A NOTE ON THE PRIMES IN A BANACH ALGEBRA OF MEASURES

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1. Introduction. Let V denote the family of all finite complex-valued and conuntably additive set functions on the Borel subsets of $R_+ = [0\ \infty)$ (hereafter called measures); $L^1(R_+)$ the set of all complex-valued functions on R_+ which are integrable in the sense of Lebesgue, identifying functions which are 0 almost everywhere; and A the elements in V which are absolutely continuous with respect to Lebesgue measure. For each $\mu \in V$ there exists an $f \in L^1(R_+)$ such that

$$\mu(E) = \int_E f(x)dx \tag{1.1}$$

for each Borel subset E of R_+. And, conversely, if $f \in L^1(R_+)$ the set function μ defined by (1.1) is a measure.

We introduce a norm into V by the formula

$$\|\mu\| = \sup \Sigma \,|\, \mu(E_i) \,| \qquad (\mu \in V)\ , \tag{1.2}$$

the supremum being taken over all finite partitions of R_+ into pairwise disjoint Borel sets E_i. It is well known ([6], p. 142 or [7]) that V becomes a commutative Banach algebra under the convolution operation

$$\nu(E) = \int_0^\infty \mu(E - x)d\lambda(x) \qquad (\mu, \lambda \in V)\ , \tag{1.3}$$

where E is any Borel subset of R_+; in symbols

$$\nu = \mu * \lambda\ . \tag{1.4}$$

The Laplace-Stieltjes transform of $\mu \in V$ will be denoted by $\hat{\mu}$:

$$\hat{\mu}(z) = \int_0^\infty e^{-zx} d\mu(x) \qquad (Re(z) \geqq 0)\ . \tag{1.5}$$

The relation (1.4) is equivalent to

$$\hat{\nu}(z) = \hat{\mu}(z)\hat{\lambda}(z) \qquad (Re(z) \geqq 0)\ . \tag{1.6}$$

The *identity* in V is the measure u such that $u(E) = 1$ if $0 \in E$ and 0 otherwise. A measure μ is *invertible* provided there exists a measure μ^{-1} such that $\mu * \mu^{-1} = u$; and the measure λ is a *divisor* of the measure μ, in symbols $\lambda \,|\, \mu$, provided there exists a measure ν such that $\mu = \lambda * \nu$. It follows from basic properties of the Laplace-Stieltjes

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transform that V is an integral domain and a semi-simple Banach algebra (see for example [6], p. 149).

The central problem under consideration here is that of determining the prime measures, that is, those noninvertible measures μ such that

(i) $\mu = \lambda * \nu$ always implies that one of the measures λ, ν is invertible.

It is clear that every prime measure μ satisfies the condition

(ii) $V * \mu \subset V * \lambda$ implies that either λ is invertible or $\mu \mid \lambda$.

And (i) follows from (ii) since V is an integral domain. Here $V * \mu$ denotes the ideal $\{\nu * \mu \mid \nu \in V\}$.

We give a partial solution by showing that all measures of the form

$$\mu_a = \frac{1}{1+a} u - \eta \qquad (Re(a) > 0)\,, \tag{1.7}$$

where $d\eta(x) = e^{-x}dx$, are primes. Stated in terms of the ideal structure of V, the result is that the maximal ideals $m_a = \{\mu \mid \hat{\mu}(a) = 0\}$, $Re(a) > 0$, are principal.

A related problem is the following: Given a fixed measure μ, for what measures λ is it true that $\lambda \mid \mu$? Climaxing a sequence of papers on this problem, notably [4] and [8], Fuchs [3] proved that $\lambda \mid \mu$ if and only if the Hausdorff method of summability $[H, \mu]$ includes the method $[H, \lambda]$. In this paper we make use of recent results on the representation of linear transformations by convolution to give a simple, and apparently unnoticed, alternative formulation in terms of the range of a convolution transform.[1]

THEOREM 1. *Every measure μ_a, $Re(a) > 0$, is a prime; and if there exists a prime μ essentially different from μ_a, $Re(a) > 0$ (two primes are essentially different if one cannot be obtained from the other by convolution with an invertible measure) then either $\hat{\mu}(z)$ has a root with real part 0 or the hull of the ideal $V * \mu$ consists only of maximal ideals in V which contain A.*

THEOREM 2. *Let T_μ, $\mu \in V$, be the linear operator from $L^1(R_+)$ into $L^1(R_+)$ defined by*

$$T_\mu f(t) = f * \mu(t) = \int_0^t f(t-x)d\mu(x) \tag{1.8}$$

for $f \in L^1(R_+)$. Let R_μ denote the range of T_μ. Then the measure λ is a divisor of the measure μ if and only if $R_\mu \subset R_\lambda$.

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2. Proofs of the Theorems.

Proof of Theorem 1. The positive result of this theorem depends on the obvious fact (see condition (ii)) that if the maximal ideal m in V is principal and μ is a generator, that is $m = V * \mu$, then μ is a prime.

Fix $Re(a) \geqq 0$ and set $h(\mu) = \hat{\mu}(a)$. It follows from (1.5) and (1.6) that h defines a multiplicative linear functional on V. Hence $m_a = \{\mu \in V \mid \hat{\mu}(a) = 0\}$ is a maximal ideal in V. That $V * \mu_a \subset m_a$ follows from (1.4), (1.6) and the fact that $\hat{\mu}_a(z) = (1 + a)^{-1} - (1 + z)^{-1}$ vanishes at a.

The reverse inclusion requires that if $\mu \in m_a$, then $\mu = \nu * \mu_a$ for some $\nu \in V$. To this end we use a device suggested by [9] and define

$$\nu = (1 + a)\mu + (1 + a)^2\theta_a \tag{2.1}$$

where

$$d\theta_a = \int_0^x e^{a(x-t)} d\mu(t)dx = -\int_x^\infty e^{-a(t-x)} d\mu(t)dx = f(x)dx \, . \tag{2.2}$$

The equality of the two integrals is a consequence of $\hat{\mu}(a) = 0$. In case $\sigma = Re(a) > 0$, an application of the Fubini theorem using the second integral in (2.2) yields

$$\begin{aligned}\int_0^\infty |f(x)|\,dx &= \int_0^\infty \left| \int_x^\infty e^{-a(t-x)} d\mu(t) \right| dx \leqq \int_0^\infty \int_x^\infty e^{-\sigma(t-x)} d\,|\mu(t)|\,dx \\ &= \int_0^\infty \int_0^t e^{-\sigma(t-x)} dx\,d\,|\mu(t)| = \frac{1}{\sigma}\int_0^\infty [1 - e^{-\sigma t}]d\,|\mu(t)| \\ &\leqq \frac{1}{\sigma}\int_0^\infty |\,d\mu(t)\,| < \infty \, .\end{aligned}$$

This proves $f \in L^1(R_+)$ so that, in view of (1.1), $\theta_a \in A$ when $Re(a) > 0$.

It remains to verify that

$$\begin{aligned}\mu = \nu * \mu_a &= (1 + a)[\mu + (1 + a)\theta_a] * [(1 + a)^{-1}u - \eta] \\ &= (1 + a)[(1 + a)^{-1}\mu - \mu * \eta + \theta_a - (1 + a)\theta_a * \eta] \, .\end{aligned}$$

But integration by parts yields the relation

$$\int_0^t e^{-(t-x)} \int_x^\infty e^{-a(y-x)} d\mu(y)dx = (1 + a)^{-1}\left[\int_0^t e^{-(t-y)} d\mu(y) + \int_t^\infty e^{-a(y-t)} d\mu(y)\right]$$

which, together with the fact that $d(\phi * \gamma)(x) = (f * \gamma)(x)dx$ whenever $d\phi(x) = f(x)dx, f \in L^1(R_+)$ and $\gamma \in V$, shows that $(1 + a)\theta_a * \eta = -\mu * \eta + \theta_a$. This establishes the result.

If μ is a prime essentially different from μ_a, $Re(a) > 0$, and $\hat{\mu}(z)$ has no roots with real part 0, then $\hat{\mu}(z)$ has no roots. To see this note that $\hat{\mu}(a) = 0$ for $Re(a) > 0$ implies that $V * \mu \subset V * \mu_a = m_a$. Hence

$\mu = \nu * \mu_a$ for some $\nu \in V$ which, because of condition (ii), forces ν to be invertible; so μ is not essentially different from μ_a. Thus $V * \mu$ is not contained in m_a for any a, $Re(a) \geqq 0$. Phillips ([6], p. 148 or [7]) has shown that in the space Δ of maximal ideals in V, $\Delta_1 = \{m_a \mid Re(a) \geqq 0\}$ is precisely those maximal ideals which *omit* an element of A so that $\Delta_2 = \Delta - \Delta_1$ consists of all those maximal ideals which contain A. It is clear, then, that the hull of $V * \mu$, i.e., all maximal ideals which contain it, must be a subset of Δ_2.

Proof of Theorem 2. First suppose that $\lambda \mid \mu$. Then $\mu = \nu * \lambda$ for some $\nu \in V$ and, therefore,

$$L^1(R_+) * \mu = L^1(R_+) * \nu * \lambda \subset L^1(R_+) * \lambda \ ,$$

i.e., $R_\mu \subset R_\lambda$.

For the converse we note that the inclusion $R_\mu \subset R_\lambda$ implies that for each $f \in L^1(R_+)$ there exists a $g \in L^1(R_+)$ such that

$$f * \mu = g * \lambda \ . \tag{2.1}$$

But the fact that V is an integral domain insures the uniqueness of g. Hence the relation (2.1) defines a mapping T: $f \rightarrow g$ which is linear, commutes with convolution in the sense that $T(f * \gamma) = T(f) * \gamma$ for $f \in L^1(R_+)$, $\gamma \in V$, and, via an application of the closed graph theorem, bounded in the norm topology of $L^1(R_+)$. It follows using the type of argument given in [2], that every such mapping has the form $T(f) = f * \nu$ for some measure ν. Thus

$$f * \mu = (f * \nu) * \lambda = f * (\nu * \lambda) \tag{2.2}$$

for every $f \in L^1(R_+)$. A second application of the fact that V is an integral domain yields $\mu = \nu * \lambda$, that is $\lambda \mid \mu$, and the theorem is proved.

3. A remark and a question. Let $Re(a) > 0$, $Re(b) > 0$. It is easy to verify that $(z + 1)/(z + b)$ is the Laplace-Stieltjes transform of an invertible measure. Consequently the measure defined by

$$\hat{\mu}(z) = \frac{z - a}{z - b} = \hat{\mu}_a(z) \frac{(1 + a)(z + 1)}{z + b} \ (Re(z) \geqq 0) \tag{3.1}$$

is a prime not essentially different from μ_a. The primes given by relation (3.1) coincide with those given in [4]. Existence of other primes remains an open question.

Repeated application of Theorem 1 yields the relation

$$V * \mu_{a_1} * \mu_{a_2} * \cdots * \mu_{a_n} = \bigcap_{i=1}^{n} m_{a_i} \ , \qquad n = 2, 3, \cdots \tag{3.2}$$

where $Re(a_i) > 0$, $i = 1, 2, 3, \cdots$. On the other hand, it is known [1] that the closed ideal $m = \bigcap_{i=1}^{\infty} m_{a_i}$ is not trivial in case $\sum_{i=1}^{\infty} 1/|a_i| < \infty$. A natural question to ask is the following: Does there exist a measure μ such that $V * \mu = m$?

REFERENCES

1. T. Carleman, *Über die approximation analytischer funktionen durch lineare aggregate von vorgegeben potenzen*, Arkiv för mat., astronomi och fysik, 17, **9** (1923), 1-30.
2. R. E. Edwards, *Representation theorems for certain functional operators*, Pacific J. Math., **7** (1957), 1333-1339.
3. W. H. J. Fuchs, *A theorem on Hausdorff's methods of summation*, Quarterly J. of Math., **16** (1945), 64-77.
4. H. L. Garabedian, Einar Hille and H. S. Wall, *Formulations of the Hausdorff inclusion problem*, Duke Math. J., **8** (1941), 193-213.
5. Einar Hille and J. D. Tamarkin, *Questions of relative inclusion in the domain of Hausdorff means*, Proc. Nat. Acad. Sci., **19** (1933), 573-577.
6. Einar Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Coll. Publ., XXXI, 1957.
7. R. S. Phillips, *Spectral theory for semi-groups of operators*, Trans. Amer. Math. Soc., **71** (1951), 393-415.
8. W. W. Rogosinski, *On Hausdorff's method of summability*, Proc. Cambridge Phil. Soc., **38** (1942), 166-192.
9. L. L. Silverman and J. D. Tamarkin, *On the generalization of Abel's theorem for certain definitions of summability*, Math. Z., **29** (1928), 161-170.