FOURIER SERIES WITH LINEARLY DEPENDENT COEFFICIENTS

HARRY HOCHSTADT

I. Introduction. The following problem is posed and solved in this article. A function $H(\theta)$ is defined over the interval $(0, \pi)$, but is as yet unknown over the interval $(-\pi, 0)$. Furthermore it is supposed that the function can be expressed as a Fourier series, with certain constraints on the coefficients. In particular

$$H(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

where

$$\alpha a_n + \beta b_n = c_n\ , \qquad n = 0, 1, 2, \cdots .$$

α and β are prescribed constants and the c_n a prescribed sequence. The question which can now be raised is whether these constraints automatically continue the function into the interval $(-\pi, 0)$. It will be shown that under certain conditions the continuation of $H(\theta)$ is unique almost everywhere.

There are two trivial special case namely if either α or β are allowed to become infinite. In these cases the proper continuation is as an odd or even function respectively.

A different, but equivalent, formulation is the following. Does the definition of $H(\theta)$ and the constraints on the Fourier coefficients a_n and b_n allow one to evaluate these coefficients? In order to be able to use the standard integral formulas for the coefficients $H(\theta)$ would have to be defined over an interval of length 2π. Over the interval $(0, \pi)$ the trigonometric functions are not orthogonal so that such integral formulas do not exist. One can show then that an equivalent statement is that the nonorthogonal set of functions $\{\sin (nx - \tan^{-1}\alpha/\beta)\}$ is complete in $L_2(0, \pi)$, for $|\alpha| \neq |\beta|$. The case $|\alpha| = |\beta|$ requires some additional stipulations.

One can also formulate a similar problem involving a function defined over the interval $(0, \infty)$, and constraints on the Fourier cosine and sine transforms.

In both of these case one can show that a unique continuation exists in the space of square-integrable functions for $|\alpha| \neq |\beta|$. In the case of the problem of the infinite interval one can explicitly demonstrate nonunique continuations in the space of nonintegrable functions.

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The proof in both cases is accomplished by reducing the problem to the solution of a singular Fredholm integral equation of the second kind. An analysis of the spectrum of the resulting linear operator shows that the lowest eigenvalue is outside the region of interest.

II. Statement of the theorems.

THEOREM A. *Suppose the periodic function $H(\theta)$ possesses the Fourier series*

$$H(\theta) = \frac{a_0}{2} + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta)$$

where the Fourier coefficients are linearly dependent. They satisfy the relationship

$$\alpha a_n + \beta b_n = c_n \,, \qquad n \geqq 0.$$

where α and β are prescribed real constants and the sequence $\{c_n\}$ is square-summable. If $H(\theta)$ is defined as a square-integrable function over the interval $(0, \pi)$, there exists a unique (a.e.) square-integrable continuation of $H(\theta)$ into the interval $(-\pi, 0)$, provided $|\alpha| \neq |\beta|$.

When $\alpha = \beta$, one also requires that the function

$$K(\theta) = H(\theta) - \alpha^{-1}[c_0/2 + \sum_1^\infty c_n \cos n\theta]$$

be such that

$$\sum_0^\infty k_n^2 < \infty, \text{ and } \sum_1^\infty |k_n| \ln n \prec \infty$$

where

$$k_n = \int_0^\pi (\cot \theta/2)^{1/2} K(\theta) \cos n\theta \, d\theta \,.$$

When $\alpha = -\beta$ the cot $\theta|2$ *is to be replaced by* tan $\theta|2$ *in the above integral.*

Theorem B is a companion theorem to A.

THEOREM B. *Suppose the function $H(\theta)$ can be represented by the Fourier Integral*

$$H(\theta) = \int_0^\infty (a(\omega) \cos \omega\theta + b(\omega) \sin \omega\theta) d\omega$$

where the Fourier cosine and sine transforms are linearly dependent. They satisfy the relationship

$$\alpha a(\omega) + \beta b(\omega) = c(\omega) , \qquad \omega \geqq 0$$

where α and β are prescribed real constants and the function $c(\omega)$ is square integrable. If $H(\theta)$ is defined as a square integrable function over the interval $(0, \infty)$ *there exists a unique* (a.e.) *square integrable continuation of $H(\theta)$ into the interval* $(-\infty, 0)$, *provided* $|\alpha| \neq |\beta|$.

When $\alpha = \beta$ one also requires that the function

$$K(\theta) = H(\theta) - \frac{1}{\alpha}\int_0^\infty c(\omega) \cos \omega\theta \, d\theta$$

be such that

$$\int_0^\infty k^2(\omega) d\omega < \infty, \text{ and } \int_0^\infty | k(\omega) | \ln \omega \, d\omega < \infty .$$

where

$$k(\omega) = \int_0^\infty \theta^{-1/2} K(\theta) \cos \omega\theta \, d\theta .$$

When $\alpha = -\beta$, $\theta^{-1/2}$ is to be replaced by $\theta^{1/2}$ in the above integral.

Equivalent formulations of these theorems are the following.

THEOREM A′. *A function $H(\theta)$ in $L_2(0, \pi)$ can be represented in the form*

$$H(\theta) = \sum_{n=0}^\infty k_n \sin (n\theta + \phi)$$

where ϕ is a fixed phase angle. For $\phi = \pm\pi/4$ one must impose additional restrictions on $H(\theta)$ as in Theorem A.

THEOREM B′. *A function $H(\theta)$ in $L_2(0, \infty)$ can be represented in the form*

$$H(\theta) = \int_0^\infty k(\omega) \sin (\omega\theta + \phi) d\omega$$

where ϕ is a fixed phase angle. For $\phi = \pm\pi/4$ one must impose additional restrictions on $H(\theta)$ as in Theorem B.

However the former formulation is preferable because that is the direct form in which the theorems are proved.

III. Reduction of the proofs to the analysis of integral equations. One can in the ensuing analysis replace the c_n by zero without loss of generality since the general expansion can be rewritten in the following form after a_n is eliminated.

$$H(\theta) - 1/\alpha\left[\frac{c_0}{2} + \sum_1^\infty c_n \cos n\theta\right]$$
$$= -\sum_1^\infty b_n(\beta/\alpha \cos n\theta - \sin n\theta) .$$

Let $h(-\theta)$ denote the continuation of $H(\theta)$ in the interval $(-\pi, 0)$, and a_n, b_n denote the Fourier coefficients of the resultant function. Then

$$\int_{-\pi}^0 h(-\theta)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} n\theta\, d\theta + \int_0^\pi H(\theta)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} n\theta\, d\theta = \pi\begin{Bmatrix}a_n\\ b_n\end{Bmatrix}$$

and let d_n and e_n be defined by

$$\int_0^\pi H(\theta)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} n\theta\, d\theta = \pi\begin{Bmatrix}d_n\\ e_n\end{Bmatrix} .$$

Thus one can solve for the corresponding integrals for $h(\theta)$ and

$$\int_0^\pi h(\theta)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} nx\, dx = \pi\begin{Bmatrix}a_n - d_n\\ e_n - b_n\end{Bmatrix} .$$

From these two equations the unknown coefficients a_n and b_u can be eliminated by use of the relationship

$$\alpha a_n + \beta b_n = 0 .$$

It follows that

$$\int_0^\pi h(\theta)(\alpha\cos n\theta - \beta\sin n\theta)dx = \pi(-\alpha d_n - \beta e_n),\ n = 0, 1, \cdots \tag{1}$$

One can now multiply the above equation first by α cos ny and then by β sin ny and take the difference of the resultant equations, to obtain

$$\frac{\alpha^2+\beta^2}{2}\int_0^\pi h(\theta)\cos n(\theta-\phi)d\theta + \frac{\alpha^2-\beta^2}{2}\int_0^\pi h(\theta)\cos n(\theta+\phi)d\theta$$
$$-\alpha\beta\int_0^\pi h(\theta)\sin n(\theta+\phi)d\theta = (\pi(-\alpha d_n - \beta e_n)(\alpha\cos n\phi - \beta\sin n\phi) .$$

One can now apply the summation formulas

$$\frac{1}{2} + \sum_1^N \cos nx = \frac{\sin\left(N - \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

$$\sum_1^N \sin nx = \frac{1}{2}\cot\frac{x}{2} - \frac{\cos\left(N - \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

to the above equation and then pass to the limit as N tends to infinity. One then obtains the integral equation

$$h(\phi) - \frac{\lambda}{2\pi}\int_0^{\pi} h(\theta)\cot\frac{\theta+\phi}{2}\,d\theta = f(\phi) \tag{2}$$

where

$$\lambda = \frac{2\,\alpha\beta}{\alpha^2+\beta^2}$$

$$f(\phi) = \frac{2}{\alpha^2+\beta^2}\left\{\frac{-\alpha^2 d_0}{2} + \sum_1^{\infty}(-\alpha d_n - \beta e_n)(\alpha\cos n\phi - \beta\sin n\phi)\right\}.$$

To convert the Fourier integral case to an integral equation one defines $d(\omega)$ and $e(\omega)$ by

$$\int_0^{\infty} H(\theta)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\omega\theta\,d\theta = \frac{\pi}{2}\begin{Bmatrix}d(\omega)\\ e(\omega)\end{Bmatrix}$$

and proceeds in a similar fashion as in the previous case. There is an alternative procedure. The period is changed from π to T by a formal change of variable and by a passage to the limit as T tends to infinity one obtains

$$h(\phi) - \frac{\lambda}{\pi}\int_0^{\infty}\frac{h(\theta)}{\theta+\phi}\,d\theta = f(\phi) \tag{4}$$

where

$$\lambda = \frac{2\alpha\beta}{\alpha^2+\beta^2}$$

$$f(\phi) = \frac{1}{\alpha^2+\beta^2}\int_0^{\infty}(-\alpha d(\omega) - \beta e(\omega))(\alpha\cos\omega\phi - \beta\sin\omega\phi)\,d\omega\,.$$

IV. Analysis of the integral equations. The integral equations corresponding to both problems are singular integral equation of the Fredholm type of the second kind. It will be shown that both equations have unique solutions in the space of square-integrable functions provided that the eigenvalue parameter λ satisfies

$$|\lambda| < 1\,.$$

But since

$$\lambda = \frac{2\alpha\beta}{\alpha^2+\beta^2}$$

and the latter function is bounded by unity it is evident that the integral equations alway have unique solutions in the space of square-integral functions. The case $|\lambda| = 1$ will be treated separately.

Equation (4) is discussed in detail in [3], and the same method can be adopted for equation (2).

We now consider equation (2) and expand the kernel in terms of an orthonormal system of functions over the interval $(0, \pi)$. We find that with the kernel we can associate the quadratic form

$$\sum_{n,k=1}^{\infty} a_{n,k} x_n x_k$$

where the $a_{n,k}$ are given by

$$\begin{aligned} a_{n,k} &= \frac{2}{\pi}\int_0^{\pi}\int_0^{\pi} \cot\frac{\theta+\phi}{2} \sin n\theta \sin k\phi \, d\theta \, d\phi \\ &= \frac{2[1-(-)^{n+k}]}{n+k} = 0 \,, \qquad n+k \text{ even} \\ &\qquad\qquad = \frac{4}{n+k} \,, \qquad n+k \text{ odd} \,, \end{aligned}$$

if the selected orthonormal system is $\{(2/\pi)^{1/2} \sin n\theta\}$.

We now consider the analytic function

$$F(z) = \sum_1^{\infty} x_n z^{n-1}$$

and suppose $\{x_n\}$ to be a square-summable sequence. A direct calculation shows that

$$\int_{-r}^{r} zF^2(z)dz = \frac{1}{2}\sum_{n,k=1}^{\infty} a_{n,k} r^{n+k} x_n x_k, \qquad 0 \leqq r < 1 \,.$$

One can also show that the quadratic form is bounded over the space of square summable sequences.

$$\begin{aligned} &\left|\int_{-r}^{r} zF^2(z)dz\right| = \left|\int_0^{\pi} r^2 e^{2i\varphi} F^2(re^{i\varphi})d\varphi\right| \\ &\leqq r^2\int_0^{\pi} |F(re^{i\varphi})|^2 d\varphi = r^2\int_0^{\pi}[(\Sigma x_n r^{n-1}\cos(n-1)\varphi)^2 \\ &+ (\Sigma x_n r^{n-1}\sin(n-1)\varphi)^2]d\varphi = \pi\Sigma x_n^2 r^{2n} \\ &\leqq \pi\Sigma x_n^2 \,. \end{aligned}$$

By letting r tend to unity one finds

$$\left|\sum_{n,k=1}^{\infty} a_{n,k} x_n x_k\right| \leqq 2\pi\sum_1^{\infty} x_n^2 \,.$$

In order for equation (2) to have a unique solution in the space of square-integrable functions it is necessary and sufficient that the quadratic form

$$Q(x) = \sum_1^\infty x_n^2 - \frac{\lambda}{2\pi} \sum_{n,k=1}^\infty a_{n,k} x_n x_k$$

be positive definite. We see that this form can be written as

$$Q(x) = \lim_{r\to1} \left[\frac{1}{\pi} \int_0^\pi r^2 \mid F(re^{i\varphi}) \mid^2 d\varphi - \frac{\lambda}{\pi} \int_{-r}^r zF^2(z) dz \right].$$

This expression must be real, and writing

$$F(z) = R(r, \varphi) e^{i\theta(r,\varphi)}$$

we obtain

$$Q(x) = \lim_{r\to1} \left[\frac{1}{\pi} \int_0^\pi r^2 R^2(r, \varphi) d\varphi - \frac{\lambda}{\pi} \int_0^\pi r^2 R^2(r, \varphi) \sin \{2\theta(r, \varphi) + 2\varphi\} d\varphi \right].$$

Evidently this is positive definite if $|\lambda| < 1$.

The preceding type of argument was first used by Fejer & F. Riesz [1], to discuss the bounds of such operators. But one can show still more, namely that the bound of the operator is not attained for any vector x. If it were $Q(x)$ would vanish, in which case

$$\sin \{2\theta(1, \varphi) + 2\varphi\} = 1 , \qquad \text{a.e.}$$

In this case the real part of the function $z^2F^2(z)$, is a harmonic function, which vanishes a.e. on $|z| = 1$. Such a harmonic function can be represented by a Poisson Integral [2]. In follows therefore that since it vanishes a.e. on $|z| = 1$ it must vanish identically and it follows that the function $zF(z)$ must also vanish identically. Therefore $Q(x)$ does not vanish for any x. One can infer from this that the homogeneous integral equation has only the trivial solution, so that the inhomogemeous equation will have a unique solution provided a solution exists even in the case $|\lambda| = 1$. But the existence of a solution depends on the nature of the inhomogeneous term. This case will be discussed in the next section.

It follows that for $|\lambda| < 1$ the integral operator is a contraction operator so that the solution can be obtained by successive iterations of the operator.

A similar analysis can be carried out for equation (4) using as an orthonormal set over $(0, \infty)$ Laguerre polynomials. The rest of the analysis is similar and details may be found in [3]. However one can approach this problem also by the use of Fourier integrals. This analysis can be found in Titchmarsh [4]. The substitutions

$$\phi = e^{\eta},\ \theta = e^{\xi},\ e^{(1/2)\eta}h(e^{\eta}) = \Phi(\eta)$$
$$,\ e^{(1/2)\xi}f(e^{\xi}) = \Psi(\xi)$$

reduces equation (4) to the form

$$\Phi(\eta) - \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} \frac{\Phi(\xi)}{\cosh\frac{1}{2}(\eta - \xi)} d\xi = \Psi(\eta) .$$

Let

$$F(\omega) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \Phi(\eta)e^{i\eta\omega}\, d\eta$$
$$G(\omega) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \Psi(\eta)e^{i\eta\omega}\, d\eta$$

and it is known that

$$\int_{-\infty}^{\infty} \frac{e^{i\eta\omega}}{\cosh\frac{1}{2}} d\eta = \frac{2\pi}{\cosh \pi\omega} .$$

One finds immediately that

$$F(\omega) = \frac{G(\omega)}{1 - \dfrac{\lambda}{\cosh \pi\omega}}$$

so that

$$\Phi(\eta) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \frac{G(\omega)e^{-i\eta\omega}}{1 - \dfrac{\lambda}{\cosh \pi\omega}} d\omega$$

From the expression it is evident that the integral equation need not have unique solutions. The solutions of the homogeneous equation must be of the form $e^{v\eta}$, where v is a zero of $\cos(\pi v) - \lambda$. Thus equation (4) has unique solutions in the space of square-integrable functions, but is only determined to within a nonintegrable term of the form $cy^{v-1/2}$, c being arbitrary.

V. The case $|\lambda| = 1$. When $|\lambda| = 1$ we have either $\alpha = \beta$ or $\alpha = -\beta$. We will consider the case $\alpha = \beta$ in detail and the other case can be reduced to this one by replacing θ by $\pi - \theta$. The given function $H(\theta)$ is to be represented in the form

$$H(\theta) = \sum_{1}^{\infty} b_n(\cos n\theta - \sin n\theta)$$

and we introduce the function

$$f(z) = \sum_1^\infty b_n z^n = u(r, \theta) + iv(r, \theta) .$$

Evidently

$$H(\theta) = u(1, \theta) - v(1, \theta) \qquad 0 < \theta < \pi .$$

Let $U(r, \theta)$ be a harmonic function defined by

$$U(r, \theta) = u(r, \theta) - v(r, \theta) ,$$

whose conjugate harmonic function is given by

$$V(r, \theta) = v(r, \theta) + u(r, \theta) .$$

Since the b_n are taken to be real u will be even and v will be odd in θ. Then

$$\begin{aligned} U(1, \theta) &= H(\theta) & 0 < \theta < \pi \\ V(1, \theta) &= H(-\theta) & -\pi < \theta < 0 . \end{aligned}$$

In order to determine the continuation of $H(\theta)$ into the interval $(-\pi, 0)$ it is necessary to determine $U(r, \theta)$ for all θ. We now define the function

$$F(z) = U + iV$$

and introduce the function

$$G(z) = e^{-i\pi/4}\left(\frac{1+z}{1-z}\right)^{1/2}$$

with the boundary values

$$\begin{aligned} G(e^{i\theta}) &= (\cot \theta/2)^{1/2} , & 0 < \theta < \pi \\ &= -i(\cot -\theta/2)^{1/2} , & -\pi < \theta < 0 . \end{aligned}$$

The function $G(z)F(z) = T(z)$ is an analytic function whose real part is defined for the whole boundary.

$$\begin{aligned} Re\, G(z)F(z) &= (\cot \theta/2)^{1/2} H(\theta) , & 0 < \theta < \pi \\ &= (\cot -\theta/2)^{1/2} H(-\theta) , & -\pi < \theta < 0 . \end{aligned}$$

Thus $T(z)$ is explicitly given by

$$T(z) = ic + \sum_1^\infty k_n z^n$$

where

$$k_n = \frac{2}{\pi}\int_0^\pi (\cot \theta/2)^{1/2} H(\theta) \cos n\theta \, d\theta$$

and c is a real, but otherwise arbitrary constant of integration. $F(z)$ is now fully determined and it follows that

$$\begin{aligned} U(1,\theta) &= Re\frac{T(e^{i\theta})}{G(e^{i\theta})} = H(\theta)\,, && 0<\theta<\pi \\ &= -(\tan -\theta/2)^{1/2}\Big[c+\sum_1^\infty k_n \sin n\theta\Big]\,, && -\pi<\theta<0\,. \end{aligned}$$

Here $U(1,\theta)$ is not uniquely specified, but if one requires that $U(1,\theta)$ be square integrable the constant c must be set equal to zero. Furthermore it is not enough to require

$$\Sigma k_n^2 < \infty\,,$$

but one also needs

$$\Sigma\,|\,k_n\,|\,ln\,n < \infty$$

in order for

$$\int_0^\pi \tan\theta/2[\Sigma k_n \sin n\theta]^2 d\theta < \infty\,.$$

The Fourier integral case be treated in an analogous fashion or by formal limiting processes.

VI. Proof of Theorems A and B. To prove Theorem A it is still necessary to show that the periodic function, which is given by $h(-\phi)$ for $-\pi<\phi<0$ and $H(\phi)$ for $0<\phi<\pi$ has the required properties. From the definitions of the coefficients d_n and e_n it follows that

$$\begin{aligned} &\frac{1}{2}\alpha^2 d_0 + \sum_1^\infty(\alpha^2 d_n \cos n\phi - \beta^2 e_n \sin n\phi \\ &\quad + \alpha\beta\, e_n \cos n\phi - \alpha\beta\, d_n \sin n\phi) \\ &\quad = \frac{\alpha^2-\beta^2}{2}H(\phi) + \frac{\alpha\beta}{2}\int_0^{*\pi} H(\theta)\cot\frac{\theta-\phi}{2}d\theta\,. \end{aligned}$$

$\int^*$ denotes the principal value of the integral. One can by the use of this summation formula now rewrite equation (2) to read

$$\begin{aligned} &\frac{\alpha^2+\beta^2}{2}h(\phi) + \frac{\alpha^2-\beta^2}{2}H(\phi) + \frac{\alpha\beta}{2\pi}\int_{-\pi}^0 h(-\theta)\cot\frac{\theta-\phi}{2}d\theta \\ &\quad + \frac{\alpha\beta}{2\pi}\int_0^{*\pi} H(\theta)\cot\frac{\theta-\phi}{2}d\theta = 0\,. \end{aligned}$$

To complete the proof one merely observes that

$$\left.\begin{aligned} H(\phi) &= \frac{a_0}{2} + \sum_1^\infty (a_n \cos n\phi + b_n \sin n\phi) \\ h(\phi) &= \frac{a_0}{2} + \sum_1^\infty (a_n \cos n\phi - b_n \sin n\phi) \end{aligned}\right\} \phi > 0$$

$$\frac{1}{2\pi}\int_{-\pi}^{*\pi} \cot\frac{\theta - \phi}{2} \begin{Bmatrix}\sin \\ \cos\end{Bmatrix} n\theta\, d\theta = \begin{Bmatrix}\cos \\ -\sin\end{Bmatrix} n\phi \,.$$

Then the previous equation reduces to

$$\alpha^2 \frac{a_0}{2} + \alpha^2 \sum_1^\infty a_n \cos n\phi - \beta^2 \sum_1^\infty b_n \sin n\phi$$
$$+ \alpha\beta \sum_1^\infty b_n \cos n\phi - \alpha\beta \sum_1^\infty a_n \sin n\phi$$
$$= 0$$

which evidently shows that

$$\alpha a_n + \beta b_n = 0 \,, \qquad n \geqq 0 \,,$$

and thus completes the proof of Theorem A.

The proof of Theorem B is completely analogous and will therefore be omitted.

The statements of the theorems can be considerably strengthened if one assumes that the original function $H(\theta)$ defined for $0 < \theta < \pi$ is continuous and bounded and the $\{c_n\}$ are such that the inhomogeneous terms in (2) and (4) are also continuous and bounded. In this case it follows from the existence of the Neumann series that the function $h(\theta)$ is also continuous and bounded for $0 < \theta < \pi$. Then the resultant periodic function is continuous and bounded at all points with the exception of points of the form $n\pi$.

References

1. L. Fejér, and F. Riesz, *Über einige funktionentheoretische Ungleichungen*, Math. Zeit, **11** (1921).

2. G. M. Golusin, *Geometrische Funktionentheorie*, Deutscher Verlag der Wissenschaften, 1957. Chapter IX.

3. W. Schmeidler, *Integralgleichungen mit Anwendungen in Physik und Technik*, Akademische Verlagsgesellschaft, (1955), 485.

4. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 2nd Ed; Oxford (1948), 310.

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