SOME CLASSES OF EQUIVALENT GAUSSIAN PROCESSES ON AN INTERVAL

JACOB FELDMAN

1. Introduction. Let T be an index set, R, S real-valued nonnegative definite functions of two variables in T, and m, n real-valued functions on T. Let Ω be the set of all real-valued functions on T, and $\mathscr{S}$ the Borel field of cylinder sets. There are then unique measures μ, ν on $\mathscr{S}$ such that the functions x_t on Ω defined by $x_t(\omega) = \omega(t)$ form Gaussian stochastic processes, with means respectively m and n, and covariances respectively R and S. It is shown in [2] that μ and ν are either mutually absolutely continuous or totally singular, and a necessary and sufficient condition for equivalence is given.

Suppose now that T is a subset of the real line, and $R(s, t) = \mathscr{r}(s-t)$, $S(s, t) = \mathscr{s}(s-t)$, where $\mathscr{r}$ and $\mathscr{s}$ are continuous nonnegative-definite functions, and hence can be written as inverse Fourier transforms of finite measures $d\rho, d\sigma$. Thus, using respectively the measures μ and ν on $\Omega, x_t - m(t)$ and $x_t - n(t)$ are the restrictions to T of stationary Gaussian processes on the real line. For simplicity, only the case $m = n = 0$ will be considered.

When T is the entire real line, then it is easy to see, by looking at $d\rho$ and $d\sigma$, exactly when $\mu \sim \nu$, as is essentially known (see [3]). The precise conditions are:

a. ρ and σ must have *identical* non-atomic parts.

b. Their points of positive mass be the same, and if the masses are a_i and b_i at x_i, then $\sum\{(a_i/b_i) - 1\}^2$ must be finite.

Now suppose T is a finite interval. The problem of determining from knowledge of ρ and σ whether μ and ν are equivalent becomes much more difficult. We here discuss only a certain class of cases. Because of stationarity, one need only consider an interval symmetric about 0. Continuity of $\mathscr{r}$ and $\mathscr{s}$ implies that the Gaussian process is continuous with probability one at any given point, so that it makes no difference whether the interval is open or closed. There is no essential loss of generality, then, in considering only the closed interval $[-\pi, \pi]$. The following facts will then be proven:

THEOREM. *Let $d\rho(x) = \{dx/(1 + x^2)^u\}$, where u is an integer $\geqq 1$, and let σ be some other finite nonnegative measure on the real line. Write $\tau = \sigma - \rho$. The following conditions are necessary and sufficient that the Gaussian processes induced on $[-\pi, \pi]$ by the Fourier trans-*

Received December 21, 1959, Research partly supported by Contract NONR-222 (60).

forms of ρ and σ have equivalent measures on path space:

(a) *if k_n is a sequence of C_∞ functions with support in $]-\pi, \pi[$ and K_n is the Fourier transform of k_n, then $\int |K_n|^2 d\sigma \to 0$ implies $\int |K_n|^2 d\rho \to 0$.*

(b) *The Fourier transform (in the sense of Schwartz distributions) of $(1 + x^2)^u d\tau(x)$ agrees on $]-2\pi, 2\pi[$ with a function ψ such that*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} | \psi(s-t) |^2 ds\, dt < \infty .$$

REMARK 1. It will be seen that sufficiency still holds if (a) is weakened to:

(a′) *$\int |K_n|^2 d\sigma \to 0$ and $K_n \to K$ in $\mathscr{L}_2(\rho)$ implies that $K = 0$ on some set of positive ρ-measure.*

REMARK 2. As a consequence of Remark 1, it is clear that if σ has a component which is absolutely continuous with respect to ρ, then Condition (a) automatically satisfied.

Retaining the notation of the theorem:

COROLLARY 1. *If $d\sigma = \Phi d\rho$, where Φ is a function such that $\Phi - 1$ is a finite linear combination of functions in various $L_a(-\infty, \infty)$ classes, $1 \leqq a \leqq 2$, then the Gaussian processes induced by ρ and σ have equivalent measures on path space.*

One direction of the following corollary was proven by *D.* Slepian in [5], using techniques of G. Baxter in [1]:

COROLLARY 2. *If A_j and B_j are polynomials, with degrees respectively a_j and b_j, $j = 1, 2$, and $b_j > a_j$, then the Gaussian processes whose spectral measures are $| A_j(x)/B_j(x) |^2\, dx$ have equivalent measures on path space if and only if*

(a) $b_1 - a_1 = b_2 - a_2$

(b) *the ratio of the leading coefficients of A_1 and B_1 has the same absolute value as the ratio of the leading coefficients of A_2 and B_2.*

The author wishes to thank J. F. Trèves for several useful discussions about distributions.

2. Some preliminaries on functions of exponential type. First, some notation. Functions will be complex-valued functions of a real variable, unless otherwise stated. $\hat{F}$ will mean the Fourier transform of F (in various degrees of generalization, depending on context), and $\check{F}$ the conjugate Fourier transform. sup (f) will mean the points where $f \neq 0$. $\mathscr{E}_a = \{F \mid F$ extends to an entire function of exponential type $\leqq a\pi\}$. $\mathscr{H}_a = \mathscr{E}_a \cap \mathscr{L}_2(-\infty, \infty)$, or, by the Payley-Wiener theorem,

$$= \{\hat{f} \mid f \in \mathscr{L}_2(-\infty, \infty), \sup(f) \subset [-a\pi, a\pi]\} .$$

$$\hat{\mathscr{D}}_a = \{ f \mid f \in \mathscr{C}_\infty, \overline{\sup(f)} \subset]-a\pi, a\pi[\}, \quad \mathscr{D}_a = \{\check{f} \mid f \in \hat{\mathscr{D}}_a\} .$$

u will be a fixed integer $\geqq 1$, and $p(x) = (i + x)^u$. ρ is the measure $d\rho(x) = \{1/|p(x)|^2\}dx$. $\mathscr{K}$ will denote the completion of $\mathscr{D}_1$ in the inner product $\langle F, G\rangle = \int F\bar{G}d\rho$.

Naturally, $\mathscr{K}$ really consists of equivalence classes of functions; but it will turn out that there is a continuous, in fact entire, member in each class. H_1 will denote a fixed function of $\mathscr{D}_1$ such that $h_1 = \hat{H}_1$ is nonnegative and has integral 1. For $a > 0$, $h_a(s)$ will be $(1/a)h_1(s/a)$, $H_a(x) = H_1(ax)$, so that $h_a = \hat{H}_a$, and $H_a \in \mathscr{D}_a$. Then H_a vanishes faster than any polynomial, $|H_a(x)| \leqq 1$ for all x, and $\lim_{a\to 0} H_a(x) = 1$ uniformly on any finite interval.

LEMMA 1. *If* $F \in \mathscr{E}_1$ *and* $\int |F|^2 d\rho < \infty$, *then* $F \in \mathscr{K}$.

Proof. If $(1/2) < c < 1$, then

$$\left(\int |F(cx) - F(x)|^2 d\rho(x)\right)^{1/2} \leqq \left(\int_{-b}^{b} |F(cx) - F(x)|^2 d\rho(x)\right)^{1/2} + \left(\int_{|x|>b} |F(cx)|^2 d\rho(x)\right)^{1/2} + \left(\int_{|x|>b} |F(x)|^2 d\rho(x)\right)^{1/2}.$$

Now,

$$\int_{|x|>b} |F(cx)|^2 d\rho(x) = \frac{1}{c}\int_{|x|>bc} |F(x)|^2 \frac{1}{\left|p\left(\frac{x}{c}\right)^2\right|} dx$$

$$\leqq 2\int_{|x|<(b/2)} |F(x)|^2 \frac{1}{|p(x)|^2} dx .$$

Choosing b large, and then choosing c close enough to 1 to make $|F(cx)-F(x)|$ small on $[-b, b]$, we see that it suffices to show that the function $G: x \to F(cx)$ is in $\mathscr{K}$. Notice that $G \in \mathscr{E}_1$, as $c < 1$.

H_aG is square-integrable, since H_a vanishes faster than $(1/|p|^2)$. So H_aG is in $\mathscr{H}_{a+c}$, its Fourier transform being some g' in $\mathscr{L}_2(-\infty, \infty)$ with support in $[-(a+c)\pi, (a+c)\pi]$. Thus $h_a * g' \in \mathscr{D}_{2a+c}$, and $H_a^2 G \in \mathscr{D}_{2a+c}$. Choosing a small causes $H_a^2 G$ to be in $\mathscr{D}_1$, and simultaneously causes $\int |H_a^2 G - G|^2 d\rho$ to get small. This proves the lemma.

Let $\mathscr{H} = \{pF \mid F \in \mathscr{H}_1\}$, and $\mathscr{D} = \{pF \mid F \in \mathscr{D}_1\}$. Lemma 1 tells us $\mathscr{H} \subset \mathscr{K}$.

LEMMA 2. $\mathscr{H}$ *is precisely the closure of* $\mathscr{D}$ *in* $\mathscr{K}$.

Proof. First, we see that $\mathscr{H}$ is closed. If $F_n \in \mathscr{H}_1$ and

$$\int | pF_n - G |^2 d\rho \to 0, \text{ then } \int | F_n(x) - F_m(x) |^2 dx \to 0 .$$

Since $\mathscr{H}_1$ is complete, there is some $F \in \mathscr{H}_1$ with $\int | F_n(x) - F(x) |^2 dx \to 0$. So some subsequence of the pF_n converges almost everywhere to pF. Thus $pF = G$ almost everywhere.

To approximate elements pF in $\mathscr{H}$ by elements in $\mathscr{D}$, just approximate F in $\mathscr{L}_2(-\infty, \infty)$ by elements in $\mathscr{D}_1$, using the technique of Lemma 1.

LEMMA 3. *$\mathscr{K} \ominus \mathscr{H}$ is precisely the finite-dimensional space $\mathscr{L}$ of functions of the form $x \to e^{ix\pi} q(i - x)$, where q is a polynomial of degree $\leqq u - 1$.*

Proof. Suppose $F \in \mathscr{K} \ominus \mathscr{H}$. Then $\int F \overline{pG} d\rho = 0$ for all G in $\mathscr{D}_1$, i.e. $\int \{F(x)/p(x)\} \overline{G(x)}\, dx = 0$ for all G in $\mathscr{D}_1$. Now, (F/p) is in $\mathscr{L}_2(-\infty, \infty)$, so it has a Fourier transform k which is likewise square-integrable, and, by Plancherel's theorem, $\int k(s) \overline{g(s)} ds = 0$ for all g in $\hat{\mathscr{D}}_1$. So k vanishes in $]-\pi, \pi[$.

Since $F \in \mathscr{K}$, F can be approximated in $\mathscr{K}$ by functions F_n in $\mathscr{D}_1$. Each F_n is in $\mathscr{D}_{a_n}$ for some $a_n < 1$, since $\overline{\sup(F_n)} \subset]-\pi, \pi[$, and hence $\subset]-a_n\pi, a_n\pi[$ for some $a_n < 1$. Let k_n be the Fourier transform of F_n/p. Then $k_n \to k$ in $\mathscr{L}_2(-\infty, \infty)$, and k_n is in the domain of the $\mathscr{L}_2$-differential operator $p(-iD) = i^u(I - D)^u$. So $p(-iD)k_n = f_n$, where f_n is the Fourier transform of F_n. Since f_n vanishes outside some $[-a_n\pi, a_n\pi]$, $a_n < 1$, k_n must be of the form $\sum_j a_i^{(n)} s^j e^s$ in $]-\infty, -\pi[$ and $\sum_j b_j^{(n)} s^j e^s$ in $]\pi, \infty[$, where j ranges between 0 and $u - 1$. Since k_n is in $\mathscr{L}_2(-\infty, \infty)$, the $b_j^{(n)}$ are zero, and, letting φ be the indicator of $]-\infty, -\pi[$, we get $\varphi k_n = \varphi \sum_j a_j^{(n)} s^j e^s$. This converges in $\mathscr{L}_2(-\infty, \infty)$, so the limit is of the form $\varphi \sum_j a_j s^j e^s$. Then $k_n \to 0$ in $]\pi, \infty[$, 0 in $[-\pi, \pi]$, and $\sum_j a_j s^j e^s$ in $]-\infty, -\pi[$, so $k = \varphi \sum_j a_j s^j e^s$. F/p is then a linear combination of terms like $\int_{-\infty}^{-\pi} e^{-ixs} s^j e^s ds$, $0 \leqq j \leqq u - 1$, which is a linear combination of terms like $e^{ix\pi}(i + x)^{-j}$, $1 \leqq j \leqq u$. Multiplying by p gives the result.

Combining information from lemmas 1, 2, 3 we get a description of $\mathscr{K}$:

PROPOSITION. $\mathscr{K}$ is the orthogonal direct sum of $\mathscr{H}$ and $\mathscr{L}$.

LEMMA 4. $\mathscr{D} = \mathscr{H} \cap \mathscr{D}_1$.

Proof. $\mathscr{D} \subset \mathscr{H}$, by definition, since $\mathscr{D}_1 \subset \mathscr{H}_1$. Also $\mathscr{D} \subset \mathscr{D}_1$, since $\mathscr{D}_1$ is closed under multiplication by polynomials (because $\hat{\mathscr{D}}_1$ is closed under differentiation). So $\mathscr{D} \subset \mathscr{H} \cap \mathscr{D}_1$, and it remains to show $\mathscr{D} \supset \mathscr{H} \cap \mathscr{D}_1$.

Suppose $G \in \mathscr{H}$. Then G is a $\langle\, , \rangle$ limit of elements G_n in $\mathscr{D}$, by Lemma 2. G_n then has the form pF_n, F_n in $\mathscr{D}_1$. Thus F_n is an $\mathscr{L}_2(-\infty, \infty)$ Cauchy sequence, hence has a limit F. Then $pF = G$.

Suppose G is also in $\hat{\mathscr{D}}_1$. Then G is infinitely differentiable. Since $\hat{G} = p\hat{F} = p(-iD)\hat{F}$, we conclude that $\hat{F}$ is infinitely differentiable. Now it must be shown that $\hat{F}$ vanishes outsides some interval $[-a\pi, a\pi]$, $0 < a < 1$. But $\hat{G} = p(-iD)\hat{F}$ vanishes outside such on interval, so $\hat{F}$ is analytic outside $[-a\pi, a\pi]$. Also, $\hat{F}$ vanishes outside $[-\pi, \pi]$, since each $\hat{F}_n$ has support in $]-\pi, \pi[$. Therefore, $\hat{F}$ vanishes outside $[-a\pi, a\pi]$. So $\hat{F}$ is in $\mathscr{D}_1$, and F is in $\mathscr{D}_1$.

LEMMA 5. *$\mathscr{D}_1/\mathscr{D}$ is finite dimensional.*

Proof. $\mathscr{D}_1/\mathscr{D} = \mathscr{D}_1/\mathscr{D}_1 \cap \mathscr{H} \approx (\mathscr{D}_1 + \mathscr{H})/\mathscr{H} \subset \mathscr{K}/\mathscr{H} \approx \mathscr{L}$.

3. **Proof of theorem.** In [2] it is shown that a necessary and sufficient condition for equivalence of μ and ν is that there be an *equivalence operator* from the closed linear span of $\{x_t \mid t \in T\}$ in $\mathscr{L}_2(\mu)$ to their closed linear span in $\mathscr{L}_2(\nu)$, sending the μ-equivalence class of x_t to the ν-equivalence class of x_t for each $t \in T$. (An equivalence operator, as defined in [2], is a linear homeomorphism H between two Hilbert spaces such that $I - H^*H$ is Hilbert Schmidt). Actually, we shall want the condition in *complex* $\mathscr{L}_2$, while the proof in [2] is for real $\mathscr{L}_2$; however, the transition from the one to the other is immediate.

Under this condition, H would map $\int_{-\pi}^{\pi} f(x_t)dt$ as an $\mathscr{L}_2(\mu)$— valued integral to $\int_{-\pi}^{\pi} f(t)x_t dt$ as an $\mathscr{L}_2(\nu)$-valued integral, for each $f \in \hat{\mathscr{D}}_1$; and conversely, if H had this effect on all such $\int_{-\pi}^{\pi} f(t)x_t dt$, then by choosing a sequence of f approximating a delta function, one could verify that H sent the equivalence class of x_t in $\mathscr{L}_2(\mu)$ to the equivalence class of x_t in $\mathscr{L}_2(\nu)$. Therefore, putting inner products $(\, ,)$ and $(\, ,)^\cdot$ on $\hat{\mathscr{D}}_1$ by the rules

$$(f, g) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \gamma(s - t)f(s)\overline{g(t)}dsdt ,$$

$$(f, g)^\cdot = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \rho(s - t)f(s)\overline{g(t)}dsdt ,$$

and noting that $(f, g) = \int\left(\int_{-\pi}^{\pi} f(s)x_s ds\right)\left(\int_{-\pi}^{\pi} g(t)x_t dt\right)d\mu$ and

$$(f, g)^\cdot = \int\left(\int_{-\pi}^{\pi} f(s)x_s ds\right)\left(\int_{-\pi}^{\pi} g(t)x_t dt\right)d\nu,$$

it follows that a necessary and sufficient condition for the equivalence of μ and ν is the existence of an equivalence operator from the $(\, ,)$ completion of $\hat{\mathscr{D}}_1$ to its $(\ ,\ )^{\cdot}$ completion, and sending the $(\ ,\ )$-equivalence class of f to its $(\ ,\ )^{\cdot}$-equivalence class.

Now let $\langle F, G\rangle^{\cdot} = \int F\bar{G}d\sigma$, where F and G are in $\mathscr{D}_1$ (and hence continuous and bounded, so that the integral exists). Let $\dot{\mathscr{K}}$ be the closure of $\mathscr{D}_1$ in $\mathscr{L}_2(\sigma)$. Let J be the map assigning to F in $\mathscr{D}_1$ its equivalence class in $\dot{\mathscr{K}}$. Since $\langle F, G\rangle = (\hat{F}, \hat{G})$, and $\langle F, G\rangle^{\cdot} = (\hat{F}, \hat{G})^{\cdot}$, the necessary and sufficient condition for the equivalence of μ and ν in the theorem is that J be the restriction to $\mathscr{D}_1$ of an equivalence map from $\mathscr{K}$ to $\dot{\mathscr{K}}$.

To prove sufficiency of the conditions in the theorem, suppose first that $|p(x)|^2\ d\tau(x)$ has a generalized Fourier transform (see [4]) which agrees on $]-2\pi, 2\pi[$ with a function ψ such that $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\psi(s-t)|^2\, dsdt = a^2 < \infty$. We extend ψ by making it 0 outside $]-2\pi, 2\pi[$.

LEMMA 6. *If* $F \in \mathscr{D}$, *then* $\langle F, F\rangle^{\cdot} \leqq (1 + a)\langle F, F\rangle$.

Proof. Write $F = pG$, $G \in \mathscr{D}_1$. Then $\int |F|^2 d\sigma = \int |F|^2 d\rho + \int |G|^2 |p|^2 d\tau$. Now, $\hat{G}$ is in $\hat{\mathscr{D}}_1$, so $\hat{G} * \hat{G}$ is infinitely differentiable with support in $]-2\pi, 2\pi[$. Then, by Schwartz's definition of generalized Fourier transform, we get $\int |G|^2 |p|^2 d\tau = \int_{-2\pi}^{2\pi} \hat{G}*\hat{G}(s)\psi(s)\, ds = \int_{-2\pi}^{2\pi}\int_{a(s)}^{b(s)} \hat{G}(s-t)\bar{\hat{G}}(1-t)$ $\psi(s)dtds$, where $a(s) = \max(-\pi, s-\pi)$ and $b(s) = \min(\pi, s+\pi)$. Letting $s - t = s'$, and $t = -t'$ gives $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \hat{G}(s')\hat{G}(t')\psi(s'-t')ds'dt'$, whose absolute value, by the Schwartz inequality, is

$$\leqq \left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\hat{G}(s)\overline{\hat{G}(t)}|^2\, dsdt \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\psi(s-t)|^2 dsdt\right]^{1/2}$$

$$= \left(\int_{-\pi}^{\pi} |\hat{G}(s)|^2\, ds\right)a = \left(\int |F|^2 d\rho\right)a\,.$$

Pick a complete orthonormal set (c.o.n.s.) $f_1, f_2, \cdots$ for $\mathscr{L}_2(-\pi, \pi)$ out of the dense subset $\hat{\mathscr{D}}_1$. Let $F_n = \check{f}_n$, and $G_n = pF_n$. Then the G_n form a c.o.n.s. for $\mathscr{H}$ (in $\langle\ ,\ \rangle$) consisting of elements of $\mathscr{D}$, because the F_n are a c.o.n.s. for $\mathscr{H}_1$ consisting of elements of $\mathscr{D}_1$.

LEMMA 7. $\sum_{n,m=1}^{\infty} |\langle G_n, G_m\rangle - \langle G_n, G_m\rangle^{\cdot}|^2 = a^2$.

Proof. $\int G_n(x)\overline{G_m(x)}d\tau(x) = \int_{-2\pi}^{2\pi} \hat{F}_n * \hat{F}_m(s)\psi(s)ds$. By using a change of variable as in the previous lemma, this equals $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_n(s)\overline{f_m(t)}\psi(s-t)dsdt$. But the functions $(s, t) \to \overline{(f_n s)} f_m(t)$ form a c.o.n.s. in $\mathscr{L}_2([-\pi, \pi] \times [-\pi, \pi])$, so that $\sum_{n,m=1}^{\infty} \left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_n(s)\overline{f_m(t)}\psi(s-t)dsdt\right|^2$ is exactly $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left|\psi(s-t)\right|^2 dsdt$.

Now consider the map J from $\mathscr{D}_1$ to $\dot{\mathscr{K}}$. *Lemma* 6 implies that its restriction to $\mathscr{D}$ is bounded, and, since $\mathscr{D}_1/\mathscr{D}$ is finite-dimensional (Lemma 5), J is bounded as an operator from $\mathscr{D}_1$ to $\dot{\mathscr{K}}$ (a finite-dimensional extension of a bounded operator is bounded, as is readily seen). So J extends uniquely to a bounded operator A from $\mathscr{K}$ to $\dot{\mathscr{K}}$.

LEMMA 8. *$I—A^*A$ is a Hilbert-Schmidt operator.*

Proof. Complete the o.n.s. $G_1, G_2, \cdots$ by adding to it a c.o.n.s. $G_0, G_{-1}, \cdots, G_{1-u}$ in $\mathscr{L}$. Then, letting $k = u - 1$,

$$\begin{aligned}
&\textstyle\sum_{n,m=-k}^{\infty} |\langle (I - A^*A)G_n, G_m\rangle|^2 \\
&\quad = \textstyle\sum_{n,m=1}^{\infty} |\langle (I - A^*A)G_n, G_m\rangle|^2 \\
&\qquad + \textstyle\sum_{n=-k}^{0}\sum_{m=-k}^{\infty} |\langle (I - A^*A)G_n, G_m\rangle|^2 \\
&\qquad + \textstyle\sum_{n=-k}^{\infty}\sum_{m=-k}^{0} |\langle G_n, (I - A^*A)G_m\rangle|^2 \\
&\quad = \textstyle\sum_{n,m=1}^{\infty} |\langle G_n, G_m\rangle - \langle AG_n, AG_m\rangle^{\cdot}|^2 \\
&\qquad + 2\textstyle\sum_{n=-k}^{0} |\langle I - A^*A)G_n, (I - A^*A)G_n\rangle|^2 ,
\end{aligned}$$

using Parseval's equality. But $\langle AG_n, AG_m\rangle^{\cdot} = \langle G_n, G_m\rangle^{\cdot}$ for $n, m > 0$, since such G_n are in $\mathscr{D}_1$, so that the sum is exactly

$$a^2 + 2\textstyle\sum_{n=-k}^{0}\langle (I - A^*A)G_n, (I - A^*A)G_n\rangle .$$

In order to complete the proof, it must be shown that A is a homeomorphism from $\mathscr{K}$ onto $\dot{\mathscr{K}}$. Since $I - A^*A$ is completely continuous, it will suffice to show

(1) that the range of A is dense in $\dot{\mathscr{K}}$.

(2) that A sends no nonzero element to zero.

(1) is clear, since the range of A contains the range of J, which is dense by the very definition of $\dot{\mathscr{K}}$.

We now make use of (a), or rather of the weaker (a′), to prove (2). Suppose, in fact, that $A(K)$ is zero in $\dot{\mathscr{K}}$ for some K in $\mathscr{K}$. Let K_n be a sequence of members of $\mathscr{D}_1$ converging to K in $\mathscr{K}$. Then K_n converges to zero in $\dot{K}$, since $A(K_n) = J(K_n)$. Then, by (a′), $K = 0$ on a set of positive ρ measure. But the Proposition of the previous section tells us that K is analytic. Thus $K = 0$.

To show the necessity of condition (a), suppose J has an extension to an equivalence operator from $\mathscr{K}$ to $\dot{\mathscr{K}}$, which we call A. Then (a) is immediate from the fact that A is continuously invertible.

Since $I - A^*A$ is an equivalence operator, $\sum_{n,m=1}^{\infty} |\langle G_n, G_m\rangle - \langle G_n G_m\rangle^{\cdot}|^2 < \infty$, where $G_1, G_2, \cdots$ is the c.o.n.s. in $\mathscr{D}$ for $\mathscr{H}$ previously constructed. Define an operator Z on $\mathscr{L}_2([-\pi, \pi]\times[-\pi, \pi])$ as follows: let $f_{n,m}(s, t) = f_n(s)f_m(t)$, where $G_n = p\check{f}_n$. For $Q = \sum_{n,m} a_{n,m} f_{n,m}$, Let $Z(Q) = \sum_{n,m} a_{n,m}(\langle G_n, G_m\rangle - \langle G_n, G_m\rangle^{\cdot})$. Then

$$|Z(Q)|^2 \leqq \textstyle\sum_{n,m} |a_{n,m}|^2 \sum_{n,m} |\langle G_n, G_m\rangle - \langle G_n, G_m\rangle^{\cdot}|^2 .$$

So $Z(Q)$ has the form $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} Q(s, t)\,\Psi(s, t)\,ds\,dt$ for some Ψ such that $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\Psi(s, t)|^2 ds\,dt < \infty$. In particular, consider $f, g \in \mathscr{D}_1$, and let $f = \sum_n a_n f_n$, $g = \sum b_m f_m$. Let $Q(s, t) = f(s)\overline{g(t)}$. Then $Z(Q) = \sum_{n,m} a_n b_m (\langle G_n, G_m\rangle - \langle G_n, G_m\rangle^{\cdot}) = \sum_{n,m} a_n \bar{b}_m \int (pF_n)(\overline{pF_m})\,dt = \int \check{f}\bar{\check{g}}\,|p|^2\,dt$.

Let $0 < r < 2\pi$, and let f, g have the closure of their supports in in $]-\pi+r, \pi[$. Let $f'(s) = f(s+r)$, $g'(s) = g(s+r)$. Then f', g' are in $\mathscr{D}_1$, and their inverse Fourier transforms satisfy $\check{f}'(x) = e^{irx}\check{f}(x) = e^{irx}\check{g}(x)$. Then

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f'(s)\overline{g'(t)}\Psi(s, t)\,ds\,dt = \int \check{f}'\bar{\check{g}}'\,|p|^2 dt$$
$$= \int \check{f}\bar{\check{g}}|p|^2 dt = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s)\,\overline{g(t)}\Psi(s, t)\,ds\,dt\,.$$

But

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s + r)\overline{g(t + r)}\Psi(s, t)ds\,dt = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s)\overline{g(t)}\,\psi(s - r, t - r)\,ds\,dt\,.$$

in view of the restrictions on the support of f and g. Since this holds for all such f, g, the equality $\Psi(s - r, t - r) = \Psi(s, t)$ holds for almost all (s, t) for which $s, t, s - r, t - r$ are in $]-\pi, \pi[$ (r being fixed). Thus, $\{(r, s, t) \mid s, t, s - r,$ are in $]-\pi, \pi[$ and $\Psi(s - r, t - r) \neq \Psi(s, t)\}$ has measure zero.

Applying Fubini's theorem, we get: for almost all pairs s, t in $]-\pi, \pi[$ the set $\{r \mid s - r, t - r$ lie in $]-\pi, \pi[$ and $\Psi(s - r, t - r) \neq \Psi(s, r)\}$ has measure 0. Denote by Δ the exceptional set of pairs (s, t).

Now let Γ_s be the line of slope 1 which passes through $(s, -s)$, where $-\pi < s < \pi$. Let Γ be the set of s for which $\Gamma_s \cap \Delta$ is *not* a set of measure 0. Then Γ has measure 0, again by Fubini's theorem, and by rotation-invariance of Lebesgue measure. If s is in $]-\pi, \pi[$ but not in Γ, then almost all points on that portion of L_s which lies in $]-\pi, \pi[\times]-\pi, \pi[$ assign to Ψ a common value; thus, if the function Ψ' is defined on $]-\pi, \pi[$ by $\Psi'(s, t) = \int_{a(s,t)}^{b(s,t)} \Psi(s-r, t-r)dr$, where $a(s, t) = \max(s - \pi, t - \pi)$ and $b(s, t) = \min(s + \pi, t + \pi)$, then, for (s, t) on Γ_r, $\Psi'(s, t)$ has this common value. Thus, for almost all r, $\Psi'(s, t) = \Psi(s, t)$ for almost all (in linear measure) points (s, t) with $-\pi < s, t < \pi$ and s, t on Γ_r. Then $\Psi'(s, t)$ is equal almost everywhere to $\Psi(s, t)$. Now set $\psi(r) = \Psi(-r/2, r/2)$, $-2\pi < r < 2\pi$.

Then

$$\Psi'(s, t) = \Psi'(s-(s+t)/2, t - (s + t)/2)$$
$$= \Psi'(-(t - s)/2, (t - s)/2) = \psi(t - s),$$

for s, t in $]-\pi, \pi[$. This completes the proof.

Corollary 1 is just a consequence of the fact (proven in [4]) that if

Φ is as in the statement, then $(\bar{\Phi} - 1)dx$ has a generalized Fourier transform which is a function φ square-summable in any finite interval, so that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\varphi(s-t)\right|^2 ds dt \leqq \left|\int_{-\pi}^{\pi}\int_{-2\pi}^{2\pi}\left|\varphi(r)\right|^2 dr \leqq 2\pi\int_{-2\pi}^{2\pi}\left|\varphi(r)\right|^2 dr\ .$$

To prove corollary 2: let c_j be the absolute value of the ratio of the leading terms of A_j and B_j, and let $u_j = b_j - a_j = \deg(B_j) - \deg(A_j)$. It is clear in general that equivalence of the Gaussian processes induced by given covariances is unaffected if both covariances are multiplied by the same constant. Thus, we find that the process whose spectral measure is

$$\left|\frac{A_j(x)}{B_j(x)}\right|^2 dx$$

has measure on path space equivalent to that whose spectral measure is

$$\frac{c_j}{(1+x^2)^{u_j}}dx,$$

because the quotient of

$$\left|\frac{A_j(x)}{B_j(x)}\right|^2 \quad \text{by} \quad \frac{c_j}{(1+x^2)^{u_i}}$$

is of the form: 1 plus a function in $\mathscr{L}_2(-\infty, \infty)$. So the problem is reduced to whether or not the processes with spectral measures

$$\frac{1}{(1+x^2)^{u_1}}\, dx \text{ and } \frac{c_2 c_1^{-1}}{(1+x^2)^{u_2}}\, dx$$

are equivalent. The criterion is that

$$\left(1 - \frac{c_2 c_1^{-1}}{(1+x^2)^{u_2-u_1}}\right)dx$$

have a generalized Fourier transform which agrees with a function on $]-2\pi, 2\pi[$ having certain properties. But this generalized Fourier transform is explicitly calculated (see [4]), and is of the required form when and only when $c_2 = c_1$ and $u_2 = u_1$.

BIBLIOGRAPHY

1. G. Baxter, *A strong limit theorem for Gaussian processes,* Proc. Amer. Math. Soc., (1956).
2. J. Feldman, *Equivalence and perpendicularity of Gaussian Processes,* Pacific J. Math., **8**, No. 4, (1958).

3. U. Grenander, *Stochastic processes and statistical inference,* Arkiv furti Matemak, Band 1, nr. 17.

4. L. Schwartz, *Théorie des distributions*: Hermann and C^{ie}, Paris, 1951.

5. D. Slepian, *Some comments on the detections of Gaussian signals in Gaussian noise*: I. R. E. Transactions on Information Theory, Vol. I–4, June, 1958.

UNIVERSITY OF CALIFORNIA, BERKELEY