A GENERALIZED SOLUTION OF THE BOUNDARY VALUE PROBLEM FOR $y''=f(x, y, y')$.

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1. Introduction. In a paper in 1922 Perron [16] presented a new method of attacking the boundary value problem for Laplace's equation. This method consisted of employing the existence of solutions of the boundary value problem for small circles and the existence in the large of subharmonic and superharmonic functions to demonstrate the existence of a solution of the boundary value problem in the large. Since then Perron's methods have been generalized and applied to more general elliptic partial differential equations, for example, Tautz [17], Beckenbach and Jackson [2], Inoue [11], Jackson [12].

Subharmonic functions bear the same relationship to harmonic functions that convex functions bear to solutions of $y''(x)=0$. In a paper in 1937 Beckenbach [3] introduced the idea of generalized convex functions. Since then a number of other mathematicians, for example, Bonsall [4], Green [7, 8], and Peixoto [15], have studied subfunctions with respect to solutions of second order ordinary differential equations. These subfunctions are special cases of Beckenbach's generalized convex functions and, if they have sufficient smoothness, are solutions of second order differential inequalities. Solutions of second order differential inequalities appear in many papers concerned with the existence of a solution of the boundary value problem for the equation

$$y''=f(x, y, y') , \tag{1}$$

for example, Nagumo [14], Babkin [1]. However, the Perron method of systematically exploiting the properties of subfunctions and superfunctions in studying the boundary value problem does not appear to have been applied to equation (1). This paper consists of such a study.

In § 2 we list some properties of solutions of (1) most of which are known. In § 3 we define subfunctions and superfunctions and give some of the properties of these functions that will be needed in the subsequent sections. Most of these properties are analogues of classical properties of convex functions as given for example in [9; Chapt. III]. In § 4 the Perron method is used to obtain a "generalized" solution of the boundary value problem. Finally, in § 5 some conditions are given which are sufficient to guarantee that the "generalized" solution of § 4

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is the solution of the boundary value problem in the usual sense.

2. Some basic lemmas. In this section we shall list the basic results concerning equation (1) which will be required in the subsequent sections.

Let R be the region in three dimensional Euclidean space defined by

$$R \equiv [(x, y, z): a \leqq x \leqq b, |y| + |z| < +\infty]$$

where a and b are finite. We shall assume throughout this paper that $f(x, y, z)$ is continuous on R. Various other assumptions will be made from time to time concerning $f(x, y, z)$. The first of these are as follows:

A_1: $f(x, y, z)$ is a nondecreasing function of y for each fixed x and z.

A_2: $f(x, y, z)$ satisfies a Lipschitz condition with respect to y and z on each compact subset of R.

However, unless we specifically state these or other assumptions we will be assuming only the continuity of $f(x, y, z)$.

By a solution of the boundary value problem,

$$y'' = f(x, y, y')$$
$$y(x_1) = y_1\,, \qquad y(x_2) = y_2$$

where $a \leqq x_1 < x_2 \leqq b$, we shall mean a function $y(x)$ which is of class $C^{(2)}$ and is a solution of (1) on (x_1, x_2), which is continuous on $[x_1, x_2]$, and which assumes the given boundary values at x_1 and x_2.

We shall also be interested in the special case of equation (1) when y' is not present, that is, equation

$$y'' = f(x, y)\,. \tag{2}$$

We shall always assume $f(x, y)$ is continuous on

$$R^* \equiv [(x, y): a \leqq x \leqq b, |y| < +\infty]\,.$$

LEMMA 1. *Given any $M > 0$ and $N > 0$ there is a $\delta(M, N) > 0$ such that the boundary value problem*

$$y'' = f(x, y, y')$$
$$y(x_1) = y_1\,, \qquad y(x_2) = y_2$$

has a solution of class $C^{(2)}$ on $[x_1, x_2]$ for any points (x_1, y_1) and (x_2, y_2) with $x_1, x_2 \in [a, b]$, $|x_1 - x_2| \leqq \delta$, $|y_1| \leqq M$, $|y_2| \leqq M$ and $|(y_1 - y_2)/(x_1 - x_2)| \leqq N$.

LEMMA 2. *Given any $M > 0$ there is a $\delta(M) > 0$ such that the boundary value problem*

$$y'' = f(x, y)$$
$$y(x_1) = y_1\ , \qquad y(x_2) = y_2$$

has a solution of class $C^{(2)}$ *on* $[x_1, x_2]$ *for any points* (x_1, y_1) *and* (x_2, y_2) *with* $x_1, x_2 \in [a, b]$, $|x_1 - x_2| \leqq \delta$, $|y_1| \leqq M$, *and* $|y_2| \leqq M$.

LEMMA 3. *Let* $M > 0$, $N > 0$ *be fixed and let* $\delta(M, N)$ *be as in Lemma* 1. *Then given any* $\varepsilon > 0$ *there is an* η, $0 < \eta \leqq \delta(M, N)$, *such that, for any points* (x_1, y_1) *and* (x_2, y_2) *with* $x_1, x_2 \in [a, b]$, $|x_1 - x_2| \leqq \eta$, $|y_1| \leqq M$, $|y_2| \leqq M$, *and* $|(y_1 - y_2)/(x_1 - x_2)| \leqq N$, *there is a solution* $y(x)$ *of* (1) *of class* $C^{(2)}$ *on* $[x_1, x_2]$ *with* $y(x_1) = y_1$, $y(x_2) = y_2$, *and with*

$$|y(x) - \omega(x)| \leqq \varepsilon$$

and

$$|y'(x) - \omega'(x)| \leqq \varepsilon$$

on $[x_1, x_2]$ *where* $\omega(x)$ *is the linear function with* $\omega(x_1) = y_1$ *and* $\omega(x_2) = y_2$. *An analogous statement with* N *and* $|(y_1 - y_2)/(x_1 - x_2)| \leqq N$ *omitted is valid with respect to solutions of* (2).

Proof. Lemmas 1 and 2 can be proved by using the Schauder-Tychonoff fixed point theorem [6; p. 456]. Let

$$G(x, t) \equiv \begin{cases} \dfrac{(x_1 - t)(x_2 - x)}{x_2 - x_1} & \text{on } x_1 \leqq t \leqq x \leqq x_2 \\ \dfrac{(x_1 - x)(x_2 - t)}{x_2 - x_1} & \text{on } x_1 \leqq x \leqq t \leqq x_2\ . \end{cases}$$

Let B be the Banach space $C^{(1)}[x_1, x_2]$ with norm $\|y\| \equiv \max|y(x)| + \max|y'(x)|$. For a function which satisfies a Hölder condition with exponent $0 < \alpha < 1$ on $[x_1, x_2]$ let

$$h_\alpha(g) \equiv \sup\left[\frac{|g(r_1) - g(r_2)|}{|r_1 - r_2|^\alpha} : r_1, r_2 \in [x_1, x_2], r_1 \neq r_2\right].$$

Let K be the set of all functions $u(x)$ in B which are such that $u'(x)$ satisfies a Hölder condition with exponent α on $[x_1, x_2]$, $u(x_1) = u(x_2) = 0$, and $\|u\| + h_\alpha(u') \leqq \max[M, N]$. Then K is a compact convex subset of B. It can be shown that there is a $\delta(M, N) > 0$ such that the mapping $F(u) = w$ defined by

$$w(x) = \int_{x1}^{x_2} G(x, t) f(t, u(t) + \omega(t), u'(t) + \omega'(t))dt$$

is a continuous mapping of K into itself provided $|x_1 - x_2| \leqq \delta(M, N)$, $|y_1| \leqq M$, $|y_2| \leqq M$, $|(y_1 - y_2)/(x_1 - x_2)| \leqq N$, and $\omega(x)$ is the linear function with $\omega(x_1) = y_1$ and $\omega(x_2) = y_2$. If $u_0(x)$ is the fixed point of the mapping, $y(x) \equiv u_0(x) + \omega(x)$ is a solution of the boundary value problem. Lemma 3 is an immediate consequence of the boundedness of $f(t, u(t) + \omega(t), u'(t) + \omega'(t))$ for $a \leqq t \leqq b$, $u \in K$, $|\omega(t)| \leqq M$, and $|\omega'(t)| \leqq N$.

LEMMA 4. *If $y_0(x)$ is a solution of* (1) *of class $C^{(2)}$ on $[x_1, x_2] \subset [a, b]$ and if $a < x_2 < b$, then there is a $\delta > 0$ such that $x_2 + \delta \leqq b$ and a solution $y(x)$ of* (1) *of class $C^{(2)}$ on $[x_1, x_2 + \delta]$ with $y(x) \equiv y_0(x)$ on $[x_1, x_2]$. A similar statement applies at x_1 in case $a < x_1$.*

Proof. This is an immediate consequence of well known results [5; p. 15] concerning continuation of solutions.

LEMMA 5. *If $f(x, y, z)$ satisfies condition A_2 and if $y_0(x)$ is a solution of* (1) *of class $C^{(2)}$ on $[x_1, x_2] \subset [a, b]$, then for all sufficiently small $|m|$ there are solutions $y(x)$ of* (1) *of class $C^{(2)}$ on $[x_1, x_2]$ satisfying $y(x_1) = y_0(x_1)$, $y'(x_1) = y_0'(x_1) + m$, and $|y(x) - y_0(x)| \leqq |m| e^{2k(x_2 - x_1)}$ on $[x_1, x_2]$ where k is a constant independent of m. A similar statement applies if the change in slope is made at x_2 instead of x_1.*

Proof. This lemma is an immediate consequence of well known results [5; p. 22] concerning the continuity of solutions with respect to initial conditions.

3. Subfunctions and superfunctions.

In this section we define and develop some of the properties of subfunctions and superfunctions with respect to the solutions of an arbitrary but fixed equation (1). These definitions and properties will of course apply also to equation (2), however, Theorem 2 will apply only to equation (2).

We shall use a capital letter I to represent a subinterval of the basic interval $[a, b]$. I may be open, closed, or half-open. $\bar{I}$ is the closure of I, I^0 the interior of I, and I' the complement of I.

DEFINITION 1. A real valued function s defined on I is said to be a subfunction on I in case $s(x) \leqq y(x)$ on $[x_1, x_2]$ for any $[x_1, x_2] \subset I$ and any solution y of (1) on $[x_1, x_2]$ with $s(x_1) \leqq y(x_1)$ and $s(x_2) \leqq y(x_2)$.

DEFINITION 2. A real valued function S defined on I is said to be a superfunction on I in case $S(x) \geqq y(x)$ on $[x_1, x_2]$ for any $[x_1, x_2] \subset I$ and any solution y of (1) on $[x_1, x_2]$ with $S(x_1) \geqq y(x_1)$ and $S(x_2) \geqq y(x_2)$.

We shall state our results in terms of subfunctions with obvious analogous results, which we shall not bother to state, holding for superfunctions. When we wish to refer to a result concerning superfunctions we shall simply refer to the corresponding statement concerning subfunctions.

THEOREM 1. *If s is a subfunction on I, then the right-hand and left-hand limits, $s(x_0 + 0)$ and $s(x_0 - 0)$, exist at every $x_0 \in I^0$ and the appropriate one-sided limits exist at the endpoints of $\bar{I}$. These limits may be infinite.*

Proof. It will suffice to consider one case. Assume that $s(x_0 - 0)$ does not exist at $x_0 \in I^0$. Then there exist finite real numbers c and d such that

$$\liminf_{x \to x_0 -} s(x) \leqq c < d \leqq \limsup_{x \to x_0 -} s(x)$$

We can pick two sequences $\{a_n\}_{n=1}^{\infty} \subset I$ and $\{b_n\}_{n=1}^{\infty} \subset I$ with the following properties:

(i) $\lim a_n = \lim b_n = x_0$,

(ii) $a_n < b_n < a_{n+1}$ for each $n \geqq 1$,

and

(iii) $\lim s(a_n) = \limsup_{x \to x_0 -} s(x)$

and

$$\lim s(b_n) = \liminf_{x \to x_0 -} s(x).$$

Let $\varepsilon = (d - c)/4$ and pick $N_1 > 0$ such that

$$s(a_n) > d - \varepsilon ,$$

and

$$s(b_n) < c + \varepsilon$$

for $n \geqq N_1$.

By Lemmas 1, 2, and 3 there is an $n_0 \geqq N_1$ such that the boundary value problem

$$y'' = f(x, y, y')$$
$$y(b_{n_0}) = y(b_{n_0+1}) = \frac{c + d}{2}$$

has a solution $y(x)$ with $|y(x) - (c + d)/2| < \varepsilon$ on $[b_{n_0}, b_{n_0+1}]$. Then since s is a subfunction,

$$s(b_{n_0}) < c + \varepsilon < \frac{c+d}{2} = y(b_{n_0}) \, ,$$

and

$$s(b_{n_0+1}) < c + \varepsilon < \frac{c+d}{2} = y(b_{n_0+1}) \, ,$$

it follows that we must have $s(a_{n_0+1}) \leqq y(a_{n_0+1})$. However,

$$s(a_{n_0+1}) > d - \varepsilon = \frac{c+d}{2} + \varepsilon > y(a_{n_0+1}) \, .$$

From this contradiction we conclude that $s(x_0 - 0)$ exists.

COROLLARY 1. *If s is a subfunction on I, then $s(x_0) \leqq \max [s(x_0 + 0), s(x_0 - 0)]$ at every $x_0 \in I^0$.*

Proof. If either $s(x_0 + 0) = +\infty$ or $s(x_0 - 0) = +\infty$, the given inequality obviously holds. If $s(x_0 + 0) < +\infty$ and $s(x_0 - 0) < +\infty$, the same type of argument as was used in proving the Theorem can be employed to show that $s(x_0) > \max [s(x_0 + 0), s(x_0 - 0)]$ is not possible. Since $s(x)$ has a finite real value at every point of I, this shows that we cannot have simultaneously $s(x_0 + 0) = -\infty$ and $s(x_0 - 0) = -\infty$ at any $x_0 \in I^0$.

COROLLARY 2. *If s is a bounded subfunction on I, then s has at most a countable number of discontinuities on I.*

Proof. This is a known consequence of the existence of onesided limits everywhere on I, for example, see [10; p. 300].

THEOREM 2. *If s is bounded function on I and is a subfunction with respect to the solutions of a differential equation* (2), *then s is continuous on I^0.*

Proof. By Theorem 1 $s(x_0 - 0)$ and $s(x_0 + 0)$ exist at every $x_0 \in I^0$ and $s(x_0) \leqq \max [s(x_0 + 0), s(x_0 - 0)]$. To be specific assume $s(x_0 - 0) \geqq s(x_0 + 0)$. First assume $s(x_0) < s(x_0 - 0)$ and let $|s(x)| \leqq M$ on I. Then by Lemma 2 for $[x_1, x_0] \subset I$ and $|x_1 - x_0| \leqq \delta(M)$ the boundary value problem

$$y'' = f(x, y)$$
$$y(x_1) = s(x_1) \, , \qquad y(x_0) = s(x_0)$$

has a solution $y(x)$. Then, since $s(x) \leqq y(x)$ on $[x_1, x_0]$, $s(x_0 - 0) \leqq y(x_0 - 0) = y(x_0) = s(x_0)$. Thus we have a contradiction and we conclude that $s(x_0) =$

$s(x_0 - 0)$.

Now assume $s(x_0 - 0) - s(x_0 + 0) = k > 0$. By Lemma 3 there is an $\eta > 0$ such that $[x_0 - \eta, x_0 + \eta] \subset I$, $\eta \leqq \delta(M)$, and such that, for any $[x_1, x_2] \subset I$ with $|x_1 - x_2| = \eta$, the boundary value problem

$$y'' = f(x, y)$$
$$y(x_1) = s(x_1)\ , \qquad y(x_2) = s(x_2)$$

has a solution $y(x; x_1, x_2)$ with

$$|y(x; x_1, x_2) - \omega(x; x_1, x_2)| < k/4$$

on $[x_1, x_2]$ where $\omega(x; x_1, x_2)$ is the linear function with $\omega(x_1) = s(x_1)$ and $\omega(x_2) = s(x_2)$. Now take $[x_1, x_2] \subset I$ such that $|x_1 - x_2| = \eta$, $x_1 < x_0 < x_2$, $|s(x_2) - s(x_0 + 0)| < k/4$, and $(2M/\eta)|x_2 - x_0| < k/4$. Then, since $|\omega'(x; x_1, x_2)| \leqq 2M/\eta$, it follows that

$$|\omega(x_0; x_1, x_2) - s(x_0 + 0)| < k/2.$$

Consequently,

$$|y(x_0; x_1, x_2) - s(x_0 + 0)| < 3k/4$$

which means that

$$s(x_0) = s(x_0 - 0) > y(x_0; x_1, x_2)\ .$$

This contradicts the fact that s is a subfunction and we conclude that s is continuous on I^0.

We shall see a little later that Theorem 2 is not true for equation (1) even if conditions A_1 and A_2 are assumed in addition to the continuity of $f(x, y, z)$.

For the following theorems the proofs are the same as the corresponding theorems for convex functions.

THEOREM 3. *If $\{s_\alpha : \alpha \in A\}$ is any collection of subfunctions on I bounded above at each point of I, then s_0 defined by*

$$s_0(x) \equiv \sup_{\alpha \in A} s_\alpha(x)$$

is a subfunction on I.

THEOREM 4. *Let s_1 be a subfunction on I and s_2 a subfunction on $[x_1, x_2] \subset \bar{I}$. Assume further that $s_2(x_i) \leqq s_1(x_i)$ for $i = 1, 2$ in case $x_i \in I^0$. Then s defined on I by*

$$s(x) \equiv \begin{cases} s_1(x) & \text{for } x \notin [x_1, x_2] \\ \max\,[s_1(x), s_2(x)] & \text{for } x \in [x_1, x_2] \end{cases}$$

is a subfunction on I.

For a function g and a point x_0 at which $g(x_0 - 0)$ or $g(x_0 + 0)$ exist we define

$$d^- g(x_0) \equiv \limsup_{y \to x_0 -} \frac{g(x) - g(x_0 - 0)}{x - x_0}$$

$$d_- g(x_0) \equiv \liminf_{x \to x_0 -} \frac{g(x) - g(x_0 - 0)}{x - x_0}$$

$$d^+ g(x_0) \equiv \limsup_{x \to x_0 +} \frac{g(x) - g(x_0 + 0)}{x - x_0}$$

$$d_+ g(x_0) \equiv \liminf_{x \to x_0 +} \frac{g(x) - g(x_0 + 0)}{x - x_0} .$$

THEOREM 5. *If s is a bounded subfunction on $I \subset [a, b]$ with $\bar{I} = [x_1, x_2]$, then $d^- s(x_0) = d_- s(x_0)$ for all $x_1 < x_0 \leqq x_2$ and $d^+ s(x_0) = d_+ s(x_0)$ for all $x_1 \leqq x_0 < x_2$.*

Proof. It suffices to consider one case. Assume that $x_1 \leqq x_0 < x_2$ and that $d^+ s(x_0) \neq d_+ s(x_0)$. Then there is a finite number c such that

$$d_+ s(x_0) < c < d^+ s(x_0) .$$

There is a $\delta > 0$ such that $[x_0, x_0 + \delta] \subset [x_1, x_2]$ and such that the initial value problem

$$y'' = f(x, y, y')$$
$$y(x_0) = s(x_0 + 0) , \qquad y'(x_0) = c$$

has a solution $y(x)$ of class $C^{(2)}$ on $[x_0, x_0 + \delta]$. It is clear that this leads to a contradiction of the fact that s is a subfunction on I. We conclude that $d^+ s(x_0) = d_+ s(x_0)$.

COROLLARY. *If s is a bounded subfunction on I, then s has a finite derivative almost everywhere on I.*

For a function g defined on I and $x_0 \in I^0$ we will employ the notation:

$$\bar{D} g(x_0) \equiv \limsup_{\delta \to 0} \frac{g(x_0 + \delta) - g(x_0 - \delta)}{2\delta} ,$$

$$\underline{D} g(x_0) \equiv \liminf_{\delta \to 0} \frac{g(x_0 + \delta) - g(x_0 - \delta)}{2\delta} .$$

THEOREM 6. *If s is a subfunction of class $C^{(1)}$ on I, then $\underline{D} s'(x) \geqq f(x, s(x), s'(x))$ on I^0.*

Proof. Let $x_0 \in I^0$ and choose a $\delta_0 > 0$ such that $[x_0 - \delta_0, x_0 + \delta_0] \subset I$.

Let $|s(x)| \leqq M$ and $|s'(x)| \leqq N$ on $[x_0 - \delta_0, x_0 + \delta_0]$.

Given $\varepsilon > 0$ there is a $\rho > 0$ such that

$$f(x, y, z) \geqq f(x_0, s(x_0), s'(x_0)) - \varepsilon$$

for $|x - x_0| < \rho$, $|y - s(x_0)| < \rho$, $|z - s'(x_0)| < \rho$. Now choose a $\delta_1 > 0$ such that $|\omega(x;\delta) - s(x_0)| < \rho/2$ and $|\omega'(x;\delta) - s'(x_0)| < \rho/2$ on $[x_0 - \delta, x_0 + \delta]$ for all $0 < \delta \leqq \delta_1$ where $\omega(x; \delta)$ is the linear function with $\omega(x_0 - \delta) = s(x_0 - \delta)$ and $\omega(x_0 + \delta) = s(x_0 + \delta)$.

By Lemmas 1 and 3 there is a $\delta_2 > 0$ with $2\delta_2 \leqq \min[2\delta_0, 2\delta_1, \delta(M, N)]$ such that for any $0 < \delta \leqq \delta_2$ the boundary value problem

$$y'' = f(x, y, y')$$
$$y(x_0 - \delta) = s(x_0 - \delta) , \qquad y(x_0 + \delta) = s(x_0 + \delta)$$

has a solution $y(x; \delta)$ with

$$|y(x; \delta) - \omega(x; \delta)| < \rho/2$$

and

$$|y'(x; \delta) - \omega'(x; \delta)| < \rho/2$$

on $[x_0 - \delta, x_0 + \delta]$. Hence, for $0 < \delta \leqq \delta_2$, $|y(x; \delta) - s(x_0)| < \rho$, and $|y'(x; \delta) - s'(x_0)| < \rho$ on $[x_0 - \delta, x_0 + \delta]$. Then, since s is a subfunction on I we have for any $0 < \delta \leqq \delta_2$

$$\frac{s'(x_0 + \delta) - s'(x_0 - \delta)}{2\delta} \geqq \frac{y'(x_0 + \delta; \delta) - y'(x_0 - \delta; \delta)}{2\delta} = y''(\xi; \delta)$$

where $x_0 - \delta < \xi < x_0 + \delta$. Hence,

$$\frac{s'(x_0 + \delta) - s'(x_0 - \delta)}{2\delta} \geqq f(\xi, y(\xi; \delta), y'(\xi; \delta)) \geqq f(x_0, s(x_0), s'(x_0)) - \varepsilon$$

for all $0 < \delta \leqq \delta_2$. From which we conclude

$$\underline{D}s'(x_0) \geqq f(x_0, s(x_0), s'(x_0)) .$$

Under more stringent conditions on the function $f(x, y, z)$, Peixoto [15; p. 564] gives $s''(x) \geqq f(x, s(x), s'(x))$ as a necessary and sufficient condition for a function of class $C^{(2)}$ to be a subfunction. Theorem 6 generalizes the necessity part of this result. The condition $s'' \geqq f(x, s, s')$ is not sufficient to guarantee that a function of class $C^{(2)}$ be a subfunction without having more than just continuity of $f(x, y, z)$. As a matter of fact continuity of $f(x, y, z)$ and condition A_1 are still not enough. To see this we observe that, if $s'' \geqq f(x, s, s')$ is a sufficient condition for a function of class $C^{(2)}$ to be a subfunction, then a solution of the boundary value problem when it exists is unique. The boundary value problem $y'' = |y'|^{1/3}$, $y(-1) = y(+1) = 4\sqrt{6}/45$ has both $y(x) \equiv 4\sqrt{6}/45$ and $y(x) = 4\sqrt{6}/45\,|x|^{5/2}$ as solutions.

The following Theorem gives conditions on f which are adequate to insure that a function satisfying $s'' \geqq f(x, s, s')$ is a subfunction. It embodies a maximum principle which must be known; however, since we are not aware of a reference for it in this form, we will include a proof.

THEOREM 7. *Assume that $f(x, y, z)$ satisfies conditions A_1 and A_2 and that the functions $u(x)$ and $v(x)$ satisfy the following conditions*:

(i) *u and v are both continuous on $\bar{I}$ and of class $C^{(1)}$ on I^0,*

(ii) *$\underline{D}u'(x) \geqq f(x, u(x), u'(x))$ and $\bar{D}v'(x) \leqq f(x, v(x), v'(x))$ on I^0, and*

(iii) *$u(x) - v(x) \leqq M$, where $M \geqq 0$, at the endpoints of $\bar{I}$.*

Then either $u(x) - v(x) < M$ on I^0 or $u(x) - v(x) \equiv M$ on $\bar{I}$.

Proof. We will assume $M = 0$ since the case where $M > 0$ can be reduced to this one by replacing $v(x)$ by $v(x) + M$.

Now assume that the statement of the Theorem is false. Then there are functions u and v satisfying the hypotheses of the Theorem with $u(x) - v(x) \not\equiv 0$ on $\bar{I}$ but with $u(x) - v(x) \geqq 0$ at some points of I^0. Let $N = \max\,[u(x) - v(x)]$ on $\bar{I}$. Because of the continuity of $u(x) - v(x)$ there is an $x_0 \in I^0$ and an interval $[x_1, x_2] \subset I^0$ such that $x_1 < x_0 < x_2$, $u(x_0) - v(x_0) = N$, and $u(x) - v(x) < N$ either on $x_0 < x \leqq x_2$ or on $x_1 \leqq x < x_0$. Assume that $u(x) - v(x) < N$ on $x_0 < x \leqq x_2$ to be specific.

Let $M_1 > 0$ be such that $|u(x)| + |u'(x)| \leqq M_1$ and $|v(x)| + |v'(x)| \leqq M_1$ on $[x_1, x_2]$ and let F be the set $[(x, y, z)\colon x_1 \leqq x \leqq x_2, |y| + |z| \leqq M_1]$. By hypothesis there is a $k > 0$ such that

$$|f(x, y_1, z_1) - f(x, y_2, z_2)| \leqq k[|y_1 - y_2| + |z_1 - z_2|]$$

for all (x, y_1, z_1) and (x, y_2, z_2) in F.

Define the functions $w_1(x)$ and $w_2(x)$ as follows:

$$w_1(x) = \begin{cases} \dfrac{f(x, u(x), u'(x)) - f(x, u(x), v'(x))}{u'(x) - v'(x)} & \text{for } u'(x) \neq v'(x) \\ 0 & \text{for } u'(x) = v'(x)\,, \end{cases}$$

and

$$w_2(x) = \begin{cases} \dfrac{f(x, u(x), v'(x)) - f(x, v(x), v'(x))}{u(x) - v(x)} & \text{for } u(x) \neq v(x) \\ 0 & \text{for } u(x) = v(x)\,. \end{cases}$$

Then it is clear that $|w_1(x)| \leqq k$ and $|w_2(x)| \leqq k$ on $[x_1, x_2]$. Because of the assumed condition A_1 on f, $w_2(x) \geqq 0$ on $[x_1, x_2]$.

Choose $x_3 \in I^0$ such that $x_2 < x_3$ and define $h(x; \alpha)$ by

$$h(x; \alpha) = e^{-\alpha(x-x_3)^2} - e^{-\alpha(x_0-x_3)^2}$$

where $\alpha > 0$ fixed is chosen large enough that

$$L[h] \equiv h''(x) - w_1(x)h'(x) - w_2(x)h(x) > 0$$

on $[x_1, x_2]$.

Since $u(x_2) - v(x_2) < N$ we can choose $\eta > 0$ such that $u(x_2) - v(x_2) + \eta h(x_2) < N$. Then, if $g(x) \equiv u(x) - v(x) + \eta h(x)$, we have $g(x_1) < N$, $g(x_0) = N$, and $g(x_2) < N$. It follows that $g(x)$ has a maximum $N_1 \geqq N \geqq 0$ at a point x_4 with $x_1 < x_4 < x_2$. It follows that $\underline{D}g'(x_4) \leqq 0$. However,

$$\begin{aligned}\underline{D}g'(x_4) &\geqq \underline{D}[u'(x_4) + \eta h'(x_4)] - \bar{D}v'(x_4) \\ &\geqq \underline{D}u'(x_4) - \bar{D}v'(x_4) + \eta h''(x_4) > w_2(x_4)N_1 \geqq 0 .\end{aligned}$$

We have arrived at a contradiction and the Theorem is established.

COROLLARY 1. *Let $f(x, y, y')$ satisfy conditions A_1 and A_2. Then the solution of the boundary value problem*

$$\begin{gathered} y'' = f(x, y, y') \\ y(x_1) = y_1,\ y(x_2) = y_2 \end{gathered}$$

for $[x_1, x_2] \subset [a, b]$, if it exists, will be unique.

COROLLARY 2. *If in the statement of Theorem 7 we assume only that $f(x, y, y')$ satisfies condition A_1 but strengthen the assumptions concerning u and v by assuming that at least one of the differential inequalities is a strict inequality for $x_1 < x < x_2$, then $u(x) - v(x) < M$ for $x_1 < x < x_2$.*

COROLLARY 3. *If in Theorem 7 $\bar{I} = [c, d]$, $u(c) = v(c)$, and $u(d) > v(d)$, then $u(x) - v(x)$ is nondecreasing on $[c, d]$. If $u(c) > v(c)$ and $u(d) = v(d)$, then $u(x) - v(x)$ is nonincreasing on $[c, d]$.*

THEOREM 8. *Let $s(x)$ be continuous on I and of class $C^{(1)}$ on I^0. Then, if $f(x, y, z)$ satisfies A_1 and A_2 and if $\underline{D}s'(x) \geqq f(x, s(x), s'(x))$ on I^0, it follows that $s(x)$ is a subfunction on I. If $f(x, y, z)$ satisfies condition A_1 and $\underline{D}s'(x) > f(x, s(x), s'(x))$ on I^0, $s(x)$ is a subfunction on I.*

Proof. Let $[x_1, x_2] \subset I$ and let $y(x)$ be a solution of (1) on $[x_1, x_2]$ with $s(x_i) \leqq y(x_i)$ for $i = 1, 2$. Then that $s(x) \leqq y(x)$ on $[x_1, x_2]$ follows from Theorem 7 or Corollary 2 of Theorem 7.

THEOREM 9. *Let $s(x)$ be a continuous subfunction and $S(x)$ a continuous superfunction on $[x_1, x_2]$ with $s(x_i) \leqq S(x_i)$, $i = 1, 2$. Assume that at least one of $s(x)$ and $S(x)$, say $S(x)$, is of class $C^{(1)}$ on $x_1 < x < x_2$. Then, if $f(x, y, z)$ satisfies A_1 and A_2, $s(x) \leqq S(x)$ on $[x_1, x_2]$. If $f(x, y, z)$ satisfies A_1 and $\bar{D}S'(x) < f(x, S(x), S'(x))$ on $x_1 < x < x_2$, then again $s(x) \leqq S(x)$ on $[x_1, x_2]$.*

Proof. Assume that the statement of the Theorem is false. Then $s(x) > S(x)$ for some points x with $x_1 < x < x_2$. Let $M = \max\,[s(x) - S(x)]$ on $[x_1, x_2]$ and let x_0 be such that $x_1 < x_0 < x_2$, $s(x_0) - S(x_0) = M$, and $s(x) - S(x) < M$ for $x_0 < x \leqq x_2$. By Lemma 1 there is a $\delta > 0$ such that $x_1 < x_0 - \delta < x_0 + \delta < x_2$ and such that the boundary value problem

$$\begin{gathered} y'' = f(x, y, y') \\ y(x_0 - \delta) = S(x_0 - \delta) + M,\ y(x_0 + \delta) = S(x_0 + \delta) + M \end{gathered}$$

has a solution $y_1(x)$ of class $C^{(2)}$ on $[x_0 - \delta, x_0 + \delta]$. If $f(x, y, z)$ satisfies A_1 and $\bar{D}S'(x) < f(x, S(x), S'(x))$ on $x_1 < x < x_2$, it follows from Corollary 2 of Theorem 7 that $y_1(x_0) < S(x_0) + M$. Furthermore, since $s(x)$ is a subfunction and $s(x_0 \pm \delta) \leqq y_1(x_0 \pm \delta)$, we have $s(x_0) \leqq y_1(x_0) < S(x_0) + M$. From this contradiction we conclude that $s(x) \leqq S(x)$ on $[x_1, x_2]$.

Now assume that $f(x, y, z)$ satisfies A_1 and A_2 and that we know only that $S(x)$ is of class $C^{(1)}$ on $x_1 < x < x_2$ which implies $\bar{D}S'(x) \leqq f(x, S(x), S'(x))$ on $x_1 < x < x_2$. Then, if $y_1(x)$ is again the solution of the above boundary value problem, we have $y_1(x_0) \leqq S(x_0) + M$. By Lemma 5 and Theorem 7 there is an $m > 0$ such that the initial value problem

$$\begin{aligned} y'' &= f(x, y, y') \\ y(x_0 - \delta) &= y_1(x_0 - \delta) \\ y'(x_0 - \delta) &= y_1'(x_0 - \delta) - m \end{aligned}$$

has a solution $y_2(x)$ of class $C^{(2)}$ on $[x_0 - \delta, x_0 + \delta]$ with $y_2(x) < y_1(x)$ on $(x_0 - \delta, x_0 + \delta]$ and

$$y_1(x_0 + \delta) > y_2(x_0 + \delta) > s(x_0 + \delta)\ .$$

Then we have $s(x_0) \leqq y_2(x_0) < y_1(x_0) \leqq S(x_0) + M$ which is again a contradiction. Thus we have $s(x) \leqq S(x)$ on $[x_1, x_2]$.

COROLLARY. *Let $M > 0$ be a constant and assume that $f(x, y, z)$*

satisfies A_1 *and* A_2. *Then, if* $S(x)$ *is a continuous superfunction on* I, $S(x)+M$ *is also, and, if* $s(x)$ *is a continuous subfunction on* I, $s(x)-M$ *is also.*

THEOREM 10. *Assume that* $f(x, y, z)$ *satisfies conditions* A_1 *and* A_2, *that* $y(x)$ *is a solution of* (1) *of class* $C^{(2)}$ *on* $[x_1, x_2] \subset [a, b]$, *and that* $s(x)$ *is a subfunction on* $[x_1, x_2]$. *Assume further that there is an* x_0, $x_1 < x_0 < x_2$, *at which either* $s(x_0) = y(x_0)$, $s(x_0+0) = y(x_0)$, *or* $s(x_0 - 0) = y(x_0)$. *Then, if* $s(x_1) \leqq y(x_1)$, $s(x) \geqq y(x)$ *on* $x_0 < x \leqq x_2$. *If* $s(x_2) \leqq y(x_2)$, $s(x) \geqq y(x)$ *on* $x_1 \leqq x < x_0$.

Proof. Follows immediately from Lemma 5, Theorem 7, and the definition of subfunctions.

COROLLARY. *If* $f(x, y, z)$ *satisfies conditions* A_1 *and* A_2, *if* $y(x)$ *is a solution of* (1) *of class* $C^{(2)}$ *on* $[x_1, x_2] \subset [a, b]$, *and if* $s(x)$ *is a subfunction on* $[x_1, x_2]$ *with* $s(x_i) \leqq y(x_i)$, $i = 1, 2$, *then either* $\max[s(x), s(x+0), s(x-0)] < y(x)$ *on* $x_1 < x < x_2$ *or* $s(x) \equiv y(x)$ *on* $[x_1, x_2]$.

In the papers mentioned in the introduction which deal with generalized convex functions it is assumed that for any two points (x_1, y_1), (x_2, y_2), $x_1 \neq x_2$ in the strip $a \leqq x \leqq b$, $|y| < +\infty$ the boundary value problem has a unique solution which is defined throughout $a \leqq x \leqq b$. This leads to the conclusion that subfunctions and superfunctions are continuous in the interiors of their intervals of definition. With the assumptions we make this conclusion cannot be drawn. Consider the equation

$$y'' = -18x(y')^4 \tag{3}$$

which is such that $f(x, y, z)$ is continuous everywhere and satisfies A_1 and A_2. The function g defined by $g(x) = x^{1/3} + 1$ for $0 < x \leqq 1$, $g(x) = x^{1/3}$ for $-1 \leqq x < 0$, and $g(0) = g_0$ where $0 \leqq g_0 \leqq 1$ is simultaneously a subfunction and superfunction on $[-1, +1]$ with respect to solutions of (3).

4. A generalized solution of the boundary value problem. In the previous Section the existence "in the small" of solutions for the initial value problem and the boundary value problem for (1) was used in discussing some of the properties of subfunctions and superfunctions with respect to solutions of (1). In this section we use the Perron method of using subfunctions and superfunctions to deal with the boundary value problem "in the large" for equation (1). Throughout this section we shall be dealing with the boundary value problem:

$$(4) \qquad \begin{gathered} y'' = f(x, y, y') \\ y(a) = \alpha\ , \qquad y(b) = \beta\ . \end{gathered}$$

DEFINITION 3. The function $\varphi(x)$ is said to be an under-function with respect to the boundary value problem (4) in case $\varphi(x)$ is a subfunction on $[a, b]$ with $\varphi(a) \leqq \alpha$ and $\varphi(b) \leqq \beta$.

DEFINITION 4. The function $\psi(x)$ is said to be an over-function with respect to the boundary value problem (4) in case $\psi(x)$ is a superfunction on $[a, b]$ with $\psi(a) \geqq \alpha$ and $\psi(b) \geqq \beta$.

In most of the results obtained in this section we shall require the following additional hypothesis:

A_3: $f(x, y, z)$ is such that with respect to the boundary value problem (4) there is an under-function which is continuous on $[a, b]$ and there is an over-function which is continuous on $[a, b]$ and is of class $C^{(1)}$ on (a, b).

DEFINITION 5. Let $\{\varphi\}$ represent the collection of all under-functions with respect to boundary value problem (4) which are continuous on $[a, b]$. Then we define $H(x)$ by

$$H(x) \equiv \sup\,[\varphi(x) \colon \varphi \in \{\varphi\}]$$

for each $x \in [a, b]$.

THEOREM 11. *If $f(x, y, y')$ satisfies conditions A_1, A_2, and A_3, $H(x)$ is a bounded subfunction on $[a, b]$.*

Proof. By A_3, $\{\varphi\}$ is nonnull and there is an over-function ψ_0 continuous on $[a, b]$ and of class $C^{(1)}$ on (a, b). By Theorem 9 $\varphi(x) \leqq \psi_0(x)$ on $[a, b]$ for each $\varphi \in \{\varphi\}$, consequently, $H(x) \leqq \psi_0(x)$ on $[a, b]$. By Theorem 3 $H(x)$ is a subfunction and, if $\varphi_0 \in \{\varphi\}$, $\varphi_0(x) \leqq H(x) \leqq \psi_0(x)$ so that H is bounded on $[a, b]$.

THEOREM 12. *If $f(x, y, y')$ satisfies A_1, A_2, and A_3, then $H(x)$ is a superfunction on $[a, b]$.*

Proof. Assume that H is not a superfunction. Then there exists $[x_1, x_2] \subset [a, b]$ and a solution $y(x)$ of (1) on $[x_1, x_2]$ such that $H(x_i) \geqq y(x_i)$, $i = 1, 2$, but $H(x) < y(x)$ for some x with $x_1 < x < x_2$. Let x_0, $x_1 < x_0 < x_2$, be such that $y(x_0) - H(x_0) = \varepsilon > 0$. By the definition of H there are continuous under-functions φ_1 and φ_2 such that $H(x_1) -$

$\varphi_1(x_1) \leqq \varepsilon/4$ and $H(x_2) - \varphi_2(x_2) \leqq \varepsilon/4$. Now define φ_3 on $[a, b]$ as follows:

$$\varphi_3(x) \equiv \begin{cases} \max\,[\varphi_1(x), \varphi_2(x)] & \text{for } x \notin [x_1, x_2] \\ \max\,[\varphi_1(x), \varphi_2(x), y(x) - \varepsilon/2] & \text{for } x \in [x_1, x_2]\,. \end{cases}$$

Then by Theorems 3 and 4 and the Corollary of Theorem 9 φ_3 is a continuous under-function. However, $\varphi_3(x_0) \geqq y(x_0) - \varepsilon/2 = H(x_0) + \varepsilon/2$ which is impossible. It follows that H is a superfunction on $[a, b]$.

COROLLARY. *For each* $x \in (a, b)$ $H(x) = \min\,[H(x+0), H(x-0)]$.

Proof. Since H is a superfunction, it follows from Corollary 1 of Theorem 1 that $H(x) \geqq \min\,[H(x+0), H(x-0)]$. Since H is lower semicontinuous on $[a, b]$, $H(x) \leqq \min\,[H(x+0), H(x-0)]$.

Theorem 13. *If* $f(x, y, y')$ *satisfies* A_1, A_2, *and* A_3, *then* H *is a solution of* (1) *on an open subset of* $[a, b]$ *the complement of which is of measure* 0.

Proof. Let $x_0 \in (a, b)$ be a point at which $H'(x_0)$ exists. Then there is a $\delta_0 > 0$ such that $[x_0 - \delta_0, x_0 + \delta_0] \subset [a, b]$ and such that for all δ with $0 < \delta \leqq \delta_0$ we have

$$\left|\frac{H(x_0+\delta) - H(x_0-\delta)}{2\delta}\right| \leqq |\,H'(x_0)\,| + 1\,.$$

Let $\delta(M, N) > 0$ be as in Lemma 1 with $M = \sup|\,H(x)\,|$ on $[a, b]$ and $N = |\,H'(x_0)\,| + 1$. Then for $2\delta = \min\,[2\delta_0, \delta(M, N)]$ the boundary value problem

$$y'' = f(x, y, y')$$
$$y(x_0 - \delta) = H(x_0 - \delta),\ y(x_0 + \delta) = H(x_0 + \delta)$$

has a solution $y(x)$ of class $C^{(2)}$ on $[x_0 - \delta, x_0 + \delta]$. Since H is simultaneously a subfunction and superfunction, $H(x) \equiv y(x)$ on $[x_0 - \delta, x_0 + \delta]$. The result then follows as a consequence of the Corollary of Theorem 5.

In Theorem 5 we proved that, if s is a bounded subfunction on $[a, b]$, then $d^+s(x) = d_+s(x)$ on $a \leqq x < b$ and $d^-s(x) = d_-s(x)$ on $a < x \leqq b$. In view of these equalities we introduce the additional notation:

$$Ds(x_0 + 0) \equiv \lim_{x \to x_0+} \frac{s(x) - s(x_0 + 0)}{x - x_0}$$

and

$$Ds(x_0 - 0) \equiv \lim_{x \to x_0-} \frac{s(x) - s(x_0 - 0)}{x - x_0}\,.$$

THEOREM 14. *If $f(x, y, y')$ satisfies A_1, A_2, and A_3, then $DH(x+0) = DH(x-0)$ for all $x \in (a, b)$. Let E be the set of points in (a, b) at which H does not have a finite derivative. If $x \in E$ is a point of continuity of H, either $DH(x+0) = DH(x-0) = +\infty$ or $DH(x+0) = DH(x-0) = -\infty$. If $H(x+0) > H(x-0)$, $DH(x+0) = DH(x-0) = +\infty$. If $H(x+0) < H(x-0)$, $DH(x+0) = DH(x-0) = -\infty$.*

Proof. Since for $x \in (a, b)$ and $x \notin E$ $DH(x+0) = DH(x-0) = H'(x)$, we need consider only the points of E.

First we observe that it follows from the argument used in the proof of Theorem 13 that, if $x_0 \in E$ is a point of continuity of H, $DH(x_0+0)$ and $DH(x_0-0)$ cannot both be finite. Assume that $x_0 \in E$ is a point of continuity of H, that $DH(x_0+0) = +\infty$, but that $DH(x_0-0) \neq +\infty$. It follows that there is an $N > 0$ and a $\delta_0 > 0$ such that $\omega(x) \equiv H(x_0) + N(x - x_0) < H(x)$ for $0 < x_0 - x \leqq \delta_0$. By Lemmas 1 and 4 there is a δ_1, $0 < \delta_1 \leqq \delta_0$, a $\delta_2 > 0$, and a solution $y(x)$ of (1) of class $C^{(2)}$ on $[x_0 - \delta_1, x_0 + \delta_2]$ with $y(x_0) = H(x_0)$ and $y(x_0 - \delta_1) = \omega(x_0 - \delta_1) < H(x_0 - \delta_1)$. Applying Theorem 10 we conclude that $H(x) \leqq y(x)$ on $[x_0, x_0 + \delta_2]$. This implies that $DH(x_0+0) \leqq y'(x_0)$ which contradicts the assumption that $DH(x_0+0) = +\infty$. We conclude that $DH(x_0-0) = +\infty$. The other possibilities at a point of continuity can be dealt with in a similar way.

Now assume that $H(x_0+0) > H(x_0-0)$ and that $DH(x_0+0) \neq +\infty$. Then by the same type of argument as was used above we can conclude that there exist $\delta_1 > 0$, $\delta_2 > 0$, and a solution $y(x)$ of (1) of class $C^{(2)}$ on $[x_0 - \delta_1, x_0 + \delta_2]$ satisfying $y(x_0) = H(x_0+0)$ and $y(x_0 + \delta_2) > H(x_0 + \delta_2)$. We can then again apply Theorem 10 to conclude that $H(x) \geqq y(x)$ on $x_0 - \delta_1 \leqq x < x_0$ from which it follows that $H(x_0-0) \geqq y(x_0) = H(x_0+0)$. This contradicts the assumption that $H(x_0-0) < H(x_0+0)$ and we conclude that $DH(x_0+0) = +\infty$.

The remainder of the proof concerning the points of discontinuity is similar to this and will be omitted.

Next we consider the behavior of H at the endpoints of the interval $[a, b]$.

THEOREM 15. *Assume that $f(x, y, y')$ satisfies A_1, A_2, and A_3. Then, if $DH(a+0) \neq +\infty$, $H(a+0) = H(a)$. If $H(a+0) < \alpha$, $DH(a+0) = -\infty$. If $DH(a+0)$ is finite, $H(a+0) = H(a) = \alpha$. Similar statements apply at $x = b$.*

Proof. The proof will be omitted since the methods used in it are very similar to those used in the proofs of Theorems 12 and 14.

If $f(x, y, y')$ satisfies conditions A_1, A_2, and A_3, and if the boundary value problem (4) has a solution, $H(x)$ is that solution. On the basis of the properties of the function $H(x)$ it seems reasonable to refer to

$H(x)$ as a "generalized solution" of the boundary value problem. Usually by a generalized solution of a second order differential equation on an interval one means a function which has an absolutely continuous first derivative and which satisfies the differential equation almost everywhere on the interval. The function $H(x)$ may not even be continuous on $[a, b]$. Consider the boundary value problem $y'' = -18x(y')^4$, $y(-1) = -1$, $y(+1) = +2$. Here conditions A_1 and A_2 are obviously fulfilled and, since $\psi(x) \equiv +2$ is an over-function and $\varphi(x) \equiv -1$ is an under-function, condition A_3 is satisfied. In this case $H(x) = x^{1/3}$ for $-1 \leqq x \leqq 0$ and $H(x) = x^{1/3} + 1$ for $0 < x \leqq +1$.

We terminate this Section by considering the function $H(x)$ with respect to the boundary value problem:

$$\begin{gathered} y'' = f(x, y) \\ y(a) = \alpha\,, \quad y(b) = \beta\,. \end{gathered} \tag{5}$$

THEOREM 16. *Assume that $f(x, y)$ satisfies A_1 and A_3 with the additional assumption that there is an over-function ψ with respect to the boundary value problem* (5) *such that ψ is continuous on $[a, b]$, is of class $C^{(1)}$ on (a, b), and satisfies $\bar{D}\psi'(x) < f(x, \psi(x))$ on (a, b). Then the function $H(x)$, defined in the same manner as above, is again bounded on $[a, b]$ and is simultaneously a subfunction and a superfunction with respect to solutions of* (2). *In this case $H(x)$ is of class $C^{(2)}$ and is a solution of* (2) *on $[a, b]$.*

Proof. The proof that $H(x)$ is bounded and is simultaneously a subfunction and a superfunction on $[a, b]$ is exactly as given in Theorems 11 and 12 with one exception. Since we do not now have the Corollary of Theorem 9 available, we must give a separate proof that, if $y(x)$ is a solution of (2) on $[x_1, x_2] \subset [a, b]$ and $M \geqq 0$, then $y(x) - M$ is a subfunction with respect to (2) on $[x_1, x_2]$. To see that this is the case assume that $y_1(x)$ is a solution of (2) on $[x_3, x_4] \subset [x_1, x_2]$ with

$$y(x_3) - M = y_1(x_3)\,,$$
$$y(x_4) - M = y_1(x_4)\,,$$

and

$$y(x) - M > y_1(x) \quad \text{on} \quad x_3 < x < x_4\,.$$

Because of condition A_1 we then have

$$y''(x) - y_1''(x) = f(x, y(x)) - f(x, y_1(x)) \geqq 0$$

on (x_3, x_4). This implies that $y(x) - y_1(x)$ is convex on $[x_3, x_4]$ which in turn implies that $y(x) - y_1(x) \leqq M$ on $[x_3, x_4]$. Thus it is not possible that $y(x) - y_1(x) > M$ on (x_3, x_4). It follows that $y(x) - M$ is a subfunction on $[x_1, x_2]$.

Since $H(x)$ is bounded and is simultaneously a subfunction and a superfunction, we can apply Lemma 2 to conclude that $H(x)$ is of class $C^{(2)}$ and a solution of (2) on $[a, b]$.

5. Existence theorems for a solution of the boundary value problem. In this concluding Section we consider the question of determining additional conditions on $f(x, y, y')$ which will suffice to guarantee that $H(x)$ be a solution of the boundary value problem (4). Some of the results are known and we are merely giving new proofs of them, others appear to be new.

THEOREM 17. *Assume that $f(x, y, y')$ satisfies A_1, A_2, and A_3, that $\psi(x)$ is an over-function continuous on $[a, b]$ and of class $C^{(1)}$ on (a, b), and that $\varphi(x)$ is a continuous underfunction. Assume that there is a function $h(t)$ positive and continuous for $t \geqq 0$ such that $|f(x, y, y')| \leqq h(|y'|)$ for $a \leqq x \leqq b$, $\varphi(x) \leqq y \leqq \psi(x)$, $|y'| < +\infty$ and such that*

$$\int_0^\infty \frac{tdt}{h(t)} = +\infty .$$

Then $H(x)$ is the solution of boundary value problem (4). *Nagumo* [14].

Proof. Let $x_0 \in (a, b)$ be a point at which $H'(x_0)$ exists. By Theorem 13 there is an open interval containing x_0 in which H is a solution of (1). Let $(c, d) \subset [a, b]$ be the maximal such interval. Then, if N is chosen so that

$$\int_{|H'(x_0)|}^{N} \frac{tdt}{h(t)} = \max \psi(x) - \min \varphi(x) ,$$

we will have $|H'(x)| \leqq N$ on (c, d). It follows from Theorems 14 and 15 that $c = a$, $d = b$ and that H is the solution of the boundary value problem.

THEOREM 18. *If $f(x, y)$ is continuous for $a \leqq x \leqq b$, $|y| < +\infty$, and satisfies A_1, then the boundary value problem* (5) *has a unique solution for each α and β. Babkin* [1].

Proof. Let $\omega(x)$ be the linear function with $\omega(a) = \alpha$ and $\omega(b) = \beta$. Define the functions $u(x)$ and $v(x)$ on $[a, b]$ by

$$u''(x) = |f(x, \omega(x))| + 1$$
$$u(a) = u(b) = 0 ,$$

and

$$v''(x) = -|f(x, \omega(x))| - 1$$
$$v(a) = v(b) = 0 .$$

Tnen it is not difficult to verify that $\psi_0(x) = v(x) + \omega(x)$ is an over-function of class $C^{(2)}$ satisfying $\psi_0''(x) < f(x, \psi_0(x))$ on $[a, b]$, and $\varphi_0(x) = u(x) + \omega(x)$ is an under-function of class $C^{(2)}$ satisfying $\varphi_0''(x) > f(x, \varphi_0(x))$ on $[a, b]$. The hypotheses of Theorem 16 are satisfied so that we can conclude that $H(x)$ is of class $C^{(2)}$ and is a solution of (2) on $[a, b]$. Since $\varphi_0(x) \leqq H(x) \leqq \psi_0(x)$ on $[a, b]$, $H(a) = \alpha$ and $H(b) = \beta$ so that $H(x)$ is a solution of boundary value problem (5). It follows from the proof of Theorem 16 that it is unique.

THEOREM 19. *Let $f(x, y, y')$ satisfy A_1, A_2, and A_3, and assume that there is a continuous function $g(x, y)$ such that $g(x, y) \leqq f(x, y, y')$ for all $(x, y, y') \in R$. Then $H(x)$ is of class $C^{(2)}$ for all $a < x < b$. If in addition $g(x, y)$ is nondecreasing as a function of y for each fixed x, $H(x)$ is continuous on $[a, b]$.*

Proof. Let $[x_1, x_2] \subset [a, b]$ and let $S(x)$ be a solution of

$$y'' = g(x, y) \tag{6}$$

on $[x_1, x_2]$ with $H(x_1) \leqq S(x_1)$ and $H(x_2) \leqq S(x_2)$. Then $S''(x) = g(x, S(x)) \leqq f(x, S(x), S'(x))$ on (x_1, x_2), hence by Theorem 8 $S(x)$ is a superfunction on $[x_1, x_2]$. Then from Theorem 9 and the fact that $\varphi(x_1) \leqq S(x_1)$ and $\varphi(x_2) \leqq S(x_2)$ we conclude that $\varphi(x) \leqq S(x)$ on $[x_1, x_2]$ for each continuous under-function φ. From this we conclude that $H(x) \leqq S(x)$ on $[x_1, x_2]$ and that H is a subfunction with respect to solutions of (6). By Theorem 2 $H(x)$ is continuous on (a, b).

Assume that H does not have a finite derivative at x_0, $a < x_0 < b$. Assume that $DH(x_0 + 0) = +\infty$. By Lemma 2 there is a $\delta > 0$ such that the boundary value problem

$$y'' = g(x, y)$$
$$y(x_0) = H(x_0) , \qquad y(x_0 + \delta) = H(x_0 + \delta)$$

has a solution $y(x)$ of class $C^{(2)}$ on $[x_0, x_0 + \delta]$. Since $H(x) \leqq y(x)$ on $[x_0, x_0 + \delta]$, $DH(x_0 + 0) \leqq y'(x_0)$ which contradicts the assumption that $DH(x_0 + 0) = +\infty$. Similarly $DH(x_0 - 0) = -\infty$ is not possible. It follows from Theorem 14 that $H(x)$ has a finite derivative at each point of (a, b), therefore, by Theorem 13 $H(x)$ is of class $C^{(2)}$ and is a solution of (1) on $a < x < b$.

If $g(x, y)$ is nondecreasing in y, it follows from Theorem 18 that the boundary value problem

$$y'' = g(x, y)$$

$$y(a) = \alpha , \qquad y(b) = \beta$$

has a solution $\psi(x)$ of class $C^{(2)}$ on $[a, b]$. $\psi(x)$ is an over-function with respect to the boundary value problem (4). It suffices to consider the endpoint $x = a$. Since $H(x) \leqq \psi(x)$, $H(a + 0) \leqq \psi(a) = \alpha$. If $H(a + 0) = \alpha$, $DH(a + 0) \leqq \psi'(a)$ so that $DH(a + 0) \neq +\infty$. It follows from Theorem 15 that $H(a + 0) = H(a)$. If $H(a + 0) < \alpha$, we again apply Theorem 15 to obtain $H(a + 0) = H(a)$. We conclude that $H(x)$ is continuous on $[a, b]$.

COROLLARY *If $f(x, y, y')$ satisfies conditions A_1 and A_2, if there is a continuous function $g(x, y)$ nondecreasing in y for each fixed x and satisfying $g(x, y) \leqq f(x, y, y')$ on R, and if there exists a continuous under-function $\varphi(x)$ with $\varphi(a) = \alpha$ and $\varphi(b) = \beta$, then the boundary value problem* (4) *has a unique solution.*

If $f(x, y, y')$ satisfies the hypotheses of Theorem 19 including the assumption that $g(x, y)$ is nondecreasing in y for each fixed x, then $H(x)$ is continuous on $[a, b]$ and of class $C^{(2)}$ on (a, b). Furthermore, $DH(a+0) \neq +\infty$ and $DH(b - 0) \neq -\infty$. As a consequence of Theorem 15 we could conclude that $H(x)$ is the solution of the boundary value problem if it could be shown that $DH(a + 0) \neq -\infty$ and $DH(b - 0) \neq +\infty$. This would be the case, for example, if for some $N > 0$ and some $\delta > 0$ $H''(x) < N$ on $a < x \leqq a + \delta$ and on $b - \delta \leqq x < b$.

As an illustration of these remarks consider the boundary value problem:

$$y'' = (1 + x^2)y^3 + e^{y'^2 \sin x}$$
$$y(-\pi/2) = \alpha , \qquad y(3\pi/2) = \beta .$$

The hypotheses of Theorem 19 are satisfied. $\varphi(x) \equiv \min[-1, \alpha, b]$ is an under-function and $\psi(x) \equiv \max[1, \alpha, \beta]$ is an over-function. In the intervals $-\pi/2 < x \leqq 0$ and $\pi \leqq x < 3\pi/2$ $H''(x)$ is bounded above, consequently, this boundary value problem always has a unique solution.

We conclude the paper with a final result in this direction.

THEOREM 20. *Assume that $f(x, y, y')$ satisfies A_1 and A_2, and that there is a continuous function $g(x, y)$ which is nondecreasing in y and is such that $g(x, y) \leqq f(x, y, y')$ for all $(x, y, y') \in R$. Assume further that there exist functions ψ, φ, and h such that*

(i) *$\psi(x)$ is continuous on $[a, b]$, is of class $C^{(1)}$ on (a, b), and is an over-function with respect to boundary value problem* (4),

(ii) *$\varphi(x)$ is a continuous under-function with respect to the boundary value problem,*

and (iii) *$h(t)$ is positive and continuous for $t \geqq 0$, $|f(x, y, y')| \leqq h(|y'|)$*

for $a \leqq x \leqq b$, $\varphi(x) \leqq y \leqq \psi(x)$, $|y'| < +\infty$, *and*

$$\int_c^\infty \frac{tdt}{h(t)} > \max \psi(x) - \min \varphi(x) \ ,$$

where

$$c = \max \left| \frac{|\psi(b) - \varphi(a)|}{b-a}, \frac{|\psi(a) - \varphi(b)|}{b-a} \right| \ .$$

Then $H(x)$ *is a solution of boundary value problem* (4).

Proof. By Theorem 19, $H(x)$ continuous on $[a, b]$ and is of class $C^{(2)}$ on (a, b). Since $\varphi(x) \leqq H(x) \leqq \psi(x)$ on $[a, b]$, $|(H(b) - H(a))/(b-a)| = |H'(x_0)| \leqq c$ for some $a < x_0 < b$. If N is chosen so that $\int_c^N tdt/h(t) = \max \psi(x) - \min \varphi(x)$ then $|H'(x)| \leqq N$ on (a, b). It follows from Theorem 15 that $H(x)$ is a solution of the boundary value problem.

As an illustration of this Theorem consider the boundary value problem

$$y'' = x^2y^3 + (y')^4$$
$$y(0) = \alpha \ , \qquad y(1) = \beta \ .$$

If $|\alpha| < M$ and $|\beta| < M$, the hypotheses of Theorem 20 are satisfied with $g(x, y) = x^2y^3$, $\psi(x) \equiv \max[|\alpha|, |\beta|]$, $\varphi(x) \equiv \min[-|\alpha|, -|\beta|]$ and $h(t) = t^4 + M^3$. Hence, by Theorem 20 the boundary value problem has a solution for $|\alpha| < M$ and $|\beta| < M$ if

$$\int_{2M}^\infty \frac{tdt}{t^4 + M^3} \geqq 2M \ .$$

The largest $M > 0$ for which this inequality is satisfied is the positive root of $\pi/2 - \mathrm{Arctan}\ 4M^{1/2} = 4M^{5/2}$.

There are few existence theorems for the boundary value problem that do not impose more stringent conditions than Theorem 20 does on the rate of growth of $f(x, y, y')$ with respect to y'. In the cases in which it is applicable Theorem 20 seems to give stronger results than other known theorems.

A different method of obtaining existence theorems for the boundary value problem (4) via existence theorems "in the small" was recently given by Kamenskii [13].

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