HYPONORMAL OPERATORS

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We say a bounded linear transformation T on a Hilbert space H is hyponormal if $\|Tx\| \geqq \|T^*x\|$ for all $x \varepsilon H$ or equivalently if $T^*T - TT^* \geqq 0$. The notion of hyponormality was introduced in [6], through under another name. In [8], Putnam studied properties of the operator $J_\theta = e^{i\theta}T + e^{-i\theta}T^*$, where T is hyponormal. Lemmas 1 through 6 which appear below occur as exercises in [1], and will be quoted without proof. Henceforth the term operator will mean bounded linear transformation.

LEMMA 1. *Let T be a hyponormal operator on the Hilbert space H, then $\|(T - zI)x\| \geqq \|(T^* - \bar{z}I)x\|$ for $x \varepsilon H$, i.e. $T - zI$ is hyponormal.*

LEMMA 2. *Let T be hyponormal on H; then $Tx = zx$ implies $T^*x = \bar{z}x$.*

LEMMA 3. *Let T be hyponormal on H with $Tx_1 = z_1x_1$, $Tx_2 = z_2x_2$ and $z_1 \neq z_2$. Then $(x_1, x_2) = 0$.*

LEMMA 4. *If T is hyponormal on H and $M \subset H$ is invariant under T; then $T|_M$ is hyponormal.*

LEMMA 5. *Let T be hyponormal on H, with $M \subset H$ invariant under T and let $T|_M$ be normal. Then M reduces T.*

LEMMA 6. *Let T be hyponormal on H and Let $M = \{x \varepsilon H : Tx = zx\}$. Then M reduces T and $T|_M$ is normal.*

THEOREM 1. *Let T be hyponormal on H, then $\|T\| = R_{sp}(T)$* (the *spectral radius of T*).

Proof. For $x \varepsilon H$, $\|x\| = 1$ *we have*

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \leqq \|T^*Tx\| \leqq \|T^2x\|.$$

But then $\|T\|^2 \leqq \|T^2\| \leqq \|T\|^2$ which implies $\|T\|^2 = \|T^2\|$.

Now

$$\begin{aligned} \|T^nx\|^2 &= (T^nx, T^nx) = (T^*T^nx, T^{n-1}x) \\ &\leqq \|T^*T^nx\| \cdot \|T^{n-1}x\| \leqq \|T^{n-1}x\| \cdot \|T^{n-1}x\| . \end{aligned}$$

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Thus $\|T^n\|^2 \leqq \|T^{n-1}\|\cdot\|T^{n-1}\|$, and combining this with the equality above, a simple induction argument yields $\|T^n\| = \|T\|^n$ for $n = 1, 2, \cdots$. Since $R_{sp}(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \lim_{n\to\infty} \|T\|$ the proof is finished.

COROLLARY. *The only quasi-nilpotent hyponormal operator is the transformation which is identically zero.*

THEOREM 2. *Let T be hyponormal on the Hilbert space H and let z_0 be an isolated point in the spectrum of T. Then $z_0 \varepsilon \sigma_p(T)$, the point spectrum of T.*

Proof. By Lemma 1 we may assume $z_0 = 0$. Choose $R > 0$ sufficiently small that 0 is the only point of $\sigma(T)$ contained in or on the circle $|z| = R$. Define

$$E = \int_{|z|=R} (T - zI)^{-1} dz .$$

Then E is a nonzero projection which commutes with T (see [9]; projection as used here does not necessarily mean self-adjoint).

Thus EH is invariant under T and $T|_{EH}$ is hyponormal. Also

$$\sigma(T|_{EH}) = \sigma(T) \cap \{|z| < R\}$$

by, [9] p. 421, so $\sigma(T|_{EH}) = \{0\}$.

From the last corollary we may conclude that $T|_{EH}$ is the zero transformation. In fact, it is now clear that $EH = \{x \varepsilon H : Tx = 0\}$ which implies EH actually reduces T.

THEOREM 3. If T is hyponormal on H with a single limit point in its spectrum, then T is normal.

Proof. We may assume by Lemma 1 that the limit point is 0. By Theorem 1 there exists $z_1 \varepsilon \sigma(T)$ such that $|z_1| = \|T\|$.

Let $M_1 = \{x \varepsilon H : Tx = z_1 x\}$; M_1 is not empty by theorem 2 and since M_1 reduces T, we conclude from theorem 2 that $T|_{M_1\perp}$ does not have z_1 in its spectrum. We note also by Lemma 6 that $T|_{M_1}$ is normal. We continue in this way, selecting points in $\sigma(T)$ ordered by absolute value, setting $M_i = \{x \varepsilon H : Tx = z_i x\}$.

Then $M_1 \oplus \cdots \oplus M_n$ reduces T and $T|_{|M_1\oplus\cdots\oplus M_n}$ is normal. We observe that $T|_{|M_1\oplus\cdots\oplus M_n|\perp}$ is hyponormal with its spectral radius equal to its norm. Thus, since 0 is the only limit point of $\sigma(T)$, the normal operators $T|_{M_1\oplus\cdots\oplus M_n}$ must converge to T in the uniform operator topology. Hence, T is normal.

COROLLARY 1. *If T is a hyponormal, completely continuous opera-*

tor, then T is normal.

COROLLARY 2. *If T is hyponormal* on H *with only a finite number of limit points in its spectrum; then T is normal.*

Proof. Let z_1 be a limit point of $\sigma(T)$ and choose a smooth simple closed curve G which does not intersect $\sigma(T)$ and contains only the limit point z_1 in its interior. Now define

$$E_1 = \int_G (T - zI)^{-1} dz .$$

Then T is invariant on $E_1 H$ and

$$\sigma(T) \cap [\text{Interior } G] = \sigma(T|_{E_1 H})$$

so $\sigma(T|_{E_1 H})$ can have only one limit point.

We now apply theorem 3 to $T | E_1 H$ to conclude that it is normal. Then by Lemma 5, T is reduced by $E_1 H$. We may thus turn our attention to T on $(E_1 H)_\perp$ and continue this process until the limit points are exhausted.

Theorem 1 and the first corollary to Theorem 3 have been proved independently by both T. Ando and S. Berberian, and will soon appear.

The subsequent theorem generalizes the well-known result which equates similarity equivalence of normal operators with unitary equivalence. However, there is a strong restriction on the spectrum of the operator.

THEOREM 4. *If T is a hyponormal scalar operator and $\sigma(T)$ has zero area, then T is normal.*

Proof. Since T is scalar, (see [2]), $T = QAQ^{-1}$ where Q is positive self-adjoint and A is normal. Let $A = \int z dE(z)$. For $\varepsilon > 0$, there exists a set of half-open, half-closed disjoint squares $\{R_i\}_{i=1}^k$ with each R_i of dimension $1/n \times 1/n$ such that $k/n^2 = \text{area } (U_{i=1}^k R_i) < \varepsilon$ where

$\sigma(T) \subset (U_{i=1}^k R_i)$. Now for $x_i \varepsilon E(R_i)H$; z_i the center of R_i

we have

$$\| (A - z_i I)x_i \| \leqq \left[\int_{R_i} | z - z_i |^2 d \| E(z)x_i \|^2 \right]^{1/2} \leqq \frac{1}{n} .$$

Thus

$$\begin{aligned} \| (A - z_i I)Q^2 x_i \| &= \| A^* - \bar{z}_i I)Q^2 x_i \| = \| Q(T^* - \bar{z}_i I)Qx_i \| \\ &\leqq \| (T^* - \bar{z}_i I)Qx_i \| \cdot \| Q \| \leqq \| (T - z_i I)Qx_i \| \cdot \| Q \| \end{aligned}$$

$$= \| Q(A - z_iI)x_i \| \cdot \| Q \| \leqq \| Q \|^2 \cdot \frac{1}{n} \quad \text{and}$$

$$\| Q^2(A - z_iI)x_i \| \leqq \| Q \|^2 \cdot \frac{1}{n} \,.$$

Combining these we have

$$\| (AQ^2 - Q^2A)x_i \| \leqq 2 \| Q \|^2 \cdot \frac{1}{n} \quad \text{for } x_i \varepsilon E(R_i)H.$$

Now $E(R_i)H$ is orthogonal to $E(R_j)H$ for $i \neq j$ so for $y \varepsilon H$, $\| y \| = 1$ we have

$$y = \textstyle\sum_{i=1}^k a_i x_i \text{ where } \sum_{i=1}^k | a_i |^2 = 1 \,.$$

Thus

$$\begin{aligned} \| (AQ^2 - Q^2A)y \| &= \| \textstyle\sum_{i=1}^k a_i(AQ^2 - Q^2A)x_i \| \\ &\leqq \textstyle\sum_{i=1}^k | a_i | \, \| (AQ^2 - Q^2A)x_i \| \\ &\leqq \left\{ \textstyle\sum_{i=1}^k | a_i |^2 \sum_{i=1}^k \left(2 \| Q \|^2 \cdot \frac{1}{n} \right)^2 \right\}^{1/2} = 2 \| Q \|^2 \cdot k^{1/2}/n \\ &\leqq 2 \| Q \|^2 \varepsilon^{1/2} \end{aligned}$$

implying that $AQ^2 = Q^2A$.

Noting that Q is positive we may conclude from the spectral theorem that $AQ = QA$ and thus $T = A$ which completes the proof.

The author has been unable to decide whether the condition on the area of the spectrum in the last theorem may be omitted. He would conjecture that it cannot. There is a generalization of the theorem quoted above which states that if A and B are normal operators, Q an arbitrary operator such that $AQ = QB$; then $A^*Q = QB^*$. This statement does not hold if A is normal and B semi-normal. To see this, let H be a Hilbert space with the basis $\{\phi_i\}_{i=-\infty}^{\infty}$ and define $A\phi_i = \phi_{i+1}$, all i; $B\phi_i = \phi_{i+1}$, $i \geqq 0$ $B\phi_i = 0$, $i < 0$; and $Q = B$. Then it is clear that $AQ = QB$ but $A^*Q\phi_0 = \phi_0 \neq QB^*\phi_0 = 0$.

Before going on to the next theorem we must recall some results from the literature. Let B be a normal operator of finite spectral multiplicity n, and let T be an operator commuting with B. Then there is a finite measure $v(\cdot)$ defined on Borel sets of the complex plane and vanishing outside of $\sigma(B)$ and n Borel sets $e_1, \cdots, e_n$ with e_1 the plane and $e_i \subset e_{i+1}$, such that if we define $v_i(e) = v(e \cap e_i)$ for such Borel set e; set $\hat{H} = \sum_{i=1}^n L_2(v_i)$ and define

$$\hat{B}f(s) = \hat{B}(f_1(s), \cdots, f_n(s)) = (sf_1(s), \cdots, sf_n(s)) = sf(s)$$

for $f(a) \varepsilon H$; then B and $\hat{B}$ are unitarily equivalent. Also there exist measurable functions $a_{ij}(s) i, j = 1, \cdots, n$ such that if we define

$$Tf(a) = \begin{vmatrix} a_{1,1}(s) & \cdots & a_{1,n}(s) \\ \vdots & & \vdots \\ a_{n,1}(s) & \cdots & a_{n,n}(s) \end{vmatrix} \begin{vmatrix} f_1(s) \\ \vdots \\ f_n(s) \end{vmatrix}$$

for $f(s)\varepsilon\hat{H}$; then T and T are unitarily equivalent. The foregoing may be found in [3] Chapter X theorem 5.10, in [4], [7] and in its earliest form is due to von Neumann.

Using results of Gonshor (see [5] Theorem 3 and remarks in section 6) we may define $\hat{H}$ in such a way that

$$T = \begin{vmatrix} a_{11}(s) & a_{12}(s) & \cdots & a_{1n}(s) \\ 0 & a_{22}(s) & \cdots & a_{2n}(s) \\ \vdots & 0 & & \vdots \\ 0 & & & a_{nn}(s) \end{vmatrix}$$

or roughly speaking $\hat{T}$ has super diagonal form. In what follows we will identify $\hat{T}$ with T and $\hat{B}$ with B.

THEOREM 5. *Let T be a hyponormal operator with $T^n = B$ where n is a positive integer and B is a normal operator; then T is normal.*

Proof. For $x_0\varepsilon H$, let $M = clm[B^iB^{*j}T^kx_0]$ (the closed linear manifold spanned by the iterates) for $k = 0, 1, \cdots, n-1$; $i, j = 0, 1, 2, \cdots$.

Then M reduces B and B has spectral multiplicity n on M (we will assume x_0, Tx_0, $\cdots$, $T^{n-1}x_0$ are linearly independent). Also M is invariant under T since $TB^iB^{*j}T^kx_0 = B^iB^{*j}T^{k+1}x_0$ and invariance holds under closure. The Fuglede theorem is used in obtaining the last equality.

Let us now consider $T|_M$ which we may write as

$$\begin{vmatrix} a_{11}(s) & \cdots & a_{1n}(s) \\ 0 & \cdot & \\ & \cdot & \\ & \cdot & a_{nn}(s) \end{vmatrix} .$$

Then for the vector $f_1 = (x_{\sigma(B)}, 0, \cdots, 0)$, where $x_{\sigma(B)}$ is the characteristic function of $\sigma(B)$,

we have

$$\| T|_M f_1 \|^2 = \int_{\sigma(B)} | a_{11}(s) |^2 dv_1(s)$$

and

$$\| (T|_M)^* f_1 \|^2 = \sum_{j=1}^n \int_{\sigma(B)} | a_{1j}(s) |^2 dv_j(s) .$$

But $\| T|_M f_1 \| \geqq \| (T|_M)^* f_1 \|$ since the restriction of a hyponormal operator to an invariant subspace is hyponormal. Thus $a_{1j}(s) = 0$ a.e. (v_j) for $j = 2, 3, \cdots, n$. Continuing this argument, we find that $a_{ij}(s) = 0$ a.e. (v_j) for $i \neq j$. We conclude from this that $T|_M$ is normal, or $\| T|_M y \| = \| (T|_M)^* y \|$ for $y \varepsilon M$. Therefore,

$$\| Tx_0 \| = \| T|_M x_0 \| = \| (T|_M)^* x_0 \| \leqq \| T^* x_0 \| .$$

But this with hyponormality implies that $\| Tx_0 \| = \| T^* x_0 \|$. Since x_0 is arbitrary T must be normal.

COROLLARY. *If T is hyponormal and commutes with a normal operator having finite spectral multiplicity; then T is normal.*

EXAMPLE. Let $\{\varphi_i\} i = -\infty$ be a basis for the Hilbert space H and define $T\varphi_i = a_i \varphi_{i+1}$. Then if $|a_i| \leqq |a_{i+1}|$, all i, T is hyponormal. T will be normal if and only if $|a_i| = |a_{i+1}|$ for all i. If $|a_k| = |a_{k+1}|$ for some fixed k and $|a_j| \neq |a_k|$ for some $j > k$ then T will not be subnormal. Another example of hyponormal operator which is not subnormal is given in [6].

REFERENCE

1. S. K. Berberian, *Introduction to Hilbert Space*, New York, Oxford University Press 1961.

2. N. Dunford, *Spectral operators*, Pacific J. Math., **4** (1954), 321-354.

3. N. Dunford and J. T. Schwartz, *Linear Operators II*, New York, Interscience Publishers, to be published.

4. S. R. Foguel, *Normal operators of finite multiplicity*, Comm. Pure Appl. Math, **11** (1958), 297-313.

5. H. Gonshor, *Spectral theory for a class of non-normal operators*, Can. J. Math, **8** (1956), 449-461.

6. P. R. Halmos, *Normal dilations and extensions of operators*, Summa Bras. Math, **2** (1950), 124-134.

7. M. A. Naimark and S .V. Fomin, *Continuous direct sums of Hilbert spaces and some of their applications*, Amer. Math, Soc. Trans., Vol. 5, series 2, 35-65.

8. C. R. Putnam, *On semi-normal operators*, Pacific J. Math., **7** (1957), 1649-1652.

9. F. Riesz and B. Sz.-Nagy, *Functional Analysis*, New York, Frederick Ungar, 1955.

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