ON UNIVALENCE OF A CONTINUED FRACTION

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1. Introduction. For a fixed positive integer α let K_α denote the class of functions $f(z)$ which are regular at $z = 0$ and which have C-fraction expansions of the form

$$(1.1)\qquad f(z) \sim \frac{z}{1} + \frac{a_1 z^\alpha}{1} + \frac{a_2 z^\alpha}{1} + \cdots + \frac{a_n z^\alpha}{1} + \cdots, \ |a_n| \leqq 1/4\,.$$

From an elementary convergence theorem for continued fractions [4, p.42], it follows that each function of the class K_α is regular for $|z| < 1$. This and the one-to-one correspondence between C-fractions and power series [4, p. 400] permit a replacement of the correspondence symbol in (1.1) by equality for $|z| < 1$.

The purpose of this paper is to determine for K_α the radius of univalence, $U(\alpha)$, and bounds for the starlike radius, $S(\alpha)$, and the radius of convexity, $C(\alpha)$. In the case of S-fractions it was shown by Thale [3] that $U(1) \geqq 12\sqrt{2}-16$ and Perron [2] established the fact that actual equality holds. This result is a special case of Theorem 2.1 whose proof employs value region techniques similar to those used by Thale and Perron. Moreover, the result $S(1) \geqq 8/9$ in [3] is improved in Theorem 4.2.

The developments in this depend on the following value region theorem which is an immediate consequence of a result of Paydon and Wall [1]:

THEOREM 1.1. *If* $f(z) \in K_\alpha$ *and* $|z|^\alpha = \rho^\alpha \leqq 4r(1-r)$, $0 \leqq r \leqq 1/2$, *then*

$$(1.2)\qquad \left|\frac{f(z)}{z} - \frac{1}{1-r^2}\right| \leqq \frac{r}{1-r^2}\,.$$

Moreover, for $z = \sqrt[\alpha]{4r(1-r)}\, e^{im\pi/\alpha}$, $(m = 1, 2, \cdots, \alpha)$, *there is a value of* $f(z)/z$ *on the boundary of the disc* (1.2) *if and only if there exists a* φ, $0 \leqq \varphi < 2\pi$, *such that* $f(z) \equiv f(z;\varphi)$, *where*

$$(1.3)\qquad f(z;\varphi) = \frac{z}{1} + \frac{\frac{1}{4}e^{i\varphi}z^\alpha}{1} + \frac{\frac{1}{4}z^\alpha}{1} + \cdots + \frac{\frac{1}{4}z^\alpha}{1} + \cdots\,.$$

2. Determination of $U(\alpha)$. For $f(z) \in K_\alpha$ and for a fixed positive integer n put

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$$f_{0,n}(z) = z \, , \tag{2.1}$$
$$f_{p+1,n}(z) = \frac{z}{1 + a_{n-p}z^{\alpha-1}f_{p,n}(z)} \, , \qquad (p = 0, 1, \cdots, n-1) \, ,$$

where the numbers a_j are the coefficients in the C-fraction expansion (1.1) of $f(z)$. It is easily seen that $f_{n,n}(z)$ is the approximant of (1.1) of order $n+1$, and that $f_{p,n}(z) \in K_\alpha$ for each p.

For non-negative integers s, t, and for non-zero numbers z_1, z_2, (2.1) may be used to show that

$$\begin{aligned} & z_1^s z_2^t f_{p+1,n}(z_1) - z_1^t z_2^s f_{p+1,n}(z_2) \\ & = \frac{f_{p+1,n}(z_1)f_{p+1,n}(z_2)}{z_1 z_2}\{z_1^{s+1}z_2^t - z_1^t z_2^{s+1} - a_{n-p}[z_1^{t+\alpha-1}z_2^{s+1}f_{p,n}(z_1) \\ & \quad - z_1^{s+1}z_2^{t+\alpha-1}f_{p,n}(z_2)]\} \, , \qquad (p = 0, 1, \cdots, n-1) \, . \end{aligned} \tag{2.2}$$

This identity plays a fundamental role in the proof of the following theorem.

THEOREM 2.1. *The radius of univalence of K_α is given by*

$$U(2) = 2\sqrt{2}/3 \, , \tag{2.3}$$
$$[U(\alpha)]^\alpha = \left[\frac{6\sqrt{\alpha^2 - 2\alpha + 9} - 2(\alpha + 7)}{(\alpha - 2)^2}\right] , \qquad (\alpha = 1, 3, 4, \cdots) \, .$$

There is no larger region, containing the disc $|z| < U(\alpha)$, in which all functions of K_α are univalent.

Proof. For $f(z) \in K_\alpha$ and for a fixed positive odd integer $n = 2m+1$ it follows from (2.2) that

$$\begin{aligned} & f_{n,n}(z_1) - f_{n,n}(z_2) \\ & = \frac{f_{n,n}(z_1)f_{n,n}(z_2)}{z_1 z_2}\{z_1 - z_2 - a_1[z_1^{\alpha-1}z_2 f_{n-1,n}(z_1) - z_1 z_2^{\alpha-1}f_{n-1,n}(z_2)]\} \, . \end{aligned} \tag{2.4}$$

Repeated application of (2.2) yields

$$\begin{aligned} & a_1[z_1^{\alpha-1}z_2 f_{n-1,n}(z_1) - z_1 z_2^{\alpha-1}f_{n-1,n}(z_2)] \\ & = \sum_{j=1}^{m+1}(z_1 z_2)^{(j-1)\alpha+1}(z_1^{\alpha-1} - z_2^{\alpha-1})\prod_{p=1}^{2j-1} a_p \frac{f_{n-p,n}(z_1)f_{n-p,n}(z_2)}{z_1 z_2} \\ & \quad - \sum_{j=1}^{m}(z_1 z_2)^{j\alpha}(z_1 - z_2)\prod_{p=1}^{2j} a_p \frac{f_{n-p,n}(z_1)f_{n-p,n}(z_2)}{z_1 z_2} \, . \end{aligned} \tag{2.5}$$

For z_1 and z_2 in the disc $|z| < 1$, r can be chosen with $0 < r < 1/2$ such that $|z_i|^\alpha \leqq 4r(1-r)$, $(i = 1, 2)$, and by Theorem 1.1, $|f_{p,n}(z_i)/z_i| \leqq 1/(1-r)$, $(i = 1, 2; p = 0, 1, \cdots, n)$. When the triangle inequality is applied to the right member of (2.5) and the indicated bounds are used, there results

$$\begin{aligned} &| a_1 | \, | z_1^{\alpha-1} z_2 f_{n-1,n}(z_1) - z_1 z_2^{\alpha-1} f_{n-1,n}(z_2) | \\ &\leqq | z - z_2 | \left[\sum_{j=1}^{m+1} (\alpha - 1) \left(\frac{r}{1-r} \right)^{2j-1} + \sum_{j=1}^{m} \left(\frac{r}{1-r} \right)^{2j} \right] \\ &< | z_1 - z_2 | \frac{r}{1-2r} [\alpha - 1 - (\alpha - 2) r] \, . \end{aligned}$$

This inequality and (2.4) give

$$\begin{aligned} &| f_{n,n}(z_1) - f_{n,n}(z_2) | \\ &\geqq \frac{| f_{n,n}(z_1) f_{n,n}(z_2) |}{| z_1 z_2 |} | z_1 - z_2 | \left\{ 1 - \frac{r[\alpha - 1 - (\alpha - 2) r]}{1 - 2r} \right\} . \end{aligned} \tag{2.6}$$

Since Theorem 1.1 shows that neither of the factors $| f_{n,n}(z_i)/z_i |$, $(i=1, 2)$, is zero, it follows from (2.6) that $f_{n,n}(z_1) \neq f_{n,n}(z_2)$ for $z_1 \neq z_2$ if r is such that $1 - 2r > r[\alpha - 1 - (\alpha - 2)r]$. This is equivalent to the condition $r < r_0(\alpha)$ where

$$\begin{aligned} r_0(2) &= 1/3 \\ r_0(\alpha) &= \frac{\alpha + 1 - \sqrt{\alpha^2 - 2\alpha + 9}}{2(\alpha - 2)} \, , \qquad (\alpha = 1, 3, 4, \cdots) \, , \end{aligned}$$

and it is easily seen that $f_{2m+1,2m+1}(z)$ is univalent for $| z |^\alpha < [U(\alpha)]^\alpha = 4r_0(\alpha)[1 - r_0(\alpha)]$.

If the function $f(z)$ has a non-terminating C-fraction (1.1), the univalence of $f(z)$ for $| z | < U(\alpha)$ is an immediate consequence of the fact that $f(z)$ is the uniform limit of its sequence of even approximants, $f_{2m+1,2m+1}(z)$, for $| z | \leqq \rho < 1$. The case where $f(z)$ has a C-fraction expansion (1.1) terminating with an odd number of partial quotients may be reduced to the previously considered case for even approximants by adding a partial quotient, $a_{2m} z^\alpha / 1$ with $a_{2m} = 0$, and noting that $f_{2m-1,2m-1}(z) = f_{2m,2m}(z)$ in this case.

In order to complete the proof that the radius of univalence of K_α is the value $U(\alpha)$ given in (2.3), it suffices to exhibit a function of K_α which is not univalent in $| z | < \rho$ for any $\rho > U(\alpha)$. Such a function is the function $f(z, \pi)$ of (1.3), that is,

$$f(z, \pi) = \frac{2z}{3 - \sqrt{1 + z^\alpha}} \, ,$$

where the branch of the radical with positive real part for $| z | < 1$ is used. This function is not univalent at the points $e^{im\pi/\alpha} U(\alpha)$, $(m = 1, 2, \cdots, \alpha)$, where its derivative vanishes.

The final statement in Theorem 2.1 may be verified by applying to the function $f(z, \pi)$ the observation that, for every real θ, $e^{-i\theta} f(e^{i\theta} z) \in K_\alpha$

whenever $f(z) \in K_\alpha$.

3. A covering theorem. The value region inequality (1.2) can be rewritten as

$$\left|\frac{f(z)}{z} - \frac{4}{2+\rho^\alpha + 2\sqrt{1-\rho^\alpha}}\right| \leqq \frac{2(1-\sqrt{1-\rho^\alpha})}{2+\rho^\alpha+2\sqrt{1-\rho^\alpha}}, \tag{3.1}$$

where $|z| = \rho$ and $f(z) \in K_\alpha$. Thus for $|z| = \rho$ the following inequalities, which provide a means of comparison between K_α and various classes of univalent functions, are obtained:

$$\frac{2}{3-\sqrt{1-\rho^\alpha}} \leqq \Re\left\{\frac{f(z)}{z}\right\} \leqq \frac{2}{1+\sqrt{1-\rho^\alpha}}, \tag{3.2}$$

$$\left|\Im\mathrm{m}\frac{f(z)}{z}\right| \leqq \frac{2(1-\sqrt{1-\rho^\alpha})}{2+\rho^\alpha+2\sqrt{1-\rho^\alpha}}, \tag{3.3}$$

$$\frac{2\rho}{3-\sqrt{1-\rho^\alpha}} \leqq |f(z)| \leqq \frac{2(1-\sqrt{1-\rho^\alpha})}{\rho^{\alpha-1}}, \tag{3.4}$$

$$\left|\arg\frac{f(z)}{z}\right| \leqq \arcsin\frac{1-\sqrt{1-\rho^\alpha}}{2}. \tag{3.5}$$

Each of the inequalities (3.2)-(3.5) is sharp. This fact follows at once from Theorem 1.1 since equality in any one of (3.2)-(3.5) depends on the attainment by $f(z)/z$ of a suitable boundary value for the disc (3.1) or (1.2).

The following theorem is an immediate consequence of (3.4) and Theorem 2.1:

THEOREM 3.1. *If $f(z) \in K_\alpha$, then the image of $|z| < U(\alpha)$ by $w = f(z)$ contains the disc*

$$|w| < \frac{2U(\alpha)}{3-\sqrt{1-[U(\alpha)]^\alpha}}, \tag{3.6}$$

and is contained in the disc

$$|w| < 2\frac{1-\sqrt{1-[U(\alpha)]^\alpha}}{[U(\alpha)]^{\alpha-1}}. \tag{3.7}$$

These results are sharp.

4. A lower bound for $S(\alpha)$. An upper bound for $S(\alpha)$, the starlike radius for the class K_α, is evidently the value $U(\alpha)$ determined in § 2. In this section a lower bound for $S(\alpha)$ is found by determining a number

$\rho_1(\alpha)$ such that every function of K_α is starlike in the disc $|z| < \rho_1(\alpha)$.

LEMMA 4.1. *If* $f(z) \in K_\alpha$ *and* $|a| \leqq 1/4$, *then*

$$w(z) = -\frac{az^{\alpha-1}f(z)}{1 + az^{\alpha-1}f(z)} \tag{4.1}$$

satisfies

$$\left| w - \frac{r^2}{1-r^2} \right| \leqq \frac{r}{1-r^2} \tag{4.2}$$

whenever $|z|^\alpha \leqq 4r(1-r)$, $0 \leqq r \leqq 1/2$.

Proof. The lemma is obvious when $a = 0$. For $0 < |a| \leqq 1/4$, (4.1) yields

$$\frac{f(z)}{z} = \frac{1}{az^\alpha} \cdot \frac{-w(z)}{1 + w(z)} ,$$

and the desired result is easily obtained by applying the inequality $|f(z)/z| \leqq 1/(1-r)$, which is a consequence of Theorem 1.1.

LEMMA 4.2. *If* α *is a positive integer and if for fixed* r, $0<r<1/2$, c *and* d *are numbers such that*

$$0 \leqq c \leqq \frac{1 + (\alpha - 2)r^2}{1 - 2r^2} , \quad 0 < d = \frac{1 + (\alpha - 2)r}{1 - 2r} - c , \tag{4.3}$$

then $\sigma = 1$ *satisfies*

$$|\sigma - c| \leqq d . \tag{4.4}$$

Moreover, if w *is a parameter satisfying* (4.2) *and if* σ_0 *satisfies* (4.4), *then* σ_1 *satisfies* (4.4) *where*

$$\sigma_1 = 1 + w(\sigma_0 + \alpha - 1) . \tag{4.5}$$

Proof. It is obvious that $1 - c \leqq d$ holds for all r, $0 < r < 1/2$, and that $-d \leqq 1 - c$ holds provided

$$c \leqq \frac{2 + (\alpha - 4)r}{2(1 - 2r)} .$$

The fact that $\sigma = 1$ satisfies (4.4) may be verified by noting that the upper bound of c in this last inequality exceeds the upper bound on c in (4.3) for all r, $0 < r < 1/2$.

The proof of the second statement is obtained by using (4.2), (4.3), (4.4), (4.5), and the triangle inequality to show that

$$\begin{aligned}|\sigma_1 - c| &\leqq \left|1 - c + \frac{(c+\alpha-1)r^2}{1-r^2}\right| \\ &\quad + (c+\alpha-1)\left|w - \frac{r^2}{1-r^2}\right| + |w||\sigma_0 - c| \\ &\leqq \frac{1+(\alpha-2)r^2-(1-2r^2)c}{1-r^2} + \frac{(c+\alpha-1)r}{1-r^2} + \frac{rd}{1-r^2} = d\,.\end{aligned}$$

LEMMA 4.3. *If* (4.3) *holds for* $0 < r < 1/2$, *there is a value of* c *satisfying* $c \geqq d$ *if and only if* $0 < r \leqq r_1(\alpha)$, *where* $r_1(\alpha)$ *is the smallest positive root of*

$$1 - (\alpha+2)r + 2(\alpha-1)r^2 - 2(\alpha-2)r^3 = 0\,. \tag{4.6}$$

Proof. By (4.3) the inequality $c \geqq d$ holds if and only if

$$\frac{1+(\alpha-2)r^2}{1-2r^2} \geqq \frac{1+(\alpha-2)r}{2(1-2r)}\,,$$

which is equivalent to the statement that the left member of (4.6) is nonnegative. Clearly $r_1(\alpha) < 1/2$.

THEOREM 4.1. *If* $f(z) \in K_\alpha$ *and* c, d *satisfy* (4.3), *where* $|z|^\alpha = \rho^\alpha \leqq 4r(1-r)$, *then*

$$\left|z\frac{f'(z)}{f(z)} - c\right| \leqq d\,. \tag{4.7}$$

Proof. For the functions $f_{p,n}(z)$ of (2.1) put

$$\sigma_{p,n} = z\frac{f'_{p,n}}{f_{p,n}}\,, \quad w_{p,n} = -\frac{a_{n-p}z^{\alpha-1}f_{p+1,n}}{1 + a_{n-p}z^{\alpha-1}f_{p+1,n}}\,,$$

and note by differentiation that $\sigma_{p+1,n} = 1 + w_{p,n}(\sigma_{p,n} + \alpha - 1)$. For $|z| = \rho$ inductive application of Lemmas 4.1 and 4.2 shows that (4.7) holds for $f_{n,n}$, and the validity of (4.7) in this case for $|z| \leqq \rho$ follows from the maximum property for harmonic functions. Inasmuch as $f_{n,n}$ is the $(n+1)$th approximant of (1.1) the theorem holds for functions of K_α having terminating C-fraction expansions. The validity of the theorem in the case of non-terminating C-fractions (1.1) is an immediate consequence of the uniform convergence of $f_{n,n}$ to f on any closed subset of $|z| < 1$.

THEOREM 4.2. *The starlike radius of* K_α *satisfies* $S(\alpha) \geqq \rho_1(\alpha)$ *where*

$[\rho_1(\alpha)]^\alpha = 4r_1(\alpha)[1 - r_1(\alpha)]$ *and where* $r_1(\alpha)$ *is the smallest positive root of* (4.6).

Proof. For $r \leqq r_1(\alpha)$ Lemma 4.3 shows that Theorem 4.1 can be applied to any function $f(z) \in K_\alpha$ with $c \geqq d$, and hence that

$$\Re\, z\frac{f'(z)}{f(z)} \geqq 0, \quad |z| \leqq \rho_1(\alpha)\ .$$

Since this inequality insures that $f(z)$ is starlike for $|z| < \rho_1(\alpha)$ the proof is complete.

In particular, $r_1(1) = (\sqrt{3} - 1)/2$ and $S(1) \geqq 4\sqrt{3} - 6$ which improves the lower bound of 8/9 obtained for $S(1)$ in [3].

5. **A lower bound for** $C(\alpha)$. It is clear that $S(\alpha)$ and $U(\alpha)$ are upper bounds for $C(\alpha)$, the radius of convexity of K_α. In this section a lower bound for $C(\alpha)$ is found by determining a number $\rho_2(\alpha)$ such that every function of K_α is convex for $|z| < \rho_2(\alpha)$.

LEMMA 5.1. *Let* α *denote a positive integer and let* $r_2(\alpha)$ *be the smallest positive root of the equation:*

$$(5.1) \qquad 1 - (\alpha^2 + 2\alpha + 6)r + 6(\alpha^2 + \alpha + 2)r^2 - 4(3\alpha^2 + 2)r^3 + 12(\alpha - 1)\alpha r^4 - 4\alpha(\alpha - 2)r^5 = 0\ .$$

If for fixed r, $0 < r \leqq r_2(\alpha)$, σ_0 *and* σ_1 *are numbers which satisfy*

$$(5.2) \qquad |\sigma_0 - c| \leqq d, \ |\sigma_1 - c| \leqq d\ ,$$

where

$$(5.3) \qquad \frac{1 + (\alpha - 2)r}{2(1 - 2r)} \leqq c \leqq \frac{1 + (\alpha - 2)r^2}{1 - 2r^2}\ , \quad d = \frac{1 + (\alpha - 2)r}{1 - 2r} - c\ ,$$

and if

$$(5.4) \qquad \gamma_1 = 2(\sigma_1 - 1) + \frac{\sigma_1 - 1}{\sigma_1}\left[\gamma_0\frac{\sigma_0}{\sigma_0 + \alpha - 1} + (\alpha - 1)\frac{2\sigma_0 + \alpha - 2}{\sigma_0 + \alpha - 1}\right],$$

then $|\gamma_0| \leqq 1$ *implies* $|\gamma_1| \leqq 1$.

Proof. For $0 < r < r_1(\alpha)$, where $r_1(\alpha)$ is as determined in Theorem 4.2, $0 < d < c$ and

$$c^2 - d^2 - c \leqq -\frac{\alpha r^2[(\alpha - 1) - 2(\alpha - 2)r + 2(\alpha - 2)r^2]}{(1 - 2r)^2(1 - 2r^2)} \leqq 0\ .$$

Thus by (5.2)

$$\left|\frac{\sigma_1 - 1}{\sigma_1} - \frac{c^2 - d^2 - c}{c^2 - d^2}\right| \leqq \frac{d}{c^2 - d^2}$$

and it follows that

$$\left|\frac{\sigma_1 - 1}{\sigma_1}\right| \leqq \frac{1}{c - d} - 1 .$$

Similarly, (5.2) can be used to show that

$$\left|\frac{\sigma_0}{\sigma_0 + \alpha - 1}\right| \leqq \frac{c + d}{c + d + \alpha - 1} ,$$

$$\left|(\alpha - 1)\frac{2\sigma_0 + \alpha - 2}{\sigma_0 + \alpha - 1}\right| \leqq (\alpha - 1)\frac{2(c + d) + \alpha - 2}{c + d + \alpha - 1} .$$

For $|\gamma_0| \leqq 1$ application to (5.4) of the triangle inequality, (5.2) and the bounds determined above lead to the inequality

$$(5.5)\quad |\gamma_1| \leqq 2(c + d - 1) + \left[\frac{1}{c - d} - 1\right]\frac{(2\alpha - 1)(c + d) + (\alpha - 1)(\alpha - 2)}{c + d + \alpha - 1} .$$

The desired inequality, $|\gamma_1| \leqq 1$, will hold for those values of $r < r_1(\alpha)$ for which the right member of (5.5) does not exceed 1, or equivalently, for which

$$(5.6)\quad c - d \geqq \frac{(2\alpha - 1)(c + d) + (\alpha - 1)(\alpha - 2)}{(2\alpha - 1)(c + d) + (\alpha - 1)(\alpha - 2) + [3 - 2(c + d)][c + d + \alpha - 1]} = D .$$

Since $2c = (c + d) + (c - d)$, (5.3) shows that the existence of a value of c satisfying (5.6) is insured for all $r < r_1(\alpha)$ for which

$$(5.7)\quad 2\frac{1 + (\alpha - 2)r^2}{1 - 2r^2} \geqq (c + d) + D .$$

This last inequality is equivalent to the requirement that the polynomial in the left member of (5.1) be non-negative.

The proof of the lemma will be completed by establishing the existence of a smallest positive zero, $r_2(\alpha)$ of (5.1) for which $r_2(\alpha) < r_1(\alpha)$. Since the equation (4.7) determining $r_1(\alpha)$ is equivalent to

$$2\frac{1 + (\alpha - 2)r^2}{1 - 2r^2} = c + d ,$$

and since $D > 0$ for $r = r_1(\alpha)$, it follows that (5.7) fails to hold for $r = r_1(\alpha)$. The desired conclusion about $r_2(\alpha)$ is then easily obtained by noting that (5.7) holds with strict inequality for $r = 0$.

THOREM 5.1. *The radius of convexity of K_α satisfies*

$$[C(\alpha)]^\alpha \geqq 4r_2(\alpha)[1 - r_2(\alpha)] = [\rho_2(\alpha)]^\alpha \tag{5.8}$$

where $r_2(\alpha)$ is the smallest positive root of (5.1)

Proof. For the functions $f_{p,n}(z)$ of (2.1) put

$$\sigma_{p,n} = z\frac{f'_{p,n}}{f_{p,n}}, \qquad \gamma_{p,n} = z\frac{f''_{q,n}}{f'_{p,n}}.$$

It is easily verified from (2.1) that

$$\gamma_{p+1} = 2(\sigma_{p+1} - 1) + \frac{\sigma_{p+1} - 1}{\sigma_{p+1}}\left[\frac{\gamma_p\sigma_p}{\sigma_p + \alpha - 1} + (\alpha - 1)\frac{2\sigma_p + \alpha - 2}{\sigma_p + \alpha - 1}\right]$$

where the subscript n has been omitted. Theorem 4.1 and the fact that $\gamma_{0,n} = 0$ show that the hypotheses of Lemma 5.1 are satisfied, and inductive application of the lemma yields $|\gamma_{n,n}| \leqq 1$. It follows that

$$\Re e[1 + \gamma_{n,n}] \geqq 0\ , \ |z| \leqq \rho_2(\alpha)\ ,$$

which insures the convexity of the $(n + 1)$th approximant of any C-fraction (1.1) for $|z| < \rho_2(\alpha)$, and the proof of the theorem may be completed, as in Theorem 4.1, by reference to uniform convergence.

It is found that $\rho_2(1) > .641$. An upper bound for $C(\alpha)$ can be obtained by finding for the function $f(z, \pi)$ of (1.3) the zeros of $zf''(z, \pi) + f'(z, \pi)$ with smallest modulus. For $\alpha = 1$ this smallest modulus is approximately .707.

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