

NON-EXISTENCE OF ALMOST-COMPLEX STRUCTURES ON QUATERNIONIC PROJECTIVE SPACES

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1. **Introduction.** In [3] F. Hirzebruch proved that the n -dimensional quaternionic projective space (which we denote by $P_n(\mathbf{H})$) does not admit any almost structure in case $n \neq 2$ or 3. According to Hirzebruch's lecture at the 1958 International Congress [5], Milnor has since proved that $P_2(\mathbf{H})$ and $P_3(\mathbf{H})$ do not admit almost complex structures. At the time of this writing, Milnor's proof has not yet been published.

It is the purpose of this note to give a short proof of this theorem of Milnor's making use of the theory of the ring $K(X)$ of complex vector bundles over a space X due to Atiyah and Hirzebruch together with certain facts that are readily available in the literature. From the brief description given in Hirzebruch's lecture (loc. cit.) it seems that our method is quite different from Milnor's. Our method may be applicable in other cases to prove the existence or nonexistence of almost complex structures on a manifold.

2. **Summary of some known facts.** We will make use of the following results:

(a) The cohomology ring $H^*(P_n(\mathbf{H}), \mathbb{Z})$ is a truncated polynomial ring generated by a 4-dimensional cohomology class u and subject to the single relation $u^{n+1} = 0$.

(b) Let τ_n denote the tangent bundle to $P_n(\mathbf{H})$. The total Pontrjagin class of τ_n is given by the formula

$$(1) \quad p(\tau_n) = (1 + u)^{2n+2}(1 + 4u)^{-1}$$

for appropriate choice of the generator u (Borel and Hirzebruch [2], 15.5 or Hirzebruch [3]).

(c) We will use the following notation: If ξ is a real n -plane bundle, then $\xi \otimes \mathbb{C}$ denotes its complexification, while if ξ is a complex n -plane bundle, then $\xi_{\mathbb{R}}$ denotes the real $2n$ -plane bundle obtained by "restriction of coefficients" to the reals. Also, ξ^* denotes the complex conjugate bundle. We then have the following relation for any complex vector bundle ξ :

$$(2) \quad \xi_{\mathbb{R}} \otimes \mathbb{C} = \xi + \xi^* \quad (\text{Whitney sum}).$$

(see Hirzebruch, [4], p. 68, proof of Theorem 4.5.1). Moreover, for any

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real vector bundle η ,

$$(3) \quad p_i(\eta) = (-1)^i c_{2i}(\eta \otimes C),$$

(Hirzebruch [4], p. 67).

(d) We will also need to make use of the properties of the ring $K(X)$ as summarized for example in § 2 of Atiyah and Todd [1]. We will use their notation and results without any further comment.

3. The ring $K(P_n(H))$. By proposition 2.3 of [1] the Chern character

$$ch: K(P_n(H)) \rightarrow H^*(P_n(H), \mathbb{Q})$$

is a monomorphism.

LEMMA 1. *The image of this homomorphism, denoted by $chP_n(H)$, is the subring of $H^*(P_n(H), \mathbb{Q})$ generated by*

$$2 \cosh \sqrt{v} = 2(1 + v/2! + v^2/4! + \dots + v^n/(2n)!)$$

where v is an appropriately chosen generator of

$$H^*(P_n(H), \mathbb{Z}) \subset H^*(H, \mathbb{Q}).$$

Proof. Consider the well-known principal fibre bundle $\pi_n: S^{4n+3} \rightarrow P_n(H)$ with group $Sp(1)$; let η_n denote the associated bundle with fibre a quaternionic vector space of dimension 1. We assert that the total symplectic Pontrjagin class of this bundle is

$$e(\eta_n) = 1 + e_1(\eta_n) = 1 + v$$

where v is an appropriately chosen generator of $H^*(P_n(H), \mathbb{Z})$. This follows from the fact that $\pi_n: S^{4n+3} \rightarrow P_n(H)$ is a universal bundle for the group $Sp(1)$ (up to the dimension $4n+2$), and that the integral cohomology ring of the classifying space for $Sp(1)$ is a polynomial ring generated by the symplectic Pontrjagin class e_1 (see Borel and Hirzebruch, [2], § 9.6).

Let ξ_n denote the complex vector bundle obtained from η_n by “restriction of coefficients” to the complex numbers; the associated principal bundle is a $U(2)$ -bundle which is the extension of $\pi_n: S^{4n+3} \rightarrow P_n(H)$ under the standard inclusion $Sp(1) \subset U(2)$. By § 9.6 of [2],

$$e_1(\eta_n) = -c_2(\xi_n),$$

hence

$$c(\xi_n) = 1 - v.$$

The Chern character (see [2], § 9.1) of ξ_n is

$$ch(\xi_n) = e^{y_1} + e^{y_2}$$

where

$$(1 + y_1)(1 + y_2) = c(\xi_n) = 1 - v.$$

Hence

$$y_1 + y_2 = 0, \quad y_1 y_2 = -v.$$

From this we conclude that

$$\begin{aligned} y_1 &= -y_2, \quad v = y_1^2, \\ y_1 &= \sqrt{v}, \quad y_2 = -\sqrt{v}, \\ ch(\xi_n) &= \exp(v) + \exp(-v) = 2 \cosh \sqrt{v}. \end{aligned}$$

Hence $ch(P_n(H))$ contains the subring generated by $2 \cosh \sqrt{v}$; it remains to show that it is exactly equal to this subring. This is done by induction on n exactly as in the proof of Proposition 3.1 of [1]. The details may be left to the reader.

Note that $ch P_n(H)$ may be equivalently described as the subring of $H^*(P_n(H), \mathbb{Q})$ generated by

$$w = 2 \cosh \sqrt{v} - 2 = v + 2v^2/4! + \cdots + 2v^n/(2n)!.$$

For many purposes this description of $ch P_n(H)$ is more convenient; note that $w^{n+1} = 0$, and $\{1, w, w^2, \dots, w^n\}$ is a basis of $ch P_n(H)$ over the integers.

Lemma 1 and equation (1) above are both stated in terms of ‘‘appropriately’’ chosen generators, v and u respectively, of the infinite cyclic group $H^4(P_n(H), \mathbb{Z})$. Therefore $u = \pm v$. We assert that $u = +v$. To prove this, it obviously suffices to show that $ch(\tau_n \otimes \mathbb{C})$ belongs to the subring of $H^*(P_n(H), \mathbb{Q})$ generated by $2 \cosh \sqrt{u}$, but that it does not belong to the subring of $H^*(P_n(H), \mathbb{Q})$ generated by $2 \cosh \sqrt{-u}$. This we will now do by an essentially straightforward, but rather lengthy, computation.

LEMMA 2. *The Chern character of $\tau_n \otimes \mathbb{C}$ is given by*

$$ch(\tau_n \otimes \mathbb{C}) = (4n + 4) \cosh \sqrt{u} - 4 \cosh^3 \sqrt{u}.$$

Proof. It follows from equations (1) and (3) that the total Chern class of $\tau_n \otimes \mathbb{C}$ is given by

$$c(\tau_n \otimes \mathbb{C}) = (1 - u)^{2n+2} (1 - 4u)^{-1}.$$

To compute the Chern character of $\tau_n \otimes \mathbb{C}$, we may proceed as follows: Write the total Chern class as a formal product

$$c(\tau_n \otimes \mathbb{C}) = \prod_{i=1}^{4n} (1 + x_i),$$

where the x_i 's have degree 2. Then

$$ch(\tau_n \otimes C) = \sum_{i=1}^{4n} \exp(x_i),$$

To actually carry out the computation, take the logarithm of both sides of the equation

$$\prod (1 + x_i) = (1 - u)^{2n+2}(1 - 4u)^{-1},$$

and use the MacLaurin series expansion of $\log(1 + z)$ and $\log(1 - z)$. The result is

$$(4) \quad \sum_{k>0} (-1)^{k+1} s_k / k = - \sum_{k>0} (2n + 2 - 4^k) u^k / k,$$

where

$$s_k = \sum_{i=1}^{4n} x_i^k.$$

Since each x_i is of degree 2, while u is of degree 4, we conclude from equation (4) that

$$\begin{aligned} s_k &= 0 && \text{for } k \text{ odd,} \\ s_{2k} &= 2(2n + 2 - 4^k) u^k. \end{aligned}$$

Therefore

$$\begin{aligned} ch(\tau_n \otimes C) &= \sum_{i=1}^{4n} \exp(x_i) = \sum_{i=1}^{4n} \sum_k x_i^k / k! \\ &= 4n + \sum_{k>0} s_k / k! \\ &= 4n + \sum_{k>0} 2(2n + 2 - 4^k) u^k / (2k)! \\ &= (4n + 4) \cosh \sqrt{u} - 2 \cosh \sqrt{4u} - 2 \\ &= (4n + 4) \cosh \sqrt{u} - 4 \cosh^2 \sqrt{u}, \end{aligned}$$

as was to be proved.

It is obvious from this formula that $ch(\tau_n \otimes C)$ belongs to the subring of $H^*(P_n(H), \mathbb{Q})$ generated by $2 \cosh \sqrt{u}$; we must now prove that $ch(\tau_n \otimes C)$ does *not* belong to the subring generated by

$$2 \cosh \sqrt{-u} = 2 \cos \sqrt{u}.$$

Assume the contrary; then there exist integers a_0, a_1, \dots, a_n such that

$$ch(\tau_n \otimes C) = \sum_{k=0}^n a_k (2 \cos \sqrt{u} - 2)^k,$$

that is,

$$\begin{aligned}
& 2 \sum_{k=1}^n (2n + 2 - 4^k) u^k / (2k)! \\
&= \sum_{k=1}^n a_k (-u + 2u^2/4! - 2u^3/6! + \cdots \pm 2u^n/(2n)!)^k.
\end{aligned}$$

If we compare coefficients of u , u^2 , u^3 , and u^4 in this equation, we obtain

$$\begin{aligned}
a_1 &= -2n + 2, \\
a_2 &= (n - 4)/3, \\
a_3 &= (7 - n)/18, \\
a_4 &= (5n - 47)/504.
\end{aligned}$$

For $n = 2$ or 3 , a_2 is not an integer; for $n < 7$, a_3 is not an integer; and for any value of n , it is impossible that both a_3 and a_4 are integers. For, if a_3 is an integer, then

$$n \equiv 7 \pmod{18} \text{ or } 5n \equiv 35 \pmod{18},$$

while if a_4 is an integer, then

$$5n \equiv 47 \pmod{18}$$

which is a contradiction.

This completes the proof that $u = +v$.

4. Proof of the theorem. Assume τ_n admits an almost complex structure θ_n ; we will show that this leads to a contradiction.

θ_n is a complex $2n$ -plane bundle over $P_n(\mathbf{H})$ such that $\tau_n = \theta_{nR}$. Then by equation (2)

$$\tau_n \otimes C = \theta_{nR} \otimes C = \theta_n + \theta_n^*.$$

Next, recall that

$$ch_i(\theta_n^*) = (-1)^i ch_i(\theta_n)$$

where ch_i denotes the component of ch of degree $2i$. However, since the base space of the bundle θ_n is $P_n(\mathbf{H})$,

$$ch(\theta_n^*) = ch(\theta_n).$$

Therefore

$$ch(\tau_n \otimes C) = 2ch(\theta_n).$$

It follows from Lemma 2 that

$$\begin{aligned}
ch(\theta_n) &= (2n + 2) \cosh \sqrt{u} - 2 \cosh^2 \sqrt{u} \\
&= -\frac{1}{2} (2 \cosh \sqrt{u} - 2)^2 + (n - 1) (2 \cosh \sqrt{u} - 2) + 2n.
\end{aligned}$$

This is the desired contradiction, since the Chern character of any complex

vector bundle is an *integral* linear combination of the powers of $w = (2 \cosh \sqrt{u} - 2)$.

REMARK. We have actually proved a slightly stronger theorem, in that we have shown that for any integer $n > 1$, $P_n(\mathbf{H})$ does not admit a "generalized almost complex structure" as defined by Hirzebruch in his lecture [5]. As Hirzebruch remarks, this can be proved easily by induction on n , once the case $n = 2$ is taken care of. However, the above computations of $ch(P_n(\mathbf{H}))$ and $ch(\tau_n \otimes C)$ may be of some independent interest.

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