i) $W(\phi) = 2 \cos n\phi$, $n = 1, 2, \cdots$

(ii) $W(\phi) = 2 \sin n\phi$, $n = 1, 2, \cdots$

(iii) Let $a_{jk} = w_{k-j}$ for $k - j \geq 1$ and $a_{jk} = 0$ otherwise.

Further suppose that the $\{w_n\}$ are real, that $A_0 = [a_{jk}]$ is bounded, and that 0 is not an eigenvalue of the Hankel matrix $[w_{j+k+1}]$, $j, k = 0, 1, 2, \cdots$.

For case (i) Putnam gives a more complete spectral analysis. He applies the perturbation theory propounded in [13] to prove the following result:

1.2 $T(2 \cos n\phi)$ is unitarily equivalent to $2T_n(\frac{1}{2}T(2 \cos \phi))$. Here T_n is the nth degree Tchebichef polynomial, $n = 1, 2, \cdots$.

In §§ 2 and 3 we prove that every Toeplitz operator is AC. The method of proof first involves deriving a generating function formula for the resolvent of T. This formula appears in the work [2] of Calderon, Spitzer, and Widom. However, we shall offer a different derivation, one that points out an interesting connection between T and the Szegö kernel function. Next we shall apply a result from the Aronszajn-Donoghue [1] theory of exponential representations of analytic functions, and consequently deduce that T is absolutely continuous. We conclude with § 4 where 1.2 is generalized. We elaborate on Putnam's method. By severely restricting W we are able to employ Kato's generalization [7], [8] of [13] to exhibit a multiplication operator M_{AC} on an L^2 space such that T is unitarily equivalent to M_{AC}.

2. T and the Szegö kernel function. We first set down some notation. We shall ambiguously employ $"F"$ to denote

(a)   the element $\{f_n\}_0^\infty$ of l^2;

(b)   the element $F(e^{i\phi})$ of $L^2(-\pi, \pi)$ that has the Fourier series $\sum_{n=0}^\infty f_n e^{in\phi}$; and

(c)   the holomorphic function $F(u) = \sum_{n=0}^\infty f_n u^n$, $|u| < 1$.

Let $\langle\ ,\ \rangle$ be the l^2 inner product and suppose $*$ is the symbol of complex conjugation, used so $F^*(e^{i\phi}) \sim \sum_{n=0}^\infty f_n^* e^{in\phi}$ and $[F(e^{i\phi})]^* \sim \sum_{n=0}^\infty f_n^* e^{-in\phi}$. Then

$$(2.1) \qquad \langle F, G \rangle = \sum_{n=0}^\infty f_n g_n^* = \frac{1}{2\pi} \int_{-\pi}^\pi F(e^{i\phi}) [G(e^{i\phi})]^* \, d\phi \, .$$

We suppose that u, v are complex numbers such that $|u| < 1$, $|v| < 1$, and define $U = \{u^n\}_0^\infty \in l^2$, $V = \{v^n\}_0^\infty \in l^2$. Note that $U(e^{i\phi}) = (1 - ue^{i\phi})^{-1}$ and $V^*(e^{i\phi}) = (1 - v^* e^{i\phi})^{-1}$.

Select λ so that $1 + \lambda \leqq$ ess inf W. Let $l^{2,\lambda}$ be the inner product space formd of elements $F \in l^2$ such that

$$[F, F] = \frac{1}{2\pi} \int_{-\pi}^\pi |F(e^{i\phi})|^2 (W(\phi) - \lambda) \, d\phi < \infty \, .$$

Since

$$[F, F] \geqq \langle F, F \rangle$$

it follows that $l^{2,\lambda}$ is a (complete) Hilbert space. Define the linear functional L_v on $l^{2,\lambda}$ by $L_v(F) = \langle F, V^* \rangle$. L_v is bounded since

$$|L_v(F)|^2 \leqq \langle F, F \rangle \langle V^*, V^* \rangle \leqq [F, F] \langle V^*, V^* \rangle \, .$$

Hence by the Frechet-Riesz representation theorem ([12], p. 61) there exists a unique element $K_v \in l^{2,\lambda}$ such that $[F, K_v] = L_v(F)$. Thus

2.2 $\qquad\qquad\qquad F(v) = \langle F, V^* \rangle = [F, K_v]$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\phi})[K_v(e^{i\phi})]^*(W(\phi) - \lambda)\, d\phi$$

for all v, $|v| < 1$.

It follows from 2.2 that $K_v(u) = \langle K_v, U^* \rangle$ is the Szegö kernel function associated with the Hilbert space of holomorphic functions F such that $[F, F]$ is finite. From ([3], p. 51);

2.3 $$K_v(u) = (1 - uv^*)^{-1}[g(v)]^*\, g(u),$$

where

2.4 $$g(u) = \exp - \frac{1}{4\pi} \int_{-\pi}^{\pi} \log\, (W(\phi) - \lambda)(e^{i\phi} + u)(e^{i\phi} - u)^{-1}\, d\phi\, .$$

We next turn our attention to the Toeplitz matrix T_0. We define the transformation T_1 to be the restriction of T_0 to the subset $\mathscr{D}_1$ of l^2 consisting of elements F that have only a finite number of non-zero components. Then if $F \in \mathscr{D}_1$, and δ is the Kronecker symbol,

2.5 $$\langle (T - \lambda)F, F \rangle = \sum\, (w_{k-j} - \lambda\delta_{j,k})f_j f_k^* = [F, F]\, .$$

Since $[F, F] \geqq \langle F, F \rangle$ we are in a situation to which the Friedrichs extension theory is applicable ([12], p. 328–333). Upon applying this theory we note that:

(a) There exists a unique self-adjoint operator T that is an extension of T_1 and whose domain $\mathscr{D}$ is contained in $l^{2,\lambda}$. $\mathscr{D}$ is a independent of the choice of $\lambda + 1 \leqq$ ess inf W. Notice that T is a quite convenient self-adjoint extension of T_1 since it preserves the analytic nicety 2.5 for all $F \in \mathscr{D}$.

(b) $(T - \lambda)^{-1}$ is a bounded positive definite operator that maps l^2 into $l^{2,\lambda}$, and furthermore

2.6 $$\langle F, G \rangle = [F, (T - \lambda)^{-1}G] \text{ for all } G \in l^2 \text{ and } F \in l^{2,\lambda}\, .$$

THEOREM 1. *Suppose* $\lambda + 1 \leqq$ *ess inf* W. *Then* $(T - \lambda)^{-1}$ *exists, is bounded, and* $\langle (T - \lambda)^{-1}V^*, U^* \rangle = K_v(u)$.

Proof. Suppose $F \in l^{2,\lambda}$. Then by 2.1 and 2.2, $\langle F, V^* \rangle = \sum_{n=0}^{\infty} f_n v^n = F(v) = [F, K_v]$. But, by 2.6, $\langle F, V^* \rangle = [F, (T - \lambda)^{-1}V^*]$. Thus $K_v = (T - \lambda)^{-1}V^*$, and $K_v(u) = \langle K_v, U^* \rangle = \langle (T - \lambda)^{-1}V^*, U^* \rangle$, as asserted.

As commented before, Theorem 1 can be derived from results in Calderon, Spitzer, and Widom's paper [2].

3. Exponential representation. We list some of the results of the Aronszajn-Donoghue theory of exponential representations of holomorphic functions in

THEOREM 2. *Suppose* R *is a function holomorphic in the upper*

half plane and there having a non-negative imaginary part.   Then:

(i)   ([1], p. 325).   *There exists a positive measure μ and real numbers $\alpha' \geq 0$ and β such that*

$$3.1 \qquad R(\lambda) = \alpha'\lambda + \beta + \int_{-\infty}^{\infty} [(t - \lambda)^{-1} - t(t^2 + 1)^{-1}] \, d\mu .$$

$\alpha' \beta, \mu$ are uniquely determined by R, and $(t^2 + 1)^{-1}$ is integrable with respect to μ.   If $|t|(t^2 + 1)^{-1}$ is integrable with respect to μ, then

$$3.2 \qquad R(\lambda) = \alpha'\lambda + \beta' + \int_{-\infty}^{\infty} (t - \lambda)^{-1} \, d\mu, \text{ where}$$

$$\beta' = \beta - \int_{-\infty}^{\infty} t(t^2 + 1)^{-1} \, d\mu$$

(ii)   ([1], p. 331).   *There exists a Lebesgue measurable function α with $0 \leq \alpha \leq 1$ and a real number σ such that*

$$3.3 \qquad R(\lambda) = \exp \sigma \exp \int_{-\infty}^{\infty} [(t - \lambda)^{-1} - t(t^2 + 1)^{-1}] \, \alpha(t) \, dt$$

α is determined by 3.3 modulo a set of Lebesgue measure zero.

(iii)   ([1], p. 386).   *A sufficient condition for μ to be AC is that for all real x*

$$3.4 \qquad \omega(x) = \lim \{\omega(a, b) : a \uparrow x, b \downarrow x\} < 1,$$

where

$$\omega(a, b) = \sup \{\alpha(d) - \alpha(c) : a < c < d < b\} .$$

We next reframe 2.3 in a form suitable for application of the preceding theorem.   Let χ_t be the characteristic function of $\{\phi : W(\phi) \leq t, -\pi < \phi \leq \pi\}$.   Put

$$P(\phi, u, v) = \frac{1}{4\pi} [(e^{i\phi} + u)(e^{i\phi} - u)^{-1} + (e^{-i\phi} + v^*)(e^{-i\phi} - v^*)^{-1}] ,$$

so if

$$v = re^{i\psi}, P(\phi, v, v) = \frac{1}{2\pi} (1 - r^2)(1 - 2r \cos (\phi - \psi) + r^2)^{-1}$$

is the Poisson kernel.   Let

$$\sigma(u, v) = - \frac{1}{2} \int_{-\pi}^{\pi} \log [1 + (W(\phi))^2] P(\phi, u, v) \, d\phi ,$$

and $\alpha(t, u, v) = \int_{-\pi}^{\pi} \chi_t(\phi) P(\phi, u, v) \, d\phi$.   Notice that $\alpha(\cdot, u, v)$ is of bounded variation, with $\alpha(t, u, v) = 0$ or 1 according to whether $t <$ ess inf W or $t >$ ess sup W respectively.   Also note that $\alpha(\cdot, v, v)$ is monotone increasing with $0 \leq \alpha(t, v, v) \leq 1$.

LEMMA 1.   *If $\Im \lambda \neq 0$ or $\lambda <$ ess inf W, then*

3.5 $(1 - uv^*) \langle (\boldsymbol{T} - \lambda)^{-1} V^*, U^* \rangle$

$$= \exp \, \sigma(u, v) \exp \int_{-\infty}^{\infty} [(t - \lambda)^{-1} - t(t^2 + 1)^{-1}] \, \alpha(t, u, v) \, dt \, .$$

Proof.   Temporarily assume that
(*) $\lambda + 1 \leqq \text{ess inf } W$.   By 2.3 and Theorem 1

$$(1 - uv^*) \langle (\boldsymbol{T} - \lambda)^{-1} V^*, U^* \rangle = \exp - \int_{-\pi}^{\pi} \log \, (W(\phi) - \lambda) P(\phi, u, v) \, d\phi$$

$$= \exp \sigma(u, v) \exp - \int_{-\pi}^{\pi} \log \, [(W(\phi) - \lambda)((W(\phi))^2 + 1)^{-1/2}] \, P(\phi, u, v) \, d\phi$$

$$= \exp \sigma(u, v) \exp - \int_{-\infty}^{\infty} \log \, [(t - \lambda)(t^2 + 1)^{-1/2}] \, d_t \, \alpha(t, u, v) \, .$$

We integrate by parts to obtain 3.5 under assumption (*).   An analytic continuation argument enables us to relax (*).
    We now apply Theorem 2.

LEMMA 2.   *Suppose* $|v| < 1$.   *Then* $\langle \boldsymbol{E}(\cdot) V^*, V^* \rangle$ *is* AC.

Proof.   Consider $R(\lambda) = (1 - |v|^2) \langle (\boldsymbol{T} - \lambda)^{-1} V^*, V^* \rangle$.   This is a holomorphic function of the type described in Theorem 2. 3.5 assures us that it has the exponential representation 3.3 with $\alpha(t) = \alpha(t, v, v)$. We shall show that α satisfies 3.4 and from this it will follow that $\mu(\cdot) = \langle \boldsymbol{E}(\cdot) V^*, V^* \rangle$ is AC.   Now,

$$\omega(a, b) = \sup \left\{ \int_{-\pi}^{\pi} [\chi_d(\phi) - \chi_c(\phi)] \, P(\phi, v, v) \, d\phi : a < c < d < b \right\}$$

$$\leqq \int_{-\pi}^{\pi} [\chi_{b-}(\phi) - \chi_{a+}(\phi)] \, P(\phi, v, v) \, d\phi$$

since $P(\cdot, v, v)$ is positive.   Thus

$$\omega(x) \leqq \int_{-\pi}^{\pi} [\chi_{x+}(\phi) - \chi_{x-}(\phi)] \, P(\phi, v, v) \, d\phi = h(r, \psi) \, , \quad \text{where} \quad v = re^{i\psi} \, .$$

Since $P(\phi, v, v)$ is the Poisson kernel, h is a non-negative harmonic function in $|v| < 1$. W is not equivalent to a constant, so h is not a constant function.   Thus by the maximum principle, $h(r, \psi) < 1$ if $r < 1$. We invoke 3.4 to complete the proof.
    Now we can settle

THEOREM 3. $\boldsymbol{T}$ *is* AC.

Proof.   From now on let ν be real Lebesgue measure as restricted to the real Borel sets $\mathscr{B}$.   Assume $\nu(\varDelta) = 0$.   Lemma 2 assures us that if $|v| < 1$. then $\langle \boldsymbol{E}(\varDelta) V^*, V^* \rangle = 0$.   Suppose now that $F \in l^2$.   We use the Schwarz inequality and the fact that $\boldsymbol{E}(\varDelta)$ is a projection to

note that

$$| \langle E(\varDelta)V^*, F \rangle | \leqq ||E(\varDelta)V^*|| \, ||F|| = [\langle E(\varDelta)V^*, \, E(\varDelta)V^* \rangle]^{1/2} ||F||$$
$$= [\langle E(\varDelta)V^*, \, V^* \rangle]^{1/2} \, ||F|| = 0 \; .$$

Thus $\langle E(\varDelta)V^*, F \rangle = 0$ for all $v, |v| < 1$. But the set $\{V^* : |v| < 1\}$ is fundamental in l^2 since $\langle G, V^* \rangle = \sum_{n=0}^{\infty} g_n v^n = 0$ for all $v, |v| < 1$ implies that the g_n all vanish. Thus $\langle E(\varDelta)F, F \rangle = 0$, and T is AC.

4. Spectral theory. Our principal goal now is to establish a spectral analysis for T. More particularly, we wish to exhibit a multiplication operator M_{AC} on an L^2 space such that M_{AC} is unitarily equivalent to T. However, we were able to achieve this goal only for a small class of $T(W(\phi))$. From now on we assume that W is even and AC, and that the derivative W' of W has an absolutely convergent Fourier series, so $\sum_n |w_n| < \infty$. Our techniques follow those of Putnam [11], but whereas he uses the theory presented by this author in [13], we use T. Kato's generalization [7], [8] of [13]. See also [9] and [10].

We start by discussing some preliminary material that we include here for completeness. A countably-additive function E on $\mathscr{B}$ to projection operators in a Hilbert space $\mathscr{L}$ is AC if $\nu(\varDelta) = 0$ implies $E(\varDelta) = 0$. E is *singular* if there exists $\beta \in \mathscr{B}$ such that $\nu(\beta) = 0$ but $E(\varDelta \cap \beta) = E(\varDelta)$ for all $\varDelta \in \mathscr{B}$. It is easy to see that a self-adjoint operator M is AC if and only if its spectral measure is AC.

We shall now establish a Lebesgue decomposition theorem for spectral measures as a corollary of the classical version of that theorem.

LEMMA 3. *Suppose $E(\cdot)$ is a spectral measure in a separable Hilbert space $\mathscr{L}$. Then:*

(i) *There exists $\gamma \in \mathscr{B}$ with $\nu(-\gamma) = 0$ and such that*

4.1 $E_{AC}(\cdot) = E(\cdot \cap \gamma)$ *is an AC and*

4.2 $E_S(\cdot) = E(\cdot - \gamma)$ *is a singular projection-valued measure.*

(ii) *If $F, G \in \mathscr{L}$, $\varDelta \in \mathscr{B}$, and E. is the resolution of the identity associated with $E(\cdot)$, then*

$$\langle E_{\mathrm{AC}}(\varDelta)F, G \rangle = \int_{\varDelta} d \langle E_x F, G \rangle / \, dx \, d\nu \; .$$

(iii) *The decomposition $E(\cdot) = E_{\mathrm{AC}}(\cdot) + E_S(\cdot)$ of $E(\cdot)$ as the sum of an AC and singular measure is unique.*

Proof. Suppose $F, G \in \mathscr{L}$, $\varDelta \in \mathscr{B}$. Then since $\langle E. F, G \rangle$ is of bounded variation it has a a derivative a.e. that is ν-summable. Also

$$\int_{\varDelta} d \langle E_x F, F \rangle / \, dx \, d\nu \leqq \int_{\varDelta} d \langle E(\cdot)F, F \rangle = \; \leqq E(\varDelta)F, F \leqq ||F||^2 \; ,$$

so the first term above represents a bounded quadratic form. Thus by ([4], p. 33), $b(F, G) = \int_{\mathcal{A}} d \langle E_x F, G \rangle / dx \, d\nu$ is a bounded bilinear functional, so there exists a bounded operator $E_{AC}(\mathcal{A})$ such that $\langle E_{AC}(\mathcal{A})F, G \rangle = b(F, G)$ for all F, G. $E_{AC}(\cdot)$ is clearly countably additive on $\mathscr{B}$, and thus so is $E_s(\cdot) = E(\cdot) - E_{AC}(\cdot)$.

Let $\{F_j\}_{j=0}^{\infty}$ be a countable dense subset of $\mathscr{L}$. By the classical version of the Lebesgue decomposition theorem as found in ([14], p. 119), corresponding to each pair j, k of non-negative integers there exists $\beta_{j,k} \in B$ such that $\nu(\beta_{j,k}) = 0$ and

$$(*) \qquad \langle E(\mathcal{A})F_j, F_k \rangle = \langle E(\mathcal{A} \cap \beta_{j,k})F_j, F_k \rangle + \langle E_{AC}(\mathcal{A})F_j, F_k \rangle$$

for all $\mathcal{A} \in \mathscr{B}$. Let β be the union of all the $\beta_{j,k}$, $j, k = 0, 1, 2, \cdots$. Then $\nu(\beta) = 0$ and (*) holds with $\beta_{j,k}$ replaced by β. Now we pass from the dense subset to all of $\mathscr{L}$. For all $F, G \in \mathscr{L}$, $\mathcal{A} \in \mathscr{B}$

$$(**) \qquad \langle E(\mathcal{A})F, G \rangle = \langle E(\mathcal{A} \cap \beta)F, G \rangle + \langle E_{AC}(\mathcal{A})F, G \rangle,$$

where the decomposition of the left hand term into singular and AC parts is unique. Put $\gamma = -\beta$. Then 4.2 holds and thus 4.1 is also true. (iii) follows from (**).

It follows from lemma 3 that $E_{AC}(\cdot) = E(\gamma)E(\cdot)E(\gamma)$ is a spectral measure in the Hilbert space $E(\gamma)\mathscr{L}$. $M_{AC} = E(\gamma)ME(\gamma)$ is the self-adjoint operator on $E(\gamma)\mathscr{L}$ having this spectral measure. M_{AC} is obviously AC.

The following simple example will play a role in what happens later. Let W be as before, even, with $\sum_n |w_n| < \infty$. Let M be the multiplication operator that maps any $F \in L^2(0, \pi) = \mathscr{L}$ into $W \cdot F \in \mathscr{L}$. Let $\chi(\mathcal{A})$ be the characteristic function of $\{\phi : W(\phi) \in \mathcal{A} : 0 \leq \phi \leq \pi\}$. Since

$$\langle MF, F \rangle = \frac{1}{\pi} \int_0^{\pi} W(\phi)|F(\phi)|^2 \, d\phi = \int_{-\infty}^{\infty} t \, d_t \frac{1}{\pi} \int_0^{\pi} \chi(\mathcal{A})(\phi)|F(\phi)|^2 \, d\phi$$

it follows that the spectral measure $E(\cdot)$ of M is defined by $E(\mathcal{A})F = \chi(\mathcal{A}) \cdot F$. Lemma 3 guarantees the existence of $\gamma \in \mathscr{B}$ such the $1/\pi \int_0^{\pi} \chi(\gamma)\chi(\cdot)(\phi)|F(\phi)|^2 \, d\phi$ is AC for all $F \in \mathscr{L}$, while $E(\cdot - \gamma)$ is singular. $E(\gamma)\mathscr{L}$ can be identified with the Hilbert space $L^2(A)$, where $F \in L^2(A)$ if and only if $\|F\|_A < \infty$, where

4.3 $$\|F\|_A = \left[\frac{1}{\pi} \int_0^{\pi} \chi(\gamma) |F(\phi)|^2 \, d\phi \right]^{1/2} = \left[\frac{1}{\pi} \int_A |F(\phi)|^2 \, d\phi \right]^{1/2},$$

and

$$A = \{\phi : W(\phi) \in \gamma, \ 0 \leq \phi \leq \pi\} \ .$$

Similarly M_{AC} can be considered to be the mapping that takes any $F \in L^2(A)$ into $W \cdot F \in L^2(A)$.

Another concept that we shall have cause to use is that of trace class. As is usual, a bounded operator on l^2 is identified with its

matrix representation. A matrix $H = [w_{j,k}]$, $i, j = 0, 1, 2, \cdots$ belongs to the *Schmidt-Hilbert class* SH if $\sum_{j,k=0}^{\infty} |w_{j,k}|^2 < \infty$. H belongs to the *trace class* TC if $H \in$ SH and $\| H \|_1 < \infty$, where $\| H \|_1$ is the sum of the absolute values of the eigenvalues of H repeated according to multiplicity.

As an example we treat the Hankel matrix $H = [w_{j+k+2}]$. As proved in [5], $H \in$ SH if and only if $\sum_{n=1}^{\infty} n \, | w_{n+1} |^2 < \infty$. This follows from the equality $\sum_{j,k=0}^{\infty} |w_{j+k+2}|^2 = \sum_{n=1}^{\infty} n|w_{n+1}|^2$, and gives a necessary condition that $H \in$ TC. Now, define $H_n = [\delta_{j+k+2,n}]$. Then $H = \sum_{n=2}^{\infty} w_n H_n$. Since $\|H_n\|_1 \leqq n$ it follows that $\|H\|_1 \leqq \sum_{n=2}^{\infty} |w_n| \, \|H_n\|_1 \leqq \sum_{n=2}^{\infty} n|w_n|$. Thus a sufficient condition $H \in$ TC is that W be AC such that W' has an absolutely convergent Fourier series. This, of course, is part of our standing hypothesis on W for this section. We do not know a useful necessary and sufficient condition for a Hankel matrix to belong to TC.

Hankel matrices enter into our picture, following an idea of Putnam's, via the following

LEMMA 4. *Let H be as as in the above example. Let $S = [s_{j,k}]$, where $s_{j,k} = 2/\pi \int_0^{\pi} W(\phi) \sin (j + 1)\phi \sin (k + 1)\phi \, d\phi$, $j, k = 0, 1, 2, \cdots$. Then $T - S = H$.*

Proof. $w_{j-k} - s_{j,k} = \dfrac{1}{\pi} \int_0^{\pi} W(\phi) \cos \ (j - k)\phi \, d\phi$

$$- \frac{2}{\pi} \int_0^{\pi} W(\phi) \sin (j + 1)\phi \sin (k + 1)\phi \, d\phi \ .$$

$$= \frac{1}{\pi} \int_0^{\pi} W(\phi) \cos (j + k + 2) \, d\phi \ .$$

We can now state a specialization of Kato's theorem in a form suitable for our application. It is understood that in the statement T and S need not necessarily be the operators we have already defined.

THEOREM 5. (Kato). *Suppose T and S are self-adjoint operators on a separable Hilbert space $\mathscr{L}$ such that $T - S = H \in$ TC and T is AC. Let γ and $E(\cdot) = E_{AC}(\cdot) + E_s(\cdot)$ be the Borel set and decomposition respectively guaranteed by Lemma 3. Then*

    (i) *as $t \to \infty$, $\exp(itT) \exp(-itS) E(\gamma)$ converges strongly to an isometric mapping U of $E(\gamma) \mathscr{L}$ onto $\mathscr{L}$.*

    (ii) *U^{-1} is the strong limit as $t \to \infty$ of $\exp(itS) \exp(-itT)$.*

    (iii) *The self-adjoint operator $S_{AC} = E(\gamma) S E(\gamma)$ on $E(\gamma)\mathscr{L}$ is unitarily equivalent to T, with $I = US_{AC} U^{-1}$.*

From this follows the following spectral analysis theorem for T.

THEOREM 6. *Suppose W is a real even AC function on $(-\pi, \pi)$ whose derivative W' has an absolutely convergent Fourier series. Then*

the Toeplitz operator $T(W(\phi))$ *is unitarily equivalent to to the multiplication operator* $M_{AC} : f \rightarrow W{\cdot}f$ *on* $L^2(A)$ *(see 4.3).*

Proof. The hypotheses of Theorem 5 are satisfied via Lemma 4, the discussion following Lemma 3, and Theorem 3. Thus T is unitarily equivalent to S_{AC}. Since $\{f_n\}_0^\infty \rightarrow 2^{1/2} \sum_{n=0}^\infty f_n \sin(n+1)\phi$ is an isometry of l^2 onto $L^2(0, \pi)$, it follows that S_{AC} is unitarily equivalent to M_{AC}. Thus T is unitarily equivalent to M_{AC}.

COROLLARY 1. *Suppose* $W(\phi) = w_0 + 2\sum_1^m w_n \cos n\phi$, *where the* w_n *are real and* m *is a positive integer. Then* $T(W(\phi))$ *is unitarily equivalent to the multiplication operator* $M : f \rightarrow W{\cdot}f$ *on* $L^2(0, \pi)$.

Proof. In this case $M = M_{AC}$. (See Putnam [11], p. 522). Now use Theorem 6.

If W' is AC and $W'' \in L^2(0, \pi)$ then $\sum_n |w_n| < \infty$. Hence a W satisfying Theorem 6 can have intervals of constancy. If such is the case, then M has an infinite number of eigenvectors. Thus one cannot validly replace "M_{AC}" and "$L^2(A)$" by "M" and "$L^2(0, \pi)$" respectively in the statement of Theorem 6, since T has no point spectra.

We can easily deduce 1.2 from Corollary 1. $T(W(2 \cos n\phi))$ is unitarily equivalent to multiplication by $2 \cos n\phi$ on $L^2(0, \pi)$, $n = 1, 2, \cdots$, and hence to $2 \cos (n \text{ arc } \cos \frac{1}{2} T(2 \cos \phi)) = 2 T_n(\frac{1}{2} T(2 \cos \phi))$ on l^2.

It would be of great interest to evaluate the limits in Theorem 5 (ii) and (iii) so one could exhibit the unitary transformation of Theorem 6. One could then have a super-abundance of new unitary operators. We pose this as an unsolved problem.

5. **Appendix.** C. R. Putnam has extended the theory he set forth in [11] in his recent article "*On Toeplitz matrices, absolute continuity and unitary equivalence*", Pacific J. Math., **9** (1959), 837–846. He proves that T is AC provided that A_0 is bounded and $M = M_{AC}$. whence, using [13], he proves Theorem 6 under the added hypothesis that $M = M_{AC}$.

It is interesting to compare our proof that T is AC with Putnam's weaker version of that result. He applies his abstract theory of commutators, while we exhibit the resolvent of T and employ the rather deep function-theoretic results of [1].

REFERENCES

1. N. Aronszajn and W. F. Donoghue, Jr., *On exponential representations of analytic-functions in the upper halfplane with positive imaginary part*, Journal D'Analyse Mathématique, **5** (1956–57), 321–388.
2. A. Calderon, F. Spitzer, and H. Widom, *Inversion of Toeplitz matrices*. Cornell University contract report, March, 1958.
3. Ulf Grenander and Gabor Szegö, *Toeplitz forms and their applications*, University of

California, 1958.

4. Paul R. Halmos, *Introdution to Hilbert space and the theory of spectral multiplicity*, New York, 1951.

5. P. Hartman and A. Wintner, *On the spectra of Toeplitz's matrices*, Amer. J. Math., **72** (1950), 359–366.

6. ————, *The spectra of Toeplitz's matrices* Amer. J. Math. **76** (1954), 867–882.

7. Tosio Kato, *On finite-dimensional perturbations of selfadjoint operators*, Journal Math Soc. Japan, **9** (1957), 239–249.

8. ————, *Perturbation of continuous spectra by trace class operators*, Proc. Japan Acad., **33** (1957), 260–264.

9. Shige Toshi Kuroda, *On the existence and the unitary property of the scattering operator*, mimeographed notes from Dept. of Physics, Univ. of Tokyo, Jan., 1959.

10. ————, *Perturbation of continuous spectra by unbounded operators*, to be published.

11. C. R. Putnam, *Commutators and absolutely continuous operators*, Trans. Amer. Math. Soc., **87** (1958). 513–525.

12. F. Riesz, and B. Sz.-Nagy, *Leçons d'analyse fonctionelle*, Budapest, 1952.

13. M. Rosenblum, *Perturbation of the continuous spectrum and unitary equivalence* Pacific J. Math., **7** (1957), 997–1010.

14. S. Saks, *Theory of the integral*, Warszawa–Lwow, 1937.

UNIVERSITY OF VIRGINIA