GROUPS WITH FINITELY MANY AUTOMORPHISMS

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1. Introduction. The connection between the structure of a group and the structure of its endomorphisms or group of automorphisms is a very interesting question but one to which there is as yet a scarcity of answers. We present here two theorems pertaining to this problem. We first show that a finitely generated group has a finite automorphism group if and only if it is a finite and central extension of a cyclic group. The restriction to finitely generated groups is essential. Indeed, there exist indecomposable torsion-free abelian groups of rank the cardinal of the continuum whose automorphism groups are cyclic of order two (see [2], p. 180, 18(b)). The second result which we present is a new and much simpler proof of a theorem to be found in a paper of Baer [1]; namely, a group possessing only finitely many endomorphisms is itself finite.

Before discussing these theorems we shall describe the notation to be used in this paper. Throughout, G will denote a group with center Z. Let coset representatives $g_\alpha, g_\beta, \cdots$ of Z in G be chosen for all $\alpha, \beta, \cdots$ elements of G/Z such that $g_1 = 1$. Let $M = \{m_{\alpha,\beta}\}$ be the corresponding factor set. For the theory of such factor sets, essential to the following, the reader should consult Kurosh [4] or M. Hall [3]. Finally, for any group H, let $\mathrm{Aut}(H)$ be the automorphism group of H.

2. Preliminary lemmas. The first lemma follows immediately from the definition of equivalence of factor sets.

LEMMA 1. *Let N and R be factor sets from the group B into the abelian group $A_1 \times A_2$. Let N_1, N_2, R_1, R_2 be the factor sets from B into A_1 and A_2 obtained by taking components of N and R. Then N and R will be equivalent if and only if N_1 is equivalent with R_1 and N_2 with R_2.*

The remaining lemmas are, I believe, entirely or in part scattered throughout the literature. We include proofs for the convenience of the reader.

LEMMA 2. *Let ψ be an endomorphism of Z such that the factor sets M and $\psi(M) = \{\psi(m_{\alpha,\beta})\}$ of G are equivalent. Then ψ may be extended to an endomorphism of G and if ψ is an automorphism of Z then it may be extended to an automorphism of G.*

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Proof. First note that it is clear from the definition of factor set that $\psi(M)$ is a factor set. Since M and $\psi(M)$ are assumed to be equivalent we may choose for each $\alpha \in G/Z$ an element $c_\alpha \in Z$ such that

$$\psi(m_{\alpha,\beta}) = c_{\alpha\beta}^{-1} m_{\alpha,\beta} c_\alpha c_\beta$$

for all $\alpha, \beta \in G/Z$. Now every element $g \in G$ can be uniquely expressed in the form $g = g_\alpha$ a where $\alpha \in G/Z$ and $a \in Z$. Thus we may define a mapping φ of G into itself according to the rule, for all $g \in G$,

$$\varphi(g) = g_\alpha c_\alpha \psi(a) .$$

Then φ is an endomorphism of G since if $g = g_\alpha\ a$, $h = g_\beta b$ are elements of G, where $\alpha, \beta \in G/Z$, $a, b \in Z$ then

$$\begin{aligned} \varphi(gh) &= \varphi(g_\alpha a\ g_\beta b) \\ &= \varphi(g_{\alpha\beta} m_{\alpha,\beta} ab) \\ &= g_{\alpha\beta} c_{\alpha\beta} \psi(m_{\alpha,\beta} ab) \\ &= g_{\alpha\beta} m_{\alpha,\beta} c_\alpha c_\beta \psi(a) \psi(b) \\ &= g_\alpha c_\alpha \psi(a)\ g_\beta c_\beta \psi(b) \\ &= \varphi(g)\, \varphi(h) . \end{aligned}$$

Since $g_1 = 1$ implies $m_{1,1} = g_1^{-1}\ g_1 g_1 = 1$ and so $c_1 = 1$, we have that φ extends ψ to G.

Finally, suppose ψ is an automorphism of G. Let $g = g_\alpha a$ be an arbitrary element of G, as above. Then $\varphi(g) = 1$ implies that $g_\alpha c_\alpha \psi(a) = 1$ and hence $\alpha = 1$ because α is the image of $g_\alpha c_\alpha \psi(a)$ in G/Z. Therefore $g_1 = c_1 = 1$ so we have $\psi(a) = 1$ and $g = g_\alpha\ a = 1$. φ is consequently a one-to-one mapping of G into itself. To conclude we prove that g is the image of some element under φ. Choose $b \in Z$ such that $\psi(b) = c_\alpha^{-1} a$. Then $\varphi(g_\alpha b) = g_\alpha c_\alpha \psi(b) = g_\alpha a = g$.

LEMMA 3. *Let A be a central and characteristic subgroup of the group H. The group K of those automorphisms of H which leave A element-wise fixed and which induce the identity automorphism on H/A is naturally isomorphic to the group Hom$(H/A, A)$ of homomorphisms of H/A into A.*

Proof. If $\sigma \in \mathrm{Hom}(H/A, A)$ define a mapping θ of H into itself by $\theta(h) = h\sigma(hA)$ for $h \in H$. θ will be an element of K, as can be verified directly, and the correspondence of σ with θ is the required isomorphism.

COROLLARY. *If* $\mathrm{Aut}(A)$, $\mathrm{Aut}(H/A)$, *and* $\mathrm{Hom}(H/A, A)$ *are finite then* $\mathrm{Aut}(H)$ *is finite.*

Proof. If $\theta \in \mathrm{Aut}(H)$ let θ_1 be the restriction of θ to A and θ_2 the automorphism of H/A induced by θ. Then the mapping sending θ to $(\theta_1, \theta_2) \in \mathrm{Aut}(A) \times \mathrm{Aut}(H/A)$ is a homomorphism with kernel K. Hence, the product of the orders of $\mathrm{Aut}(A)$, $\mathrm{Aut}(H/A)$, and $\mathrm{Hom}(H/A, A)$ is a bound for the order of $\mathrm{Aut}(H)$.

3. Finite automorphism groups.

THEOREM 1. *A finitely generated group G has a finite automorphism group if and only if it has a central cyclic subgroup of finite index.*

Proof. First we shall demonstrate the sufficiency of these conditions. In so doing, we may assume G is infinite and therefore a finite and central extension of an infinite cyclic group. The center Z of G is of finite index in G so by Schreier's subgroup theorem (3; p. 97) is finitely generated. Hence Z is the direct sum of a finite abelian group T and an infinite cyclic group. Consequently, $\mathrm{Aut}(G/Z)$ and $\mathrm{Hom}(G/Z, Z) = \mathrm{Hom}(G/Z, T)$ are finite. To conclude, we can apply the above Corollary once we prove that $\mathrm{Aut}(Z)$ if finite. However, this follows from another application of the Corollary with $A = T$ and $H = Z$.

The necessity of the condition is more difficult to prove. If $\mathrm{Aut}(G)$ is finite then G/Z is finite being isomorphic to the group of inner automorphisms of G. Again, by Schreier's theorem, Z will be finitely generated. We may assume, in contradiction to the theorem, that Z in a direct decomposition into cyclic groups, contains two or more infinite cyclic factors. In fact let $Z = W \times (a) \times (b)$ where (a) and (b) are the subgroups generated by elements a and b of infinite order and W is a subgroup of Z. We need only show that infinitely many automorphisms of Z, which are the identity on W and map the group (a, b) generated by a and b onto itself, may be extended to automorphisms of G.

For the remainder of this proof we shall use the additive notation for Z. Write the factor set M as $m_{\alpha,\beta} = w_{\alpha,\beta} + r_{\alpha,\beta}a + t_{\alpha,\beta}b$ where $w_{\alpha,\beta} \in W$, $s_{\alpha,\beta}$ and $t_{\alpha,\beta}$ are integers. Let the automorphism θ of Z be defined by $\theta(w) = w$ for w W, $\theta(a) = ma + nb$, $\theta(b) = pa + qb$ where $m, n, p,$ and q are integers such that $|mq - np| = 1$. The factor set $\theta(M)$ is then expressible as $\theta(m_{\alpha,\beta}) = w_{\alpha,\beta} + s'_{\alpha,\beta}a + t'_{\alpha,\beta}b$ where $s'_{\alpha,\beta} = ms_{\alpha,\beta} + pt_{\alpha,\beta}$ and $t'_{\alpha,\beta} = ns_{\alpha,\beta} + qt_{\alpha,\beta}$. Therefore, $S_0 = \{s_{\alpha,\beta}\}$, $T_0 = \{t_{\alpha,\beta}\}$, $S'_0 = \{s'_{\alpha,\beta}\}$, and $T'_0 = \{t'_{\alpha,\beta}\}$ are factor sets of G/Z with integral values and $S'_0 = mS_0 + pT_0$, $T'_0 = nS_0 + qT_0$. By Lemmas 1 and 2 the proof is now reduced to the following problem: Find infinitely many quadruplets (m, n, p, q) of integers such that $|mq - np| = 1$ and the factor sets $mS_0 + pT_0$ and $nS_0 + qT_0$ are equivalent with the factor sets S_0 and T_0 respectively. That is, the factor sets $(m-1)S_0 + pT_0$ and $nS_0 + (q-1)T_0$ are both equivalent to the trivial factor sets. However, if G/Z has order

h then hS_0 and hT_0 both are equivalent to the trivial factor set (3; p. 223). Thus, for any integer k the following values for m, n, p and q suffice:

$$\begin{aligned} m &= -h^2 - hk + 1 \\ n &= h \\ p &= -h^2k - hk^2 - h \\ q &= hk + 1 \,, \end{aligned}$$

and the theorem is proved.

4. Groups with finitely many endomorphisms.

THEOREM 2. *A group G which has only finitely many endomorphisms is itself finite.*

Proof. The group G will then have only finitely many inner automorphisms so Z/G will be finite, say of order n. We shall once again use multiplicative notation for Z. For any positive integer k the factor sets M and M^{kn+1} are equivalent (as above) so that we may, by Lemma 2, extend to endomorphisms of G the endomorphisms $\theta(k)$ of Z which map z to z^{kn+1} for $z \in Z$. If Z contains at least one element of infinite order then all the $\theta(k)$ will be distinct. Therefore, we shall assume that Z is a torsion group.

Suppose that we can decompose Z as the direct product $\prod_\gamma A_\gamma$ of infinitely many nontrivial factors. The elements $m_{\alpha,\beta}$ of the factor set M will have components in only finitely many of the factors A_γ, because there are only finitely many elements $m_{\alpha,\beta}$. If A_δ is any factor containing no component of any element $m_{\alpha,\beta}$ then we can extend the endomorphism ψ of Z to G where ψ is defined by

$$\begin{aligned} \psi(a) &= a \text{ if } a \in A_\rho \,, \quad \rho \neq \delta \\ \psi(a) &= 1 \text{ if } a \in A_\delta \,. \end{aligned}$$

This will give us infinitely many endomorphisms of G.

If Z is of bounded order then[1] it is the direct sum of cyclic groups so that either Z is finite or the preceding paragraph applies. If Z is of unbounded order write Z as the direct product of its Sylow subgroups. In this case either infinitely many of these subgroups are nontrivial and the above remarks pertain or for some prime p the p-Sylow subgroup is of unbounded order. Choose then elements $x_1, x_2, \cdots$ of Z such that x_i has order p^i. The endomorphisms $\theta(p^j)$ defined in the first paragraph will all be distinct. For if $n = p^r n_1$ where p and n_1 are coprime then

[1] For basic results on abelian groups see [**2**].

$\theta(p^j)(x_i) = x_i$ if and only if $i \leqq j + r$. These endomorphisms $\theta(p^j)$ may, as we said, be extended to G so G again has infinitely many endomorphisms. Thus Z is finite so G is also finite.

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