A NOTE ON ABELIAN GROUP EXTENSIONS

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In Exercise 21 page 248 of his book Abelian Groups L. Fuchs asks for a proof of the following

THEOREM. *If A is a torsion-free and C a torsion group, then* $\mathrm{Ext}(A, C)$ *is either* 0 *or contains an element of infinite order.*

Unfortunately the hint given with the exercise leads only to the conclusion that every countable subgroup of A is free. Professor Fuchs has informed me that he meant to assume A countable. The purpose of this note is to prove this theorem.

LEMMA. *If $C_1, C_2, \cdots$ is a sequence of abelian groups, ΠC_i their direct product and ΣC_i their direct sum, then* $\mathrm{Ext}(A, \Pi C_i/\Sigma C_i) = 0$ *for all torsion-free groups A.*

Proof. A special case of this lemma with all the $C_i = Z$ the group of integers is a consequence of Theorem 1 of [1]. The proof of the special case given in [4] makes no use of the fact that $C_i = Z$. This proof will be sketched here. It is enough to prove the case in which A is the rational numbers. Since $\mathrm{Ext}(A, \Pi C_i/\Sigma C_i)$ is a homomorphic image of $\mathrm{Ext}(A, \Pi C_i)$ we must show that each extension $0 \to \Pi C_i \to E \to A \to 0$ splits over $\Pi C_i/\Sigma C_i$, i.e., that there is a map $f\colon E \to \Pi C_i/\Sigma C_i$ whose restriction to ΠC_i is the canonical projection. With A the rationals we choose elements $e^1, e^2, \cdots$ in E such that e^n maps onto $1/n!$ modulo ΠC_i. Then E is generated by ΠC_i and the e's with relations

$$e^n = (n + 1)e^{n+1} + c^n \qquad n = 1, 2, \cdots$$

where $c^n \in \Pi C_i$. We choose $b^n \in \Sigma C_i$ such that the first n coordinates of $c^n + b^n$ are 0 and put

$$x^n = \textstyle\sum_{k \geqq n}(k!/n!)(c^k + b^k) .$$

Then

$$x^n = (n + 1)x^{n+1} + c^n + b^n$$

and we can define f to be the projection on ΠC_i and by $f(e^n) = x^n + \Sigma C_i$.

PROPOSITION. If C is the direct sum of infinitely many copies of

Received April 25, 1962. This work was supported by the National Science Foundation research grant NSF-G 11098.

D and if A is torsion-free with $\operatorname{Ext}(A, D) \neq 0$, then $\operatorname{Ext}(A, C)$ has an element of infinite order.

Proof. Since D is a direct summand of C we have $\operatorname{Ext}(A, C) \neq 0$. The sequence

$$\operatorname{Ext}(A, \Sigma C) \to \operatorname{Ext}(A, \Pi C) \to \operatorname{Ext}(A, \Pi C/\Sigma C) \to 0$$

is exact where ΣC is the direct sum and ΠC the direct product of countably many copies of C. By the lemma $\operatorname{Ext}(A, \Pi C/\Sigma C) = 0$ so that the left-most map in the sequence is an epimorphism. Since A is torsion-free $\operatorname{Ext}(A, C)$ is divisible and hence has elements of arbitrarily large finite order if it has nonzero elements of finite order at all. Hence $\operatorname{Ext}(A, \Pi C) \cong \Pi \operatorname{Ext}(A, C)$ has an element of infinite order. It follows that $\operatorname{Ext}(A, \Sigma C)$ also has an element of infinite order. Since C is the direct sum of infinitely many copies of D we have $\Sigma C \cong C$ so that $\operatorname{Ext}(A, \Sigma C) \cong \operatorname{Ext}(A, C)$ proving the proposition.

Now to prove the theorem we suppose that A is torsion-free, C is torsion and that $\operatorname{Ext}(A, C)$ is a nonzero torsion group. Then $\operatorname{Ext}(A, C)$ has a nonzero p-primary component for some prime p. Since $C = C' \oplus E$ where C' is the p-primary component of C and E is the sum of the other primary components we have

$$\operatorname{Ext}(A, C) = \operatorname{Ext}(A, C') \oplus \operatorname{Ext}(A, E) .$$

Multiplication by p is an automorphism of E, hence also an automorphism of $\operatorname{Ext}(A, E)$. It follows that $\operatorname{Ext}(A, C')$ is a nonzero torsion group. Hence in proving the theorem we may assume that C is p-primary.

In [3] it was shown that, for A torsion-free and C p-primary,

$$\operatorname{Ext}(A, C) \cong \operatorname{Ext}(A, M)$$

where M is a direct sum of copies of $\Sigma Z/p^n Z$, the number of copies being equal to the final rank of C. If C has bounded order, then $\operatorname{Ext}(A, C) = 0$ for all torsion-free groups A. Otherwise the final rank of C is infinite. This last case is the one to be considered. Then M is the direct sum of countably many copies of itself and the proposition shows that $\operatorname{Ext}(A, M)$ is either 0 or has an element of infinite order.

The referee has pointed out that a stronger form of the lemma in this paper has been proved by A. Hulanicki (Bull. Acad. Pol. Sci. Ser. Sci. Math. Astr. Phys., 10 (1962), 77–80.) He showed that each element of infinite height in $\Pi C_i/\Sigma C_i$ is in the maximal divisible subgroup, hence this group is algebraically compact.

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