COMPLETE HOLOMORPHS

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1. Introduction. Throughout this paper let G be an additive group, and denote the group of all automorphisms of G by $A(G)$ and the holomorph of G by $K(G)$. Then $K(G) = A(G) \times G$, where $(\alpha, a) + (\beta, b) = (\alpha\beta, a\beta + b)$ for all elements (α, a) and (β, b) of $K(G)$. We prove that if G is abelian and $x \to 2x$ is an automorphism of G, then $K(G)$ is complete if and only if $G' = 1 \times G$ is a characteristic subgroup of $K(G)$. From this it follows that if G is abelian, $x \to 2x$ is an automorphism of G, and $A(G)$ is abelian, then $K(G)$ is complete.

In § 3 we derive analogous results for ordered abelian groups. Then we show that any divisible, torsion free, abelian group can be ordered so that its o-holomorph is o-complete. It is known (see [2]) that the holomorph of a non-abelian group is not complete. In § 4 we give an example of a non-abelian o-group with an o-complete o-holomorph. Finally, we show that the lexicographically ordered direct sum of two o-complete groups is again o-complete.

2. Complete holomorphs. Recall that a group is *complete* if it has a trivial center and all of its automorphisms are inner.

In 1957, W. Peremans [3] investigated under what conditions the holomorph of an abelian group is complete. He was able to derive a necessary and sufficient condition for the holomorph to be complete when $x \to 2x$ is an automorphism of the group. Using this result he was then able to prove that if $x \to 2x$ is an automorphism of the group and if the group is either directly indecomposable, the direct sum of cyclic groups, or is divisible, then the holomorph is complete.

We derive a necessary and sufficient condition which is simpler in statement than that of Peremans. However, before this theorem can be proved some preliminary lemmas are necessary which have independent interest. Let B be a subgroup of $A(G)$, and let τ be a mapping from B into G. Then τ is a *crossed homomorphism* if for all α and β in B,

$$(\alpha\beta)\tau = (\alpha\tau)\beta + \beta\tau .$$

LEMMA 2.1. *Let G be an abelian group. If τ is a crossed homomorphism of $A(G)$ into G, then the mapping χ of $K(G)$ into itself defined by*

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$$(\alpha, a)\chi = (\alpha, \alpha\tau + a)$$

is an automorphism of $K(G)$ which induces the identity automorphism on G'. Conversely, if χ is an automorphism of $K(G)$ and if χ induces the identity automorphism on G', then there exists a crossed homomorphism τ mapping $A(G)$ into G such that for all (α, a) in $K(G)$

$$(\alpha, a)\chi = (\alpha, \alpha\tau + a) .$$

Proof. The first part of this lemma follows by an easy computation which we leave to the reader.

Suppose that χ is an element of the automorphism group of $K(G)$ and that χ induces the identity automorphism on G'. If (α, a) is an element of $K(G)$, then

$$\begin{aligned}(\alpha, a)\chi &= (\alpha, 0)\chi + (1, a)\chi \\ &= (\alpha\sigma, \alpha\tau) + (1, a) \\ &= (\alpha\sigma, \alpha\tau + a) ,\end{aligned}$$

where $1\sigma = 1$ and $1\tau = 0$.

For (α, a) and (β, b) belonging to $K(G)$ we have

$$\begin{aligned}((\alpha, a) + (\beta, b))\chi &= (\alpha\beta, a\beta + b)\chi \\ &= ((\alpha\beta)\sigma, (\alpha\beta)\tau + a\beta + b)\end{aligned}$$

and

$$\begin{aligned}(\alpha, a)\chi + (\beta, b)\chi &= (\alpha\sigma, \alpha\tau + a) + (\beta\sigma, \beta\tau + b) \\ &= (\alpha\sigma\beta\sigma, (\alpha\tau + a)\beta\sigma + \beta\tau + b) .\end{aligned}$$

Therefore,

$$(\alpha\beta)\tau + a\beta = (\alpha\tau)(\beta\sigma) + a(\beta\sigma) + \beta\tau .$$

If $\alpha = 1$, then for all a in G, $a\beta = a(\beta\sigma)$. Hence, for all β in $A(G)$, $\beta = \beta\sigma$, and thus, $\sigma = 1$. Thus, we have that $(\alpha\beta)\tau = (\alpha\tau)\beta + \beta\tau$, and $(\alpha, a)\chi = (\alpha, \alpha\tau + a)$.

LEMMA 2.2. *If G is an abelian group such that $x \to 2x$ is an automorphism of G, and if χ is an automorphism of $K(G)$ such that $G'\chi = G'$, then χ is an inner automorphism of $K(G)$.*

Proof. Since $G'\chi = G'$, there exists an inner automorphism δ of $K(G)$ such that $\chi = \delta$ on G'. Let $\chi_1 = \chi\delta^{-1}$. Then χ_1 induces the identity automorphism on G', and if we can show that χ_1 is an inner automorphism of $K(G)$, then we will also have shown that χ is an inner automorphism of $K(G)$. Hence, we will consider χ_1 instead of χ.

By Lemma 2.1 we know that $(\alpha, a)\chi_1 = (\alpha, \alpha\tau + a)$ where τ is a crossed homomorphism mapping $A(G)$ into G. Let $\bar{2}$ be the automorphism $a\bar{2} = 2a$, where a is in G. Since τ is a crossed homomorphism and $\alpha\bar{2} = \bar{2}\alpha$ for all α in $A(G)$, we have

$$2(\alpha\tau) + \bar{2}\tau = (\alpha\bar{2})\tau = (\bar{2}\alpha)\tau = (\bar{2}\tau)\alpha + \alpha\tau .$$

Hence,

$$\alpha\tau = (\bar{2}\tau)\alpha - \bar{2}\tau .$$

Then, for all (α, a) in $K(G)$,

$$\begin{aligned}(1, \bar{2}\tau) + (\alpha, a) - (1, \bar{2}\tau) &= (\alpha, (\bar{2}\tau)\alpha + a) + (1, -\bar{2}\tau) \\ &= (\alpha, (\bar{2}\tau)\alpha - \bar{2}\tau + a) \\ &= (\alpha, \alpha\tau + a) \\ &= (\alpha, a)\chi_1 .\end{aligned}$$

LEMMA 2.3. *Suppose that G is an abelian group and that D is a non-trivial subgroup of $A(G)$. Then the natural splitting extension H of G by D has a non-trivial center if and only if there exists a non-zero element a of G such that $a\alpha = a$ for all elements α of D.*

Proof. We have that $H = D \times G$ where $(\alpha, a) + (\beta, b) = (\alpha\beta, a\beta + b)$ for all (α, a) and (β, b) in H.

Suppose there exists a nonzero element a of G such that $a\alpha = a$ for all α in D. Then $(1, a)$ is an element of the center of H, and $(1, a) \neq (1, 0)$.

Now suppose that (β, b) is an element of the center of H such that $(\beta, b) \neq (1, 0)$. Then, for all (α, a) in H,

$$(\alpha, a) + (\beta, b) = (\beta, b) + (\alpha, a) .$$

Thus, for all α in D and all a in G, $\alpha\beta = \beta\alpha$, and $a\beta + b = b\alpha + a$. If $\alpha = 1$, then, for all a in G, $a\beta = a$. Thus, $\beta = 1$. Hence, $b = b\alpha$ for all α in D, and since $\beta = 1$, b must be nonzero for otherwise $(\beta, b) = (1, 0)$.

THEOREM 2.1. *If G is an abelian group in which $x \to 2x$ is an automorphism, then $K(G)$ is complete if and only if G' is characteristic in $K(G)$.*

Proof. It follows from Lemma 2.3 that the center of $K(G)$ is trivial since $x \to 2x$ leaves no point of G fixed. If $K(G)$ is complete, then every automorphism of $K(G)$ is inner, and thus, since G' is normal in $K(G)$, G' is characteristic.

Next suppose that χ is an automorphism of $K(G)$ and that G' is characteristic in $K(G)$. Then $G'\chi = G'$, and hence, by Lemma 2.2, χ is an inner automorphism of $K(G)$. Thus, $K(G)$ is complete.

If G is finite, then the theorem gives the known result that the holomorph $K(G)$ of an abelian group of odd order is complete if and only if G' is characteristic in $K(G)$. In this case the mapping $x \to 2x$ is clearly an automorphism of G.

COROLLARY 2.1. *If G is an abelian group in which $x \to 2x$ is an automorphism and if $A(G)$ is abelian, then $K(G)$ is complete.*

Proof. It is well known that the commutator subgroup of a group is always a characteristic subgroup; hence, if we can show that G' is the commutator subgroup of $K(G)$, then by theorem 2.1, $K(G)$ will be complete.

Since $K(G)/G'$ is isomorphic to $A(G)$ and $A(G)$ is abelian, G' contains the commutator subgroup. Also, for any $(1, a)$ in $K(G)$ and any b in G,

$$-(1, a) - (\bar{2}, b) + (1, a) + (\bar{2}, b) = (1, a) .$$

Thus, every element of G' is a commutator.

3. o-complete o-holomorphs. The ideas of completeness and the holomorph can be carried over into the theory of (linearly) ordered groups. An o-group is o-*complete* if its center is trivial and all of its o-automorphisms are inner. Suppose that G is an o-group and that the group $A_0(G)$ of all o-automorphisms of G can be ordered. We define the o-*holomorph* of G to be the subgroup $K_0(G) = A_0(G) \times G$ of $K(G)$. Let (α, a) be positive if α is positive or if $\alpha = 1$ and a is positive in G. Then it is easy to verify that $K_0(G)$ is an o-group with respect to this definition and that G' is a normal convex subgroup of $K_0(G)$.

It is known that an o-group is o-complete if and only if it is a direct summand in any o-group which contains it as a normal convex subgroup. The proof is a slight variation of the classical proof for non-ordered complete groups.

THEOREM 3.1. *Let G be an abelian* o-*group for which $A_0(G)$ can be ordered. If τ is a crossed homomorphism of $A_0(G)$ into G, then the mapping χ from $K_0(G)$ into $K_0(G)$ defined by*

$$(\alpha, a)\chi = (\alpha, \alpha\tau + a)$$

is an order preserving automorphism of $K_0(G)$ which induces the identity automorphism on G'. Conversely, if χ is an order preserving automorphism of $K_0(G)$ and if χ induces the identity automorphism on G', *then there exists a crossed homomorphism τ mapping $A_0(G)$ into G such that for all (α, a) in $K_0(G)$,*

$$(\alpha, a)\chi = (\alpha, \alpha\tau + a) .$$

The proof is identical with the proof of Lemma 2.1. One only need verify that the mapping $(\alpha, a)\chi = (\alpha, \alpha\tau + a)$ preserves order (when (α, a) is in $K_0(G)$ and τ is a crossed homomorphism). But if $1 < \alpha$, then (α, a) and $(\alpha, \alpha\tau + a)$ are positive, and if $\alpha = 1$ and $0 < a$, then $(\alpha, \alpha\tau + a) = (1, a)$ is positive.

COROLLARY 3.1. *Suppose that G is an abelian* o-*group in which $x \to 2x$ is an automorphism and for which $A_0(G)$ can be ordered. If χ is an order preserving automorphism of $K_0(G)$ such that $G'\chi = G'$, then χ is an inner automorphism of $K_0(G)$.*

This corollary follows at once from the proof of Lemma 2.2 and the fact that an inner automorphism of an o-group is an o-automorphism.

If G is an o-group, then a subgroup C of G is said to be o-*characteristic* if $C\delta = C$ for all δ in $A_0(G)$.

THEOREM 3.2. *Suppose that G is an abelian* o-*group in which $x \to 2x$ is an automorphism and for which $A_0(G)$ can be ordered. Then $K_0(G)$ is* o-*complete if and only if G' is* o-*characteristic in $K_0(G)$.*

The proof of this theorem is analogous to the proof of Theorem 2.1.

Suppose that G is an o-group and that C and C' are two convex subgroups of G. Then C *covers* C' if C contains C' and there is no convex subgroup of G between C and C'. Let Γ be the set of all ordered pairs (G^α, G_α) of convex subgroups such that G^α covers G_α. Define $(G^\alpha, G_\alpha) > (G^\beta, G_\beta)$ if G_α contains G^β. This orders Γ. We can regard Γ as an ordered set $\alpha, \beta, \gamma, \cdots$. The order type of Γ is the *rank* of G. The set Γ will be called the *rank set* of G. The groups G^α/G_α for α in Γ are the *components* of G.

COROLLARY 3.2. *If G is an abelian* o-*group in which $x \to 2x$ is an automorphism and for which $A_0(G)$ can be ordered, and if G has well-ordered rank, than $K_0(G)$ is* o-*complete.*

Before we prove this corollary, we shall prove a lemma concerning well-ordered subsets of an ordered set.

LEMMA 3.1. *If L is an ordered set, if W is a well-ordered convex subset of L, and if f is a one-to-one, order preserving mapping of L onto itself such that $f(\delta) = \delta$ where δ is the least element of W, then $f(\alpha) = \alpha$ for all α in W.*

Proof. Suppose α is any element of W such that $\alpha \neq \delta$. Then $[\delta, \alpha]$ is a well-ordered subset of L. Suppose $f(\alpha) \neq \alpha$. Then either $f(\alpha) < \alpha$ or $f^{-1}(\alpha) < \alpha$. Without loss of generality we may assume that $f(\alpha) < \alpha$. Then f is a one-to-one mapping of $[\delta, \alpha]$ into itself. Hence, $\alpha \leqq f(\alpha)$ which is a contradiction to our assumption. Thus, $f(\alpha) = \alpha$ for all α in W.

Proof of Corollary 2.2. The rank set of $K_0(G)$ is an ordered set. Since G has well-ordered rank, the rank set of $K_0(G)$ contains a well-ordered convex subset—the rank set of G. Now any order preserving automorphism of $K_0(G)$ induces a one-to-one, order preserving mapping of the rank set of $K_0(G)$ onto itself. By Lemma 3.1 this order preserving mapping is the identity on the rank set of G. But this means that G' is o-characteristic, and therefore by Theorem 3.2, we see that $K_0(G)$ is o-complete.

It is well known that a torsion free abelian group can be ordered, and as mentioned before, Peremans has shown that the holomorph of a divisible abelian group is complete. It does not seem likely that for every ordering of a divisible, torsion free, abelian group it will be possible to order the resulting group of order preserving automorphisms. However, Conrad [1] has proved the following useful result:

If G is an o-group of well-ordered rank each of whose components is isomorphic to the additive group of rational numbers, then $A_0(G)$ can be ordered.

This result together with Corollary 3.2 gives us the following theorem.

THEOREM 3.3. *Any divisible, torsion free, abelian group can be ordered so that*

(1) $A_0(G)$ *can be ordered and*

(2) $K_0(G)$ *is* o-*complete.*

Proof. A divisible, torsion free, abelian group G is a rational vector space. Hence we can choose a basis A for G and well-order A. If g is any nonzero element of G, then $g = r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n$ where the r_i are nonzero rational numbers and the α_i are elements of the basis A. Without loss of generality we may assume that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ in the well-ordering of A. We will say that g in G is positive if $0 < r_n$. Then G is an o-group with well-ordered rank each of whose components is o-isomorphic to the rational numbers. Thus, by the result of Conrad stated above, $A_0(G)$ can be ordered, and by Corollary 3.2, $K_0(G)$ is o-complete.

REMARK. It is well known that any torsion free abelian group is contained in a unique (to within an isomorphism) minimal divisible group which is also torsion free and abelian. Thus, any torsion free abelian group is contained is an o-complete group.

4. Examples of o-complete groups. This section will consist of several examples of o-complete groups and a theorem which concerns direct sums of o-complete groups.

A small amount of notation needs to be introduced at this time. If G and H are groups, then $\mathrm{Hom}\,(G, H)$ will denote the set of all homomorphisms mapping G into H. Throughout the examples $\mathbf{R}$ will denote the additive group of real numbers with their natural order, R will denote the additive group of rational numbers with their (unique) natural order, $\mathbf{R}'$ will denote the multiplicative group of positive real numbers, and R' will denote the multiplicative group of positive rational numbers.

EXAMPLE I. The o-automorphism group of $\mathbf{R}$ is (isomorphic to) $\mathbf{R}'$. Give $\mathbf{R}'$ its natural order. Then $K_0(\mathbf{R})$ is o-complete by Corollary 3.2. It should be noted that $K_0(\mathbf{R})$ is (isomorphic to) the multiplicative group of 2×2 matrices of the form

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$$

where a is in $\mathbf{R}'$ and b is in $\mathbf{R}$. Such a matrix is posivive if $1 < a$ or $1 = a$ and $0 < b$. Also note that $K_0(\mathbf{R})$ is of rank two.

EXAMPLE II. Let M be the additive group of all rationals of the form $m/2^n$ where m and n are integers, and let M have its natural order. Let N be the cyclic subgroup of R' generated by 2. Notice that neither M nor any of its proper subgroups are divisible; hence $\mathrm{Hom}\,(R, M) = 0$.

Let $G = R \oplus M$ where (a_1, a_2) in G is positive if $a_1 > 0$ or $a_1 = 0$ and $a_2 > 0$. Then G is an abelian o-group of rank 2. Then since $\mathrm{Hom}\,(R, M) = 0$, if ϕ is an element of $A_0(G)$ then $\phi = (p_1, p_2)$ where p_1 is in R' and p_2 is in N, and conversely, if $\phi = (p_1, p_2)$ where p_1 is in R' and p_2 is in N, then ϕ is in $A_0(G)$, i.e., $A_0(G) = R' \otimes N$. Now R' is a free abelian group of countable rank, and so is $R' \otimes N$. Thus, R' is isomorphic to $R' \otimes N$. Define an element of $R' \otimes N$ to be positive if its image in R' is positive, where R' is given its natural order. Then $R' \otimes N$ is an abelian o-group of rank one, and so by Corollary 3.2 $K_0(G)$ is o-complete and of rank three.

EXAMPLE III. Let $G = R \oplus R$ where (a_1, a_2) in G is positive if $0 < a_1$ or $0 = a_1$ and $0 < a_2$. Then it is easy to show that $A_0(G)$ is isomorphic to the group of 2×2 matrices of the form

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$$

where a and b are elements of R' and c is in R. Such a matrix is positive if $1 < a$ or $1 = a$ and $1 < b$ or $1 = a = b$ and $0 < c$. Then, $A_0(G)$ is an o-group of rank three. By Corollary 3.2, $K_0(G)$ is o-complete, and we observe that $K_0(G)$ is of rank five.

The above three examples show that there are o-complete groups of rank two, three, and five. Using the next theorem we can show that there are o-complete groups for every finite rank greater than one.

Throughout the following discussion, let D and N be o-groups. To avoid confusion let the identity element of D be denoted by θ and that of N by 0. *Whenever* $G = D \oplus N$ we will always order G as follows: (α, a) in G is positive if $\theta < \alpha$ or $\theta = \alpha$ and $0 < a$.

LEMMA 4.1. *Suppose that $G = D \oplus N$ and that the center of N is trivial. If $N' = \theta \times N$ is o-characteristic in G, then $A_0(G)$ is isomorphic to $A_0(D) \otimes A_0(N)$.*

Proof. If ϕ is in $A_0(G)$ and if (α, a) and (β, b) are in G, then

$$\begin{aligned}(\alpha, a)\phi &= (\alpha, 0)\phi + (\theta, a)\phi = (g(\alpha), h(\alpha)) + (\theta, P(a)) \\ &= (g(\alpha), h(\alpha) + P(a))\end{aligned}$$

where P is in $A_0(N)$ and $h(\theta) = 0$.

$$\begin{aligned}((\alpha, a) + (\beta, b))\phi &= (\alpha + \beta, a + b)\phi \\ &= (g(\alpha + \beta), h(\alpha + \beta) + P(a + b)) \\ (\alpha, a)\phi + (\beta, b)\phi &= ((g(\alpha), h(\alpha) + P(a)) + (g(\beta), h(\beta) + P(b)) \\ &= (g(\alpha) + g(\beta), h(\alpha) + P(a) + h(\beta) + P(b)) .\end{aligned}$$

Hence, $g(\alpha + \beta) = g(\alpha) + g(\beta)$, and it follows by an easy argument that g is an element of $A_0(D)$. Also,

$$h(\alpha + \beta) + P(a) = h(\alpha) + P(a) + h(\beta) .$$

If $\alpha = \theta$, then for all a in N and β in D,

$$h(\beta) + P(a) = P(a) + h(\beta) .$$

Therefore, $h(\beta)$ is in the center of N (which is trivial) for all β in D, and hence, $(\alpha, a)\phi = (g(\alpha), P(a))$. It follows that the mapping of ϕ upon (g, P) is an isomorphism of $A_0(G)$ onto $A_0(D) \otimes A_0(N)$.

THEOREM 4.1. *Suppose that $G = D \oplus N$. Then G is o-complete if and only if D and N are o-complete and N' is o-characteristic in G.*

Proof. Let us denote the center of a group G by $Z(G)$.

First suppose that G is o-complete. Then since $0=Z(G)=Z(D)\oplus Z(N)$, D and N have trivial centers. Consider δ in $A_0(N)$ and (α, a) in G. Define the mapping ϕ of G into itself by

$$(\alpha, a)\phi = (\alpha, a\delta) .$$

Clearly, ϕ is in $A_0(G)$, and since G is o-complete there exists (β, b) in G such that

$$\begin{aligned}(\alpha, a\delta) = (\alpha, a)\phi &= -(\beta, b) + (\alpha, a) + (\beta, b) \\ &= (-\beta + \alpha + \beta, -b + a + b) .\end{aligned}$$

Thus, $a\delta = -b + a + b$, and hence, N is o-complete. By a similar argument D is o-complete. Since G is o-complete and N' is a normal convex subgroup of G, it is clear that N' is o-characteristic in G.

Finally, suppose that D and N are o-complete and that N' is o-characteristic in G. If ϕ is in $A_0(G)$, then by Lemma 4.1, we have that ϕ is equivalent to (g, P), where g is in $A_0(D)$ and P is in $A_0(N)$. Since D and N are both o-complete there exists β in D and b in N such that for all a in N, $P(a) = -b + a + b$, and for all α in D, $g(\alpha) = -\beta + \alpha + \beta$. Therefore, for all (α, a) in G.

$$\begin{aligned}(\alpha, a)\phi &= (-\beta + \alpha + \beta, -b + a + b) \\ &= (-\beta, -b) + (\alpha, a) + (\beta, b) \\ &= -(\beta, b) + (\alpha, a) + (\beta, b) .\end{aligned}$$

Thus, ϕ is an inner automorphism. Since $Z(D)$ and $Z(N)$ are both trivial it is clear that $Z(G)$ must be trivial, and hence, G is o-complete.

The second half of Theorem 4.1 may be used to construct further examples of o-complete groups. Using the examples given in the first portion of this section we see that we can easily construct o-complete groups for any finite rank greater than one.

Suppose that G is an o-group such that $A_0(G)$ can be ordered and $K_0(G)$ is o-complete. Then $A_0(K_0(G))$ is isomorphic to $K_0(G)$, and hence, inherits an order. Since $K_0(G)$ is o-complete and since every o-complete group is a direct summand of any o-group which contains it as a normal convex subgroup, we have that $K_0(K_0(G)) = T \oplus K_0(G)$ where T is o-isomorphic to $A_0(K_0(G))$. Therefore, $K_0(K_0(G))$ is o-complete if and only if $K_0(G)$ is o-characteristic (by Theorem 4.1). In particular, if $K_0(G)$ has well-ordered rank, then $K_0(K_0(G))$ is o-complete. Thus, the second o-holomorphs of any one of the examples are o-complete.

Added in Proof. It has been pointed out to the author by Professor W. Peremans that Theorem 2.1 of this paper has previously appeared as "Satz*" on page 101 of W. Specht, *Gruppentheorie* (Springer, 1956). However the proof given by Specht is different from the one given here, and the proof given by Specht is not applicable for o-groups (c.f. Theorem 3.2)

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