THE SUBGROUPS OF A DIVISIBLE GROUP G WHICH CAN BE REPRESENTED AS INTERSECTIONS OF DIVISIBLE SUBGROUPS OF G

SAMIR A. KHABBAZ

Introduction. In [1], page 70, L. Fuchs asks the following question: Which are those subgroups of a divisible group G that can be represented as intersections of divisible subgroups of G?

The main purpose of this paper is to give an answer to this question.

NOTATION.

N1: If H is a primary p-group, let $S(H)$ denote the subgroup of elements of H whose orders are 1 or p.

N2: If G is Abelian, let $T(G)$ be the torsion subgroup of G; let G_p denote the primary p-component of $T(G)$; and, in case G is divisible, let $F(G)$ denote a maximal torsion free subgroup of G.

N3: Let the symbol $\oplus$ denote a direct sum. Let the symbol $<$ denote "properly contained in." Let $\subset$ denote "contained in." Let $N \backslash M$ denote "the set of elements in N and not in M." Let $\cong$ denote "is isomorphic to." Let $\exists$ denote "there exists (exist)." Let $\ni$ denote "such that." Let $(N_a)_{a \in A}$ denote a family of sets N_a indexed by members of the set A. Finally if Q is a subset of a group, let $\{Q\}$ denote the subgroup of that group generated by the elements of Q.

N4: Let R denote the additive group of rationals. Let P denote the set of primes. Let the group $C(p^\infty)$ be the indecomposable divisible primary p-group.

N5: Let $C = C(2^\infty) \oplus C(3^\infty) \oplus C(5^\infty) \oplus \cdots$; and if $S \subset P$, let $C_S = \bigoplus_{p \in S} C(p^\infty)$.

N6: If G is a group, let $P(G)$ be the set of $p \in P$, such that $\exists x \in G$ with order $x = p$.

N7: Finally, we recall the following convenient and succinct classification of the subgroups of R [see Kurosh I, page 208]. Let $p_1, p_2, p_3, \cdots$ be the sequence of primes in natural order. A characteristic is a sequence $a = (a_1, a_2, a_3, \cdots)$, where $a_i =$ a non-negative integer or ∞. A type is a class of equivalent characteristics, two characteristic $a = (a_1, a_2, a_3, \cdots)$ and $b = (b_1, b_2, b_3, \cdots)$ being equivalent if and only if $\sum_{i=1}^{\infty} |a_i - b_i| < \infty$, where $\infty - \infty = 0$.

$A \subset R$ has type a if and only if it is isomorphic to the subgroup

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of R consisting of those rationals whose denominators in the reduced form are divisible by no higher power of the prime p_i than the a_ith if $a_i < \infty$, and by every power of p_i if $a_i = \infty$.

Define $a \geqq b$ if and only if $a_i \geqq b_i$, for $i = 1, 2, 3, \cdots$.

N8: Let $S \subset P$. We shall say that A above has type T_S is and only if $a_i = 0$ for $p_i \in S$ and $a_i = \infty$ otherwise. Then it is well known that $R/B \subset C_S$ if and only if B contains a subgroup A of type T_S, and that the intersection of two subgroups of R containing subgroups of type T_S again contains a subgroup of type T_S.

N9: Let the symbol $\bigcap_G$ stand for the phrase "an intersection of divisible subgroups of G."

LEMMA 1. (Kulikov):

a. *A divisible group M is a minimal divisible group containing the subgroup L if and only if $H \subset M$ and $H \cap L = 0$ imply $H = 0$, H* being a subgroup.

b. *If M is a minimal divisible group containing L, then M/L is torsion and divisible.*

LEMMA 2. *Let G be divisible and L a subgroup of G. Let M be a minimal divisible subgroup of G containing L. Thus, using the notational in N2, we may write, $G = M \oplus E = M \oplus T(E) \oplus F(E)$. We have:*

a. *If M is minimal divisible containing L, then $S(M_p) = S(L_p)$ for each $p \in P$, and $T(M)$ is minimal divisible containing $T(L)$.*

b. Kulikov: *If L is torsion, then M is minimal divisible containing L if and only if $S(L_p) = S(M_p)$ for all $p \in P$.*

c. *If L is $\bigcap_G$ then $T(L)$ is $\bigcap_{T(G)}$ and hence $\bigcap_G$.*

Proof.

a. Let $x \in S(M_p) \backslash S(L_p)$. Then $\{x\} \cap S(L_p) = 0$, and therefore $\{x\} \cap L = 0$. By Lemma 1a, $x = 0$. Next, $T(M)$ is divisible, and contains $T(L)$. If $N \subset T(M)$ is divisible and contains $T(L)$, $T(M)$ can be written as $T(M) = N \oplus K$, where $K \cap T(L) = 0$; and hence $K \cap L = 0$, so that by Lemma 1a, $K = 0$; and hence, $N = T(M)$.

b. The "only if" part is contained in part a. For the "if" part, assume $N \subset M$ is divisible and contains L. Then we may write $M = N \oplus K$. Then, by hypothesis, K cannot have elements of prime order, and must therefore be 0.

c. Assume $L = \bigcap_{a \in A} M_a$, where each M_a is divisible and contains L. Then each $T(M_a)$ is divisible and contains $T(L)$. Moreover, we have $\bigcap_{a \in A} T(M_a) = T(\bigcap_{a \in A} M_a) = T(L)$. Hence $T(L)$ is $\bigcap_{T(G)}$ and hence $\bigcap_G$.

LEMMA 3. *Let G be a minimal divisible group containing the* *subgroup L, and having a representation of the form $G = \bigoplus_{a \in A} G_a$. Then G/L* (which by Lemma 1b is divisible and torsion) *contains a subgroup isomorphic to $C(p^\infty)$ if and only if for some $a \in A$, $G_a/G_a \cap L$ contains a subgroup of the same kind. In other words: $P(G/L) = \bigcup_{a \in A} P(G_a/G_a \cap L) = \bigcup_{a \in A} P(G_a + L/L)$.*

Proof. Because of the divisibility of all the groups concerned, it suffices to check the existence of elements of order p. Suppose $x \in G_a/G_a \cap L$ has order p. Then $G/L \supset G_a + L/L \cong G_a/(G_a \cap L)$. Hence, G/L has an element of order p. Conversely, suppose that for no $a \in A$ does $G_a/(G_a \cap L)$ contain an element of order p. Then no $G_a + L/L$ contains an element of order p. Hence, the subgroup of G/L generated by the $(G_a + L)/L$ contains no elements of order p. But since the G_a's generate G, the $G_a + L/L$'s generate all of G/L.

THEOREM 1. *Let G be a divisible group; let L be a subgroup of G; let M be a minimal divisible subgroup of G containing L. Suppose G has a representation of the form $G = M \oplus E$, then $L = M \cap (\bigcap_{\omega \in \Omega} M_\omega)$, where M_ω is a divisible subgroup of G containing L for each $\omega \in \Omega$, if and only if there exists homomorphisms $\underline{h_\omega\colon M \to E}$ for each $\omega \in \Omega$ such that $\bigcap_{\omega \in \Omega} \ker h_\omega = L$.*

Proof. To prove the "if" part, let I be the identity map of M; and for each $\omega \in \Omega$ let $g_\omega\colon M \to G$ be defined by $g_\omega = I + h_\omega$. Let $M_\omega = g_\omega(M)$. Then $L \subset M_\omega$ since $h_\omega(L) = 0$, and therefore M_ω is divisible since it is a homomorphic image of M. Finally, $x \in M_\omega \cap M$ implies $x \in \ker h_\omega$ (since $x = y + h_\omega(y)$ implies $h_\omega(y) = x - y \in M \cap E = 0$); and hence, $L \subset \bigcap_{\omega \in \Omega} M_\omega \cap M = \bigcap_{\omega \in \Omega} \ker h_\omega = L$.

To prove the "only if" part, suppose $L = M \cap (\bigcap_\omega M_\omega)$. It can be assumed that each M_ω is minimal divisible containing L and, therefore, also minimal divisible containing $M \cap M_\omega$. Also, M is minimal divisible containing $M \cap M_\omega$. Also, M is minimal divisible containing $M \cap M_\omega$; so there is an isomorphism $i_\omega\colon M \to M_\omega$ which is the identity on $M \cap M_\omega$. Note that $i_\omega(x) \in M \Rightarrow i_\omega(i_\omega(x)) = i_\omega(x) \Rightarrow i_\omega(x) = x \Rightarrow x \in M \cap M_\omega$. Let p: $G \to E$ be the projection determined by the decomposition $G = M \oplus E$. Let h_ω be defined by $h_\omega = pi_\omega$. Then $h_\omega(x) = 0$, $i_\omega(x) \in M$, and $x \in M \cap M_\omega$ are equivalent. Thus $\bigcap_{\omega \in \Omega} \ker h_\omega = \bigcap_{\omega \in \Omega} M \cap M_\omega = L$.

REMARK. The underlined portion of Theorem 1 may be replaced by $h_\omega\colon M \to T(E)$.

COROLLARY 1. *Let G be a divisible group; let L be a subgroup of G; and let M be a subgroup of G which is minimal with respect to being divisible and containing L. Thus, we may write $G = M \oplus E =$*

$M \oplus T(E) \oplus F(E)$, and $P(E) = P(T(E))$. Then L is $\bigcap_G$ if and only if $P(M/L) \subset P(E)$.

Proof. The condition $P(M/L) \subset P(E)$ is easily seen to be equivalent to the existence of the family $\{h_\omega\}_{\omega \in \Omega}$ of homomorphisms in Theorem 1.

REMARK. Let G be divisible and torsion free, then $L \subset G$ is $\bigcap_G$ if and only if L is divisible or, equivalently, is a direct summand of G.

COROLLARY 2. *Let G be divisible, and let L be a torsion subgroup of G. Then L is $\bigcap_G$ if and only if for each $p \in P$, $S(L_p) < S(G_p)$ whenever $S(L_p) \neq 0$, and L_p is not divisible.*

Proof. If L is $\bigcap_G$ then by Lemma 2c for each $p \in P$ obviously L_p is $\bigcap_{G_p}$; and, hence, to prove that our condition is necessary, we may assume that G is primary, and L is not divisible, in which case the necessity becomes obvious in view of the fact that otherwise G would be the only minimal divisible subgroup of itself containing L; and, consequently, $L = G$, since L is $\bigcap_G$, contrary to L being not divisible.

To Prove the "only if" part, note that $p \in P(M/L)$ implies $M_p/L_p = (M/L)_p \neq 0$, since L is torsion. Thus, by hypothesis, $p \in P(M/L) \Rightarrow S(L_p) < S(G_p) \Rightarrow S(M_p) < S(G_p) = S(M_p) \oplus S(E_p) \Rightarrow S(E_p) \neq 0 \Rightarrow p \in P(E)$.

COROLLARY 3. *Let G be divisible and $L \subset G$ be torsion, reduced and $\bigcap_G$. Then every subgroup of L is $\bigcap_G$.*

COROLLARY 4. *Let G be divisible and L be $\bigcap_G$. Let M be a minimal divisible subgroup of G containing L. Let K be a subgroup of G such that $L \subset K \subset M$. Then K is $\bigcap_G$.*

Proof. If $G = M \oplus E$, then $P(M/K) \subset P(M/L) \subset P(E)$.

COROLLARY 5. *Let K be any Abelian group of arbitrary cardinal number A. Then K can be embedded in a divisible group G of power $A\aleph_0$ in such a way that any subgroup of K can be represented as an intersection of two divisible subgroups of G.*

Proof. Let M be a minimal divisible group containing K and let E be a group isomorphic to M/K. Let G be the direct sum of M and E. The cardinality of G is clearly $\aleph_0 A$ and the isomorphism of M/K into E induces a homomorphism h: $M \to E$ with $\ker h = K$. Thus, Theorem 1 gives the required conclusion.

REMARK. Let $L \subset G$, and let $L = L_1 \oplus L_2$, where L_2 is divisible and

L_1 is reduced. Then, also, $G = L_2 \oplus K$, where K may be chosen to contain L_1. It is easy to see that L is $\bigcap_G$ if and only if L_1 is $\bigcap_K$. Thus, in order to avoid excessive wording, we may in the following theorems assume without loss of generality that L is reduced.

THEOREM 2. *Assume $L \subset G$ is reduced, then L is $\bigcap_G$ if and only if $T(L)$ is $\bigcap_G$ and $P(G/L) \subset P(G)$, equality holding if L is $\bigcap_G$.*

Proof. Let $G = M \oplus E$, where M and E are as in Theorem 1. Then $P(G) = P(T(M)) \cup P(E)$, and $P(G/L) = P(M/L) \cup P(E)$, because $G/L \cong (M/L) \oplus E$. Note that if $T(L)$ is $\bigcap_G$, then $T(L)$ is $\bigcap_{T(G)}$ and hence $P(T(M)/T(L)) \subset P(T(E)) = P(E)$, by Corollary 1. But since $T(L)$ is reduced, $P(T(M)/T(L)) = P(T(M))$. Thus the assumption that $T(L)$ is $\bigcap_G$ implies $P(G) = P(E)$ and, therefore, that the conditions $P(G/L) = P(G)$, $P(G/L) \subset P(G)$, and $P(M/L) \subset P(E)$ are equivalent. This observation, together with Lemma 2c and Corollary 1, proves Theorem 2.

COROLLARY 6. *Let G be any divisible group. Let C be as defined in* N5, *and let $\bar{G} = C \oplus G$. Then any subgroup $K \subset G$ is $\bigcap_{\bar{G}}$.*

REMARK. In Corollary 6, C may be replaced by any Abelian group containing it.

COROLLARY 7. *Any torsion free subgroup T of $\bar{G}$ above is $\bigcap_{\bar{G}}$.*

Proof. T is contained in a direct summand of $\bar{G}$ whose complimentary direct summand contains a subgroup isomorphic to C.

REMARK. The following example shows that if $L \subset G$ is $\bigcap_G$ and if $\bar{L} \subset G$ is isomorphic to L, $\bar{L}$ need not be $\bigcap_G$. Let $G = \bigoplus_{i=1}^{\infty} C_i$ where $C_i \cong C(p^\infty)$ and where p is fixed. Then,

$$S(\bigoplus_{i=1}^{\infty} C_i) \cong S(\bigoplus_{i=2}^{\infty} C_i) \ ;$$

however, $S(\bigoplus_{i=1}^{\infty} C_i)$ is not $\bigcap_G$, while $S(\bigoplus_{i=2}^{\infty} C_i)$ is $\bigcap_G$.

In this connection we have:

COROLLARY 8. *Let $L \subset G$ be $\bigcap_G$, and let $\bar{L} \subset G$ be isomorphic to L. Then if $T(\bar{L})$ is $\bigcap_G$—this is in particular the case if $\bar{L} \subset L$—$\bar{L}$ is also $\bigcap_G$. Thus, $\bar{L}$ is $\bigcap_G$ if and only if $T(\bar{L})$ is $\bigcap_G$.*

Proof. For the proof we may assume L is reduced. By Theorem 2, it suffices to show that $P(G/\bar{L}) \subset P(G)$. Let M and $\bar{M}$ be minimal divisible subgroup of G containing L and $\bar{L}$, respectively, so that $G =$

$M \oplus N = \bar{M} \oplus \bar{N}$. Then we have $M/L \cong \bar{M}/\bar{L}$ [see Kurosh I, page 168]. Thus,

$$\begin{aligned} P(G/\bar{L}) &= P(\bar{M}/\bar{L}) \cup P(\bar{N}) \\ &= P(M/L) \cup P(\bar{N}) \\ &\subset P(G/L) \cup P(G) \\ &\subset P(G), \text{ since } L \text{ is } \bigcap_G. \end{aligned}$$

THEOREM 3. *Let G be a divisible group and let $L \subset G$ be reduced. Then the following statements are equivalent*:

(a) *L is $\bigcap_G$.*

(b) *$T(L)$ is $\bigcap_G$ and for any subgroup $\bar{R}$ of G isomorphic to R, either $L \cap \bar{R}$ is zero or of type $\geqq T_{P(G)}$.*

(c) *$S(L_p) < S(G_p)$ if $p \in P(G)$ and for any subgroup $\bar{R}$ of G isomorphic to R, either $L \cap \bar{R}$ is zero or of type $\geqq T_{P(G)}$.*

(d) *$T(L)$ is $\bigcap_G$ and $L \subset (\bigoplus_{a\in A} R_a) \oplus T(G) \subset G$, where $R_a \cong R$ and each $L \cap R_a$ is either zero or of $\geqq T_{P(G)}$.*

Proof. By Theorem 2, (a) is equivalent to the conditions $T(L)$ is $\bigcap_G$ and $P(G/L) \subset P(G)$. These conditions imply (b) since G/L contains a subgroup isomorphic to $\bar{R}/\bar{R} \cap L$, so that $P(\bar{R}/\bar{R} \cap L) \subset P(G/L) \subset P(G)$ and therefore $\bar{R} \cap L$ is zero or of type $\geqq T_{P(G)}$ by N8. Properties (b) and (c) are equivalent by Lemma 2c and Corollary 2. Also (b) implies (d). Finally, suppose (d) holds. Then, $P(T(G)/(T(G) \cap L)) = P(T(G)/T(L)) \subset P(T(G)) = P(G)$ by Theorem 2. Let $G = (\bigoplus_{b\in B} R_b) \oplus (\bigoplus_{a\in A} R_a) \oplus T(G)$. Then $R_b \cap L$ is 0 for all $b \in B$, and for each $a \in A$, $R_a \cap L$ is 0 or has type $\geqq T_{P(G)}$ by hypothesis. Thus, by Lemma 3, $P(G/L) = \bigcap_{b\in B} P(R_b/(R_b \cap L)) \cup \bigcup_{a\in A} P(R_a/(R_a \cap L)) \cup P(T(G)/(T(G) \cap L)) \subset P(G)$. By Theorem 2, this implies (a).

DEFINITION. Define a subset $(x_a)_{a\in A}$ of elements of an Abelian group H to be independent if and only if

$$n_1 x_{a_1} + n_2 x_{a_2} + \cdots + n_m x_{a_m} = 0$$

implies $n_1 = n_2 = \cdots = n_m = 0$ where each $a_i \in A$ and the n_i's are integers.

COROLLARY 10. Assume $L \subset G$ is reduced, then L is $\bigcap_G$ if and only if $T(L)$ is $\bigcap_G$ and L contains a subgroup H which contains a maximal independent subset of L and which has the form $\bigoplus_{a\in A} S_a$, where each S_a is isomorphic to a subgroup of R having type $T_{P(G)}$.

Proof. Assume L is $\bigcap_G$, then by Lemma 2c $T(L)$ is $\bigcap_G$. Let $M \subset G$ be minimal divisible containing L, and assume that $M = T(M) \oplus F(M) = T(M) \oplus (\bigoplus_{a\in A} R_a)$, where $R_a \cong R$ for all $a \in A$. Then, by Theorem 3c and Lemma 1a, each $L \cap R_a$ contains a subgroup S_a of type $T_{P(G)}$. Then it is easy to see that $\bigoplus_{a\in A}S_a$ exists; and that, if $x_a \in S_a$, then $(x_a)_{a\in A}$ is a maximal independent subset of M and therefore, of L.

Next assume the condition holds and let M be as usual. For each S_a, let $R_a \supset S_a$ be a subgroup of M of type R. Since any two non-zero subgroups of R have a non-zero intersection, and since $\bigoplus_{a\in A}S_a$ exists, also $\bigoplus_{a\in A}R_a$ exists. Let $x_a \in S_a$; then $(x_a)_{a\in A}$ is a maximal independent subset of H; and since H contains a maximal independent subset of L, $(x_a)_{a\in A}$ is also a maximal independent subset of L. Thus, we must have $M = T(M) \oplus (\bigoplus_{a\in A}R_a)$. Theorem 3d and the fact that $T(L)$ is $\bigcap_G$ imply that L is $\bigcap_G$.

REMARK. Concerning the last definition given above, it is well known that H contains a maximal independent subset and that if $(x_a)_{a\in A}$ is independent, then $H/\{(x_a)_{a\in A}\}$ is torsion if and only if $(x_a)_{a\in A}$ is also maximal independent. Thus, Corollary 10 may be worded as follows: Assume L is reduced; then L is $\bigcap_G$ if and only if $T(L)$ is $\bigcap_G$ and L contains a subgroup H which has the form $\bigoplus_{a\in A}S_a$ where S_a is isomorphic to a subgroup of R of type $T_{P(G)}$ and such that L/H is torsion.

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