A CLASS OF LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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I. Introduction. The purpose of this paper is to study the following integral equation:

$$\varphi(x) = \int_x^{x+1} K(y)\varphi(y)dy \tag{1}$$

or the differential-difference equation

$$\varphi'(x) = K(x+1)\varphi(x+1) - K(x)\varphi(x) \tag{1'}$$

with the boundary condition

$$\lim_{x\to\infty} \varphi(x) = 1\ . \tag{2}$$

Equations of the type (1), (1′) have been investigated in great generality by many authors. In particular, the interested reader is referred to Yates [6], and Cooke [2], for recent developments, and a bibliography of significant earlier work. The equations of the form (1) which we shall consider are related to the class of linear differential-difference equations with asymptotically constant coefficients, a class treated thoroughly by Wright [5], and Bellman [1].

The novelty of the results below arises from the boundary condition (2) which appears not to have been studied before, and which gives reresults of an essentially different character from those of the works cited above. The system (1), (2) is of interest in some problems connected with the theory of neutron slowing down (Placzek [3]).

A further departure from previous work is the fact that no use is made of complex variable methods or the asymptotic characteristic equation of the kernel $K(y)$.

Aside from some fairly obvious theorems concerning uniqueness, boundedness and positivity, our main results are the following:

(a) necessary and sufficient conditions for the existence of a solution of (1), (2); this is achieved by constructing a minorant for the solution.

(b) proof of the existence of $\varphi(-\infty)$ under fairly general conditions.

(c) an application of Fubini's theorem to exhibit a rather surpris-

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ing relation between an integral of the solution over the real axis and its limits at $\pm\infty$. We assume

H 1 $K(x)$ is measurable ,
H 2 $0 < K(x) \leqq 1$, for almost all x ,
H 3 For $x \geqq M$, $K(x)$ increases ,
H 4 $\lim_{x\to\infty} K(x) = 1$,

throughout the paper.

To summarize the results below, we shall give necessary and sufficient conditions for the existence (Theorem 4), uniqueness (Theorem 1), boundedness (Theorem 2), and positivity (Theorem 3) of the solution; a a sufficient condition for its monotonicity (Theorem 5); a proof of the existence of $\varphi(-\infty)$(Theorem 6) and the evaluation of a definite integral involving the solution (Theorem 7).

By "solution" we shall always mean a function $\varphi(x)$ satisfying both (1) and (2). All integrals are to be understood in the sense of Lebesgue.

II. Existence and uniqueness of solutions.

THEOREM 1. *Under* H 1 – H 4, *the solution* $\varphi(x)$, *when it exists, is unique.*

Proof. If the theorem is false, there exists a function $\psi(x)$ not identically zero which satisfies (1) and for which

$$\lim_{x\to\infty} \psi(x) = 0 .$$

Then by the continuity of $\psi(x)$ there exist numbers η and x_0 such that $\eta > 0$, $|\psi(x_0)| = \eta$ and for all $x > x_0$, $|\psi(x)| < \eta$. But then

$$\eta = |\psi(x_0)| \leqq \int_{x_0}^{x_0+1} |\psi(y)|\, dy < \eta$$

a contradiction, which completes the proof.

THEOREM 2. *With* H1 – H4 *we have, for any solution* $\varphi(x)$ *of* (1), (2),

$$|\varphi(x)| \leqq 1 \qquad (-\infty < x < \infty) . \tag{3}$$

Proof. For if $|\varphi(x)| > 1$ for some x, then by (2) and the continuity of $|\varphi(x)|$ there is a $C > 1$ and an x_0 such that $|\varphi(x_0)| = C$, and for all $x > x_0$, $|\varphi(x_0)| < C$. But then

$$|\varphi(x_0)| \leqq |\int_{x_0}^{x_0+1} |\varphi(y)|\, dy$$

implies $C < C$, which is a contradiction.

THEOREM 3. *Supposing* H1 – H4, *the solution* $\varphi(x)$ *of* (1) *and* (2), *when it exists, is positive for all* x, *and is non-decreasing for* $x \geqq M$.

Proof. We prove positivity first. If $\varphi(x)$ is not >0 for all x, then by (2) and the continuity of $\varphi(x)$ there is an x_0 such that $\varphi(x_0) = 0$ and for all $x > x_0$, $\varphi(x) > 0$. Then

$$\varphi(x_0) = 0 = \int_{x_0}^{x_0+1} K(y)\varphi(y)dy ,$$

which is a contradiction by H2.

To prove the monotonicity part, we define

$$\psi_0(x) = 1 , \tag{4}$$

and

$$\psi_{n+1}(x) = \int_x^{x+1} K(y)\psi_n(y)dy . \tag{5}$$

Since $0 < K(y) \leqq 1$, $\psi_1(x) \leqq \psi_0(x)$, and since

$$\psi_n(x) - \psi_{n+1}(x) = \int_x^{x+1} K(y)[\psi_{n-1}(y) - \psi_n(y)]dy , \tag{6}$$

we see by induction that $\{\psi_n(x)\}$ is a decreasing sequence. But since $\varphi(x) \leqq 1 = \psi_0(x)$, we see by a second induction that $\psi_n(x) \geqq \varphi(x)$ for all x. Hence the $\psi_n(x)$ decrease to a limit function $\psi(x)$ satisfying (1) by Lebesgue's dominated convergence theorem, and

$$\lim_{x\to\infty} \psi(x) = 1$$

since $\varphi(x) \leqq \psi(x) \leqq 1$. Now $\psi_0(x)$ is non-decreasing for $x \geqq M$, and thus so is $\psi_1(x)$, and again by induction, $\psi_n(x)$ and hence $\psi(x)$. But by Theorem 1, $\psi(x) = \varphi(x)$, which proves the theorem.

LEMMA 1. *Under* H1 – H4 *and*

$$\text{H5:}\quad 1 - K(x) \in \mathscr{L}(M, \infty)$$

there is a function $S(x)$ *such that* $S(x) \geqq 0$, $S(x)$ *is non-decreasing*, $\lim_{x\to\infty} S(x) = 1$, *and*

$$S(x) \leqq \int_x^{x+1} K(y)S(y)dy \qquad (-\infty < x < \infty) . \tag{7}$$

Proof. Define

$$S(x) = \begin{cases} 0 & x \leqq M \\ C_n & M + \dfrac{n}{2} \leqq x < M + \dfrac{n+1}{2} \end{cases} \qquad (n = 0, 1, 2, \cdots) \tag{8}$$

where the C_n are constants to be determined, and define

$$q_n = \int_{M+(n/2)}^{M+(n+1/2)} K(y)dy \ . \tag{9}$$

Now, requiring that $S(x)$ satisfy (1) at the points $M + (n/2)$ gives

$$C_n q_n + C_{n+1} q_{n+1} = C_n$$

that is

$$C_{n+1} = C_n\left[\frac{1 - q_n}{q_{n+1}}\right],$$

and

$$C_{n+1} = \prod_{j=0}^{n}\left[\frac{1 - q_j}{q_{j+1}}\right]C_0 \ . \tag{10}$$

But since

$$\frac{1 - q_j}{q_{j+1}} - 1 = \frac{1 - (q_j + q_{j+1})}{q_{j+1}} \geqq 0$$

we see that the C_n form a non-decreasing sequence. Also

$$\frac{1 - q_j}{q_{j+1}} - 1 \leqq \frac{1 - K\{M + (n/2)\}}{K(M)}$$

since $K(y)$ increases. But then H5 implies that

$$\sum_{n=0}^{\infty}\{1 - K[M + (n/2)]\}$$

converges, and so the limit of the product in (10) exists. We can then choose C_0 so that

$$\lim_{n\to\infty} C_n = 1 \ .$$

It remains to show that (7) is everywhere satisfied. If $x_0 > M$ and $x_0 \neq M+(n/2)$ for any n, let $M + (n_0/2)$ be the largest of the $M + (n/2)$ which is less than x_0. Then

$$\begin{aligned}
&\int_{x_0}^{x_0+1} K(y)S(y)\,dy \\
&\quad = \int_{M+(n_0/2)}^{M+(n_0+2/2)} K\left(y + x_0 - M - \frac{n_0}{2}\right) S\left(y + x_0 - M - \frac{n_0}{2}\right)dy \\
&\quad \geqq \int_{M+(n_0/2)}^{M+(n_0+2/2)} K(y)S(y)dy \\
&\quad = C_{n_0} \\
&\quad = S(x_0) \ ,
\end{aligned}$$

since K and S are positive and non-decreasing.

We can now prove

THEOREM 4. *Let* H1 — H4 *hold. Then, necessary and sufficient for the existence of a solution of* (1), (2) *is* H5.

Proof. Suppose $\varphi(x)$ exists, then

$$\begin{aligned}\varphi(x) &= \int_x^{x+1} K(y)\varphi(y)dy \\ &= \int_x^{x+1} \varphi(y)dy - \int_x^{x+1} [1 - K(y)]\varphi(y)dy \ .\end{aligned}$$

Choose ε between 0 and 1 and $x_0 > M$ such that $\varphi(x) > 1 - \varepsilon$ for $x \geqq x_0$. Then

$$(1-\varepsilon)\int_{x_0}^{x_0+1}[1 - K(y)]dy \leqq \varphi(x_0+1) - \varphi(x_0)$$

since $\varphi(x)$ is non-decreasing (Theorem 3) for $x \geqq M$. Replacing x_0 by $x_0 + 1$, etc., and adding

$$\int_{x_0}^{\infty}[1 - K(y)]dy \leqq 1 - \varphi(x_0) < \infty \ .$$

On the other hand, if H5 holds, consider again the $\psi_n(x)$ of (4)-(5). Since $\{\psi_n(x)\}$ is a decreasing sequence, and

$$\psi_{n+1}(x) - S(x) \geqq \int_x^{x+1} K(y)[\psi_n(y) - S(y)]dy$$

we see that $\psi_n(x) \geqq S(x)$ for all n and x. Hence $\psi_n(x)$ decreases to a a limit $\varphi(x)$, satisfying (1), and since

$$1 \geqq \varphi(x) \geqq S(x)$$

we have (2) also.

III. Monotonicity. The solution $\varphi(x)$ of (1), (2), when it exists, need not to be monotone on the whole real axis. In this section we will first illustrate the above statement, and then give sufficient conditions for the monotonicity of the solution. A lemma that will be of use in the illustration is

LEMMA 2. *Let* $K_a(x)$ *and* $K_b(x)$ *each satisfy* H1–H5, *and in additon suppose that for all* x

$$K_a(x) \leqq K_b(x) \ .$$

Then if $\varphi_a(x), \varphi_b(x)$ are the corresponding solutions of (1), (2), we have

$$\varphi_a(x) \leqq \varphi_b(x)$$

for all x.

Proof. First,

$$\begin{aligned}\varphi_a(x) &= \int_x^{x+1} K_a(y)\varphi_a(y)dy \\ &\leqq \int_x^{x+1} K_b(y)\varphi_a(y)dy \ .\end{aligned}$$

Now let $\varphi_{a,0}(x) = \varphi_a(x)$, and define

$$\varphi_{a,n+1}(x) = \int_x^{x+1} K_b(y)\varphi_{a,n}(y)dy \ .$$

Then $\{\varphi_{a,n}(x)\} \uparrow_n$ and is bounded above by 1. Hence the sequence converges to a solution of

$$\begin{cases}\varphi(x) = \int_x^{x+1} K_b(y)\varphi(y)dy \\ \lim_{x\to\infty} \varphi(x) = 1 \ .\end{cases}$$

The result then follows from Theorem 1 .

Now consider the family

$$K_a(x) = \frac{x^2 + a}{x^2 + 1} \qquad (0 \leqq a \leqq 1) \ .$$

Clearly each $K_a(x)$ satisfies H1–H5. Let $\varphi_0(x)$ satisfy (1), (2) with $K(x) = K_0(x)$. Then

$$\varphi_0'(-1) = -K_0(-1)\varphi_0(-1) = -(1/2)\varphi_0(-1) < 0$$

by Theorem 3. Hence $\varphi_0(x)$ is not monotone. In fact we can invoke Lemma 2 to show that there exists a number $a^* \varepsilon (0, 1)$ such that for $a < a^*$ $\varphi_a(x)$ is not monotone. For if not, there exists a sequence $\{a_n\} \downarrow 0$ such that $\varphi_{a_n}(x)$ satisfies (1), (2) with $K(x) = K_{a_n}(x)$ and $\varphi_{a_n}(x)$ is monotone for each n. Since $\{\varphi_{a_n}(x)\}$ decreases to a solution of (1), (2) with $K(x) = K_0(x)$ (by Lemma 2 and Theorem 1) we must have $\varphi_0(x)$ monotone which is a contradiction.

The following theorem, however, gives a sufficient condition for the monotonicity of $\varphi(x)$:

THEOREM 5. *With* H1–H5, *suppose that for almost all* x,

$$K(x+1) \geqq K(x) \int_x^{x+1} K(y)dy \ . \tag{11}$$

Then $\varphi(x)$ is non-decreasing on the real axis.

Proof. Let $S_0(x)$ be the function $S(x)$ of (8). Define

$$S_{n+1}(x) = \int_x^{x+1} K(y)S_n(y)dy \qquad (n = 0, 1, \cdots) \ . \tag{12}$$

Then, for all n,

$$\begin{aligned} &\text{(a)} \quad 0 \leqq S_n(x) \leqq 1 \\ &\text{(b)} \quad \lim_{x\to\infty} S_n(x) = 1 \\ &\text{(c)} \quad S_n(x) \uparrow \varphi(x) \ . \end{aligned} \tag{13}$$

We show next that with (11), the subsequence $\{S_{2n}(x)\}$ is a sequence of non-decreasing functions. Clearly $S_0(x)\uparrow_x$ for all x. Now suppose that for all $k \leqq n$, $S_{2k}(x)\uparrow_x$ for all x. Then

$$S'_{2n+2}(x) = K(x+1)S_{2n+1}(x+1) - K(x)S_{2n+1}(x)$$

a.e.

Now by (13)(c),

$$S_{2n+1}(x+1) \geqq S_{2n}(x+1)$$

and since

$$S_{2n+1}(x) = \int_x^{x+1} K(y)S_{2n}(y)dy \ ,$$

it follows from the inductive hypothesis that

$$S_{2n+1}(x) \leqq S_{2n}(x+1) \int_x^{x+1} K(y)dy \ .$$

Hence

$$\begin{aligned} S'_{2n+2}(x) &\geqq \left[K(x+1) - K(x)\int_x^{x+1} K(y)dy \right] S_{2n}(x+1) \\ &\geqq 0 \quad \text{a.e.} \end{aligned}$$

by (11), which proves the theorem, since $S_{2n+2}(x)$ is absolutely continuous.

IV. Behaviour for large negative values of x. We wish now to explore the limiting behaviour of the solution $\varphi(x)$ as $x \to -\infty$. We have seen that the solution will in general oscillate. We will establish below a sufficient condition for the existence of $\varphi(-\infty)$.

THEOREM 6. *Suppose* $\varphi(x)$ *is a solution of* (1), (2). *Let* $K(x)$ *satisfy* H1–H4, *and further suppose that*

$$\lim_{x\to-\infty}\int_x^{x+1}|K(t+1)-K(t)|\,dt = 0\,. \tag{14}$$

Then

$$\lim_{x\to-\infty}\varphi(x) \equiv \varphi(-\infty) \tag{15}$$

exists.

Proof. Let m (resp. M) be the lim inf (resp. lim sup) of $\varphi(x)$ as $x \to -\infty$, and write

$$k = \limsup_{x\to-\infty}\int_x^{x+1}|\varphi'(t)|\,dt\,.$$

Let $\varepsilon > 0$ be given. Let $-x_0 > 0$ be chosen so that $\varphi(x_0) < m + \varepsilon$ and for $x \leqq x_0$, $\int_x^{x+1}|\varphi'(t)|\,dt < k + \varepsilon$. Let x_1 be the first point to the left of x_0 at which $\varphi(x_1) = M - \varepsilon$, so that $\varphi(x) < M - \varepsilon$ on the interval $x_1 < x \leqq x_0$. It follows that $x_0 < x_1 + 1$ for otherwise a "proper" maximum for $\varphi(x)$ on $x_1 \leqq x \leqq x_1 + 1$ occurs at x_1, which is impossible. For the same reason there is a point x_2 satisfying $x_1 < x_0 < x_2 \leqq x_1 + 1$ at which $\varphi(x_2) = M - \varepsilon$. Hence

$$\begin{aligned} k + \varepsilon \geqq \int_{x_1}^{x_1+1}|\varphi'(t)|\,dt &\geqq \int_{x_1}^{x_0}|\varphi'(t)|\,dt + \int_{x_0}^{x_2}|\varphi'(t)|\,dt \\ &\geqq \left|\int_{x_1}^{x_0}\varphi'(t)dt\right| + \left|\int_{x_0}^{x_2}\varphi'(t)dt\right| \\ &= (M - m - \varepsilon) + (M - m - \varepsilon)\,. \end{aligned}$$

Hence $k \geqq 2(M - m)$.
However, since

$$\varphi'(x) = K(x+1)[\varphi(x+1) - \varphi(x)] + \varphi(x)[K(x+1) - K(x)]\,,$$

we find, using (14) $k \leqq M - m$. Thus $M = m$, which proves the theorem, and incidently, $k = 0$.

REMARK. $\int_x^{x+1}|K(t+1) - K(t)|\,dt \leqq \int_x^{x+2}|1 - K(t)|\,dt$; thus in the above theorem, (14) may be replaced be $1 - K(x) \in \mathscr{L}(-\infty, \infty)$, and the conclusion is still valid.

We are now able to prove the following integral relationship.

THEOREM 7. *Suppose* $\varphi(x)$ *is a solution of* (1), (2). *Let* $K(x)$ *satisfy* H1–H4, *and suppose further*

$$1 - K(x) \in \mathscr{L}(-\infty, \infty) . \tag{16}$$

Then

$$\int_{-\infty}^{\infty} [1 - K(y)]\varphi(y)dy = \frac{1 - \varphi(-\infty)}{2} . \tag{17}$$

Proof. Put

$$F(x) = \int_0^1 \varphi(x - y)ydy .$$

Then

$$F'(x) = \int_0^1 \varphi'(x - y)ydy = -\varphi(x - 1) + \int_0^1 \varphi(x - y)dy$$
$$= \int_0^1 \varphi(x - y)[1 - K(x - y)]dy .$$

Since $\varphi(x)$ is bounded and $1 - K(x) \in \mathscr{L}(-\infty, \infty)$, it follows from Fubini's theorem (see reference 4, p. 87) that $F'(x) \in \mathscr{L}(-\infty, \infty)$, and

$$F(\infty) - F(-\infty) = \int_{-\infty}^{\infty} [1 - K(t)]\varphi(t)dt .$$

But since $\varphi(x)$ satisfies (2), $F(\infty) = (1/2)$, and by the remark following Theorem 6, $F(-\infty) = (1/2)\varphi(-\infty)$. This completes the proof.

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