CLOSED LINEAR OPERATORS AND ASSOCIATED CONTINUOUS LINEAR OPERATORS

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Introduction. Suppose X and Y are normed linear spaces. Throughout this paper, T shall be a closed linear operator with domain $D(T)$ dense in X and range $R(T) \subset Y$. For the sake of completeness, we present the classification scheme devised in [7].

As regards $R(T)$, there are the following three possibilities:

I: $R(T) = Y$,

II: $R(T) \neq Y$ but $\overline{R(T)} = Y$,

III: $\overline{R(T)} \neq Y$.

If $R(T) = Y$, we say that T is in state I, written $T \in I$. Analogous notation is used regarding II and III.

As regards T^{-1}, there are the following there possibilities:

1: T^{-1} exists and is continuous,

2: T^{-1} exists but is not continuous,

3: T^{-1} does not exist.

Here we say that T is in state 1, written $T \in 1$, to indicate that T has continuous inverse, with analogous usage concerning 2 and 3.

By combining the various possibilities from the two lists, we obtain nine possible states for T, e.g., $T \in I_3$ shall mean that $R(T) = Y$ and that T has no inverse.

This classification scheme may now be applied to the conjugate T' of T. A corresponding "state diagram" was constructed in [3] which exhibits the states which can occur for T together with T'.

The purpose of this paper is to give some insight into the reasons why the state diagram for closed linear operators is the same as that for continuous linear operators (cf. [3]). It is shown that given T closed, there corresponds a *continuous* linear operator T such that T and T' are in the same states as T and T', respectively.

In the sequel, we shall adopt the following convention: if E is a linear space and Γ is a set of linear functionals on E, then (E, Γ) is the set E with the weak topology induced by Γ (cf. [2, p. 419]). For any set $K \subset E$, $\bar{K}^{(E,\Gamma)}$ shall denote the closure of K in (E, Γ). The set Γ will be called total if $f(x) = 0$ for all $f \in \Gamma$ implies $x = 0$. If Γ is a total subspace, then (E, Γ) is a locally convex topological linear space which is also Hausdorff.

DEFINITION. Let $D(T')_1$ denote the linear space $D(T')$ with norm

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defined by $\|y'\|_1 = \|y'\| + \|T'y'\|$. It was noted by Sz.-Nagy [5], that $D(T')_1$ is a Banach space. Define T'_1 as the operator T' mapping $D(T')_1$ into X'.

Theorem 1 shows that, with the appropriate identifications, $D(T')_1$ is not only complete, but is in fact a conjugate space. Moreover, the corresponding operator T'_1 is the conjugate of a bounded linear operator.

The following lemma is due to I. Singer [6, Theorem 1].

LEMMA. *Let E be a normed linear space, V a subspace of E' and $\mathscr{J}$ the "canonical mapping" of E into V' defined by*

$$[\mathscr{J}(x)]v = v(x) \text{ for every } v \in V .$$

Denote by S_E and $S_{V'}$ the closed unit spheres in E and V', respectively. Then $\mathscr{J}S_E$ is dense in $S_{V'}$ with respect to the w^* topology, (V', V).

THEOREM 1. *Define $J: Y \to (D(T')_1)'$ by $(Jy)y' = y'y$. Let $\mathscr{J}$: $D(T')_1 \to (JY)'$ be defined by $(\mathscr{J}y')Jy = (Jy)y'$. Then*

(i) *$D(T')_1$ is linearly isometric to $(JY)'$ under the map $\mathscr{J}$ and $\|y'\|_1 = \sup_{\|Jy\|=1} |y'y|$*

(ii) *JT is a continuous linear map from normed linear space $D(T)$ into normed linear space JY. Moreover, $T'_1 = (JT)'\mathscr{J}$.*

(iii) *The states of T and T' are the same as those of continuous linear operators JT and $(JT)'$ respectively.*

Proof of (i). For convenience, Let E denote $D(T')_1$ and let V denote JY. Since $|(Jy)y'| = |y'y| \leqq \|y'\|_1\|y\|$ for all $y' \in E$, it follows that $V \subset E'$ and $\|J\| \leqq 1$. Obviously V is a total subspace of E' and both $\mathscr{J}$ and J are one-to-one. We now prove that the image of $\mathscr{J}$ is V'. By [4], the closed unit sphere S_E in E is a compact subset of (Y', Y), i.e., Y' with the w^* topology. Since (E, V) is E with the relative topology inherited from (Y', Y), S_E is also a compact subset of (E, V). Thus $S_E = \bar{S}^{(E,V)}$ since (E, V) is Hausdorff. It is easy to see that $\mathscr{J}$ is a homeomorphism from (E, V) onto $\mathscr{J}E$ with respect to the relative topology inherited from (V', V). Hence by the lemma,

$$(*)\ S_E = \mathscr{J}\bar{S}_E^{(E,V)} = \overline{\mathscr{J}S_E}^{(E,V)} \cap \mathscr{J}E = S_{V'} \cap \mathscr{J}E .$$

Therefore, $S_{V'} \cap \mathscr{J}E$ is compact and thus closed in Hausdorff space (V', V). Suppose that $\mathscr{J}E \neq V'$. Then there exists some $v' \in V'$ such that $\|v'\| = 1$ and $v' \notin E$, i.e., v' is not a member of the convex set $S_{V'} \cap \mathscr{J}E$ which we have shown closed in (V', V). By [2, theorem V. 2.10], there exists a linear functional f which is continuous on (V', V) and a constant c such that

$$Rf(v') > c \geqq Rf(S_{V'} \cap \mathscr{J}E) .$$

Thus

$$(^{**})\quad c \geqq \sup_{z' \in S_{V'} \cap \mathscr{J}E} |f(z')| ,$$

for if $u \in S_{V'} \cap \mathscr{J}E$, and $f(u) = |f(u)| e^{i\theta}$, then $e^{-i\theta}u \in S_{V'} \cap \mathscr{J}E$. Hence $c \geqq Rf(e^{-i\theta}u) = |f(u)|$. Since f is continuous on (V', V), it follows from [2, Theorem V. 3.9] that there exists some $v \in V$ such that $f(z') = z'(v)$ for all $z' \in V'$. Consequently, by (*) and (**) we infer that

$$|v'v| \geqq Rv'(v) = Rf(v') > c \geqq \sup_{z' \in S_{V'} \cap \mathscr{J}E} |f(z')| = \sup_{x \in S_E} |f(\mathscr{J}x)| = \sup_{x \in S_E} |v(x)| = \|v\| ,$$

where $\|v\|$ is the norm of $v \in E'$. Hence $v'(v/\|v\|) > 1$. This, however, is a contradiction since $\|v'\| = 1$. We have therefore shown that $\mathscr{J}$ must map E onto V'. Now from (*), $S_E = \mathscr{J}^{-1}S_{V'}$. Therefore, given any $y' \in E$,

$$\|y'\|_1 = \|\mathscr{J}^{-1}\mathscr{J}y'\| = \|\mathscr{J}y'\| \leqq \|y'\|_1$$

which shows that $\mathscr{J}$ is an isometry and

$$\|y'\|_1 = \|\mathscr{J}y'\| = \sup_{\|Jy\|=1} |(\mathscr{J}y')Jy| = \sup_{\|Jy\|=1} |y'y| .$$

REMARK. By examining closely [1, Theorem 19], one can conclude that (i) is valid after observing that S_E is compact in (E, V) [4]. The proof given above, however, is quite different from the proof given by Dixmier, and indeed, may be used to prove Theorems 19 and 17′ of Dixmer.

Proof of (ii). JT is continuous from $D(T)$ into E' since

$$|(JTx)y'| = |y'Tx| = |T'y'x| \leqq \|T'y'\| \|x\| \leqq \|T_1'\|_1 \|y'\|_1 \|x\|$$

implies that $\|JT\| \leqq \|T_1'\|$. For x in $D(T)$ and y' in E,

$$\begin{aligned}[(JT)'(\mathscr{J}y')]x &= (\mathscr{J}y')(JTx) = (JTx)y' \\ &= y'Tx = T_1'y'(x) = [(T_1'\mathscr{J}^{-1})(\mathscr{J}y')]x .\end{aligned}$$

Hence $(JT)'(\mathscr{J}y') = T_1'\mathscr{J}^{-1}(\mathscr{J}y')$ or $(JT)' = T_1'\mathscr{J}^{-1}$.

From the above result it is obvious that T_1' and $(JT)'$ are in the same state. We assert that T' and T_1' are in the same state and therefore so are T and $(JT)'$. It suffices to show that $T' \in 1$ if and only if $T_1' \in 1$. If $T' \in 1$, then T_1' has an inverse and $R(T_1') = R(T')$ is closed since T' is closed. However, T_1' is a continuous linear operator on Banach space E. Therefore, as a consequence of the interior mapping principle, $T_1' \in 1$. Conversely, if $T_1' \in 1$, then T' has an inverse and $R(T')$ is closed. By the closed graph theorem, it follows that $T' \in 1$.

It is easy to verify that T and JT are in the same "range state". Finally, to prove that T and JT are in the same state, it remains only to show that $T \in 1$ if and only if $JT \in 1$. By inspecting the state diagram in [3], and recalling that T' and $(JT)'$ are in the same state, we can conclude that $T \in 1$, $T' \in I$ and $JT \in 1$ are equivalent statements.

2. Let $\overline{JY}$ be the closure of JY in E'. $\overline{JY}$ is therefore a Banach space. Suppose X and Y are Banach spaces. Define $\widehat{JT}: X \rightarrow \overline{JY}$ as the continuous linear extension of JT. We now compare the states of T and with those of $\widehat{JT}$ and $(\widehat{JT})'$ respectively.

Clearly, $(JT)' = (\widehat{JT})'$. This implies, by the preceding results, that T' and $(\widehat{JT})'$ are in the same state. An inspection of the state diagram in [3] verifies the following assertions:

(a) $T \in I$ if and only if $T' \in 1$ if and only if $\widehat{JT} \in I$.

(b) $T \in II$ if and only if $T' \in II_2$ or III_2 if and only if $\widehat{JT} \in II$.

(c) $T \in III$ if and only if $\widehat{JT} \in III$.

(d) $T \in 1$ if and only if $T' \in I$ if and only if $\widehat{JT} \in 1$.

(e) If X is reflexive, then $T \in 2$ if and only if $T \in II_2$ or II_3 if and only if $\widehat{JT} \in 2$.

(f) If X reflexive, then $T \in 3$ if and only if $\widehat{JT} \in 3$.

We thus obtain the following

THEOREM 2. *Suppose X and Y are complete. Then*

(i) *The states of T' and $(JT)'$ are the same.*

(ii) *T and JT are in the same "range state."*

(iii) *$T \in 1$ if and only if $\widehat{JT} \in 1$.*

(iv) *If X is reflexive, then the state diagram for T and T' is the same as that for $\widehat{JT}$ and $(\widehat{JT})'$.*

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