EQUICONTINUITY OF SOLUTIONS OF A QUASI-LINEAR EQUATION

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On a bounded domain Ω of the xy-plane the equicontinuity of a family of solutions of a linear elliptic partial differential equation is usually demonstrated by showing that the first partial derivatives of solutions are uniformly bounded on compact interior subsets of Ω. Finn [2] uses this same method in showing the equicontinuity for a class of quasi-linear elliptic equations referred to by him as "equations of minimal surface type." However, Finn cites an example which demonstrates that in general bounded collections of solutions of elliptic equations do not have uniformly bounded first partial derivatives on compact interior subsets.

Here we shall consider the question of the equicontinuity of a family of solutions of the quasi-linear equation

$$(1) \qquad L[z] \equiv A(x, y, p, q)r + 2B(x, y, p, q)s + C(x, y, p, q)t = 0$$

where, as usual, $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, and $t = z_{yy}$ and where A, B and C satisfy a growth condition.

Suppose D to be a domain in the xy-plane for which

(i) $A > 0$, $AC - B^2 = 1$, and A, B, and C are continuous and have continuous first partial derivatives with respect to p and q on T defined by $T \equiv \{(x, y, p, q) : (x, y) \in D \text{ and } -\infty < p, q < +\infty\}$, and

(ii) $(A + C)^2 \leqq (1/125) \log\log(p^2 + q^2 + e) + h(x, y)$ for all $(x, y, p, q) \in T$ where $h(x, y)$ is positive and continuous on D.

Henceforth, we shall assume that conditions (i) and (ii) are satisfied whenever reference is made to the equation (1).

THEOREM 1. *Let Ω be a bounded sub-domain of D with boundary ω such that $\bar{\Omega} = \Omega + \omega \subset D$. If $\{f_\nu(x, y) : \nu \in \mathscr{A}\}$ is any collection of functions which are continuous and uniformly bounded on ω and if corresponding to each f_ν there exists a function $z(x, y; f_\nu)$ which is of class C^2 on Ω, is continuous on $\bar{\Omega}$, is a solution of* (1) *on Ω, and is such that $z(x, y; f_\nu) = f_\nu(x, y)$ on ω, then the collection $\{z(x, y; f_\nu) : \nu \in \mathscr{A}\}$ is equicontinuous on Ω.*

In proving Theorem 1 we shall employ a modification of the method used by Serrin [5] and in so doing depend heavily on the following

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principle:

Maximum Principle [3]. Let D be any plane domain and consider the function $F(x, y, z, p, q, r, s, t)$ with the following assumptions:

(i) F is continuous in all 8 variables in the region T defined by $T \equiv \{(x, y, z, p, q, r, s, t) : (x, y) \in D$ and $-\infty < z, p, q, r, s, t < +\infty\}$ and

(ii) F_z, F_p, F_q, F_r, F_s, and F_t are continuous on T, $F_s^2 - 4F_rF_t < 0$, $F_r > 0$, and $F_z \leqq 0$ on T.

Let $z_1(x, y)$ and $z_2(x, y)$ be continuous in a bounded and closed subdomain $\mathscr{X} \subset D$ and of class C^2 in the interior of $\mathscr{X}$. Furthermore, suppose $z_1(x, y) \leqq z_2(x, y)$ on the boundary of $\mathscr{X}$ and suppose that in the interior of $\mathscr{X}$

$$F(x, y, z_1, z_{1x}, z_{1y}, z_{1xx}, z_{1xy}, z_{1yy}) \geqq 0$$

and

$$F(x, y, z_2, z_{2x}, z_{2y}, z_{2xx}, z_{2xy}, z_{2yy}) \leqq 0 .$$

Then, either $z_1(x, y) < z_2(x, y)$ in the interior of $\mathscr{X}$ or

$$z_1(x, y) \equiv z_2(x, y) \quad \text{on } \mathscr{X} .$$

Suppose $M > 1$ to be a uniform bound of $|f_\nu|$, $\nu \in \mathscr{A}$ on ω. Since constants are solutions of (1) it follows from the Maximum Principle that $|z_\nu(x, y)| \equiv |z(x, y; f_\nu)| < M$ for $(x, y) \in \bar{\Omega}$ and all $\nu \in \mathscr{A}$. Also, suppose $\{z_\nu(x, y) : \nu \in \mathscr{B}\} \equiv \{z_\nu(x, y) : \nu \in \mathscr{A}$ and $z_\nu(x, y) > 0$ on $\bar{\Omega}\}$.

LEMMA 1. *Let $P_0(x_0, y_0)$ be any point of Ω and suppose $\{K_n\}_{n=0}^{n=\infty}$ is a sequence of closed circular disks each having $P_0(x_0, y_0)$ as its center and $R_n = (1/7)^n R_0$ as its radius where $R_0 \leqq 1$ and $K_0 \subset \Omega$. Then whenever $z_\nu(x, y)$ is a positive solution of* (1) *there exists a constant H, $0 < H < 1$, depending only on R_0, $\delta \equiv \max h(x, y)$ where $(x, y) \in \bar{\Omega}$, and M such that for all $\nu \in \mathscr{B}$*

$$z_\nu(x, y) > H[\delta, M, R_0] z_\nu(x_0, y_0) \qquad \text{on } 0 \leqq |P - P_0| \leqq (1/7)R_0$$

and

$$z_\nu(x, y) > H[\delta, M, (1/7)^n R_0] z_\nu(x_0, y_0) \geqq H[\delta, M, (1/7)R_0](1/n) z_\nu(x_0, y_0)$$

on $0 \leqq |P - P_0| \leqq (1/7)^{n+1} R_0$, $n = 1, 2, 3, \cdots$.[1]

Proof. Let E denote the component of the set

$$\{(x, y) \in K_0 : z_\nu(x, y) > (1/2)\, z_\nu(x_0, y_0)\}$$

[1] See Bers and Nirenberg [1] for a proof of a Harnack inequality for solutions of the uniformly elliptic equation (1).

which contains $P_0(x_0, y_0)$. We can apply the Maximum Principle to conclude that E must contain an arc of the circumference of K_0. Hence, there is a Jordan arc Γ contained in E with one end at (x_0, y_0) and the other end at a point (x_1, y_1) on the circumference of K_0 which is such that with the exception of (x_1, y_1) Γ is contained in the interior of E. Let K^2 and K^3 be the two closed disks each of which has radius $\sqrt{5}/2\, R_0$ and each of which has the points (x_0, y_0) and (x_1, y_1) on its circumference. Each point $(x, y) \in K^2 \cap K^3$ satisfies at least one of the following conditions:

(a) $(x, y) \in \Gamma \cup bdry(K^2 \cap K^3)$,

(b) (x, y) is in a subdomain of K^2 the boundary of which consists of arcs of Γ and arcs of the circumference of K^2,

(c) (x, y) is in a subdomain of K^3 the boundary of which consists of arcs of Γ and arcs of the circumference of K^3.

Let K^4 be the closed disk with center at

$$(x_4, y_4) \equiv \left(\frac{3x_0 + x_1}{4}, \frac{3y_0 + y_1}{4}\right)$$

and radius $(3/4)\, R_0$ and let (x_2, y_2) and (x_3, y_3) be the respective centers of K^2 and K^3. It is clear that

$$\{(x, y) : (x - x_2)^2 + (y - y_2)^2 \leqq \varepsilon^2(\sqrt{5}/2\, R_0)^2\} \subset \operatorname{comp} K_0\,,$$
$$\{(x, y) : (x - x_3)^2 + (y - y_3)^2 \leqq \varepsilon^2(\sqrt{5}/2\, R_0)^2\} \subset \operatorname{comp} K_0\,,$$

and

$$\{(x, y) : (x - x_4)^2 + (y - y_4)^2 \leqq \varepsilon^2(3/4\, R_0)^2\} \subset \text{interior}\,(K^2 \cap K^3)$$

where $\varepsilon = 1/10$.

Consider the function

$$v(x, y; \xi, \eta; r) \equiv \frac{N(e^{-\alpha\sigma^2} - e^{-\alpha r^2})}{1 - e^{-\alpha r^2}}$$

defined on the region

$$S(\xi, \eta; r) \equiv \{(x, y) : \varepsilon^2 r^2 \leqq \sigma^2 = (x - \xi)^2 + (y - \eta)^2 \leqq r^2\} \cap K_0$$

where $\alpha > 0$ and $N = 1/2\, z_\nu(x_0, y_0)$. In this region

$$v_x^2 + v_y^2 = \frac{4\alpha^2 N^2 \sigma^2 e^{-2\alpha\sigma^2}}{(1 - e^{-\alpha r^2})^2} \leqq \frac{4N^2}{\sigma^2} \leqq \frac{M^2}{\varepsilon^2 r^2} \qquad \text{for all } \nu \in \mathscr{B}\,.$$

Furthermore, $v < N$ on $S(\xi, \eta; r)$, $v = 0$ where $\sigma = r$, and $v > 0$ where $\sigma < r$. If A, B, and C are evaluated at $(x, y, \gamma v_x, \gamma v_y)$, the following succession of inequalities are valid in $S(\xi, \eta; r)$ where γ, $0 < \gamma < 1$, is any fixed real number.

$$\begin{aligned}
L[\gamma v](1-e^{-\alpha r^2}) &= 2\alpha\gamma Ne^{-\alpha\sigma^2}\{2\alpha[A(x-\xi)^2+2B(x-\xi)(y-\eta) \\
&\qquad + C(y-\eta)^2]-(A+C)\} \\
&\geqq 2\alpha\gamma Ne^{-\alpha\sigma^2}\left\{\frac{4\alpha(AC-B^2)\sigma^2}{(A+C)+\sqrt{(A+C)^2-4(AC-B^2)}}-(A+C)\right\} \\
&\geqq \frac{2\alpha\gamma Ne^{-\alpha\sigma^2}}{(A+C)}\{2\alpha\varepsilon^2r^2-(A+C)^2\} \\
&\geqq \frac{2\alpha\gamma Ne^{-\alpha\sigma^2}}{(A+C)}\left\{2\alpha\varepsilon^2r^2-\frac{1}{125}\log\log[\gamma^2(v_x^2+v_y^2)+e]-h(x,y)\right\} \\
&\geqq \frac{2\alpha\gamma Ne^{-\alpha\sigma^2}}{(A+C)}\left\{2\alpha\varepsilon^2r^2-\frac{1}{125}\log\log\left[\frac{M^2}{\varepsilon^2r^2}+e\right]-\delta\right\}
\end{aligned}$$

where $\delta \equiv \max h(x, y)$ for $(x, y) \in \bar{\Omega}$. Now $L[\gamma v] \geqq 0$ on $S(\xi, \eta; r)$ if one chooses

$$\alpha \geqq \frac{1}{250\varepsilon^2r^2}\left\{\log\log\left[\frac{M^2}{\varepsilon^2r^2}+e\right]+125\delta\right\} .$$

Let

$$v_2(x, y) \equiv v(x, y; x_2, y_2; \sqrt{5}/2\, R_0)$$

and

$$v_3(x, y) \equiv v(x, y; x_3, y_3, \sqrt{5}/2\, R_0) .$$

Let

$$\alpha = \frac{32}{45R_0^2}\log\log[(4M/3\varepsilon R_0)^2+e]+125\delta\}$$

and assume that (x, y) is in the interior of $K^2 \cap K^3$ and either $(x, y) \in \Gamma$ or (x, y) satisfies condition (b), then we can apply the Maximum Principle to conclude that $z_\nu(x, y) > v_2(x, y)$. Similarly, if (x, y) is in the interior of $K^2 \cap K^3$ and either $(x, y) \in \Gamma$ or (x, y) satisfies (c), we can conclude that $z_\nu(x, y) > v_3(x, y)$. Thus, for all $(x, y) \in$ interior $(K^2 \cap K^3)$ it follows that

$$z_\nu(x, y) > \min[v_2(x, y), v_3(x, y)] .$$

Now on the circle $(x-x_4)^2+(y-y_4)^2 = \varepsilon^2(3/4\, R_0)^2$

$$\begin{aligned}
\gamma_0 \equiv \min[v_2(x, y), v_3(x, y)] &= N\frac{\exp\left(-\frac{5}{4}\lambda^2\alpha R_0^2\right)-\exp\left(-\frac{5}{4}\alpha R_0^2\right)}{1-\exp\left(-\frac{5}{4}\alpha R_0^2\right)} \\
&> N(1-\lambda^2)\exp\left(-\frac{5}{4}\lambda^2\alpha R_0^2\right)
\end{aligned}$$

where $\lambda = [(\sqrt{17} + 3\varepsilon)\sqrt{5}]/10 < 1$. Another application of the Maximum Principle yields $z_\nu(x, y) > (\gamma_0/N)v_4(x, y)$ on $S(x_4, y_4, 3/4\, R_0)$ where

$$v_4(x, y) \equiv v(x, y; x_4, y_4; 3/4\, R_0)\ .$$

Now the annulus $S(x_4, y_4; 3/4\, R_0)$ contains the disk with center at (x_0, y_0) and radius $1/7\, R_0$. On this disk

$$\begin{aligned} v_4(x, y) \geqq \gamma_1 \equiv N \frac{\exp\left(-\frac{9}{16}\rho^2\alpha R_0^2\right) - \exp\left(-\frac{9}{16}\alpha R_0^2\right)}{1 - \exp\left(-\frac{9}{16}\alpha R_0^2\right)} \\ > N(1-\rho^2)\exp\left(-\frac{9}{16}\rho^2\alpha R_0^2\right) \end{aligned}$$

where $\rho = 11/21$.

Therefore, on the disk with center (x_0, y_0) and radius $1/7\, R_0$.

$$\begin{aligned} z_\nu(x, y) > \frac{\gamma_0\gamma_1}{N} &> \frac{1}{2}(1-\lambda^2)(1-\rho^2)\exp\left[-\left(\frac{5}{4}\lambda^2 + \frac{9}{16}\rho^2\right)\alpha R_0^2\right] z_\nu(x_0, y_0) \\ &> \frac{1}{2}(1-\lambda^2)(1-\rho^2)\exp\left(-\frac{45}{32}\alpha R_0^2\right) z_\nu(x_0, y_0) \\ &> \frac{1}{2}(1-\lambda^2)(1-\rho^2)\exp(-125\delta) \\ &\quad \cdot \exp\{-\log\log[(4M/3\varepsilon R_0)^2 + e]\} z_\nu(x_0, y_0) \\ &> H[\delta, M, R_0] z_\nu(x_0, y_0) \\ &\qquad \text{on } 0 \leqq |P - P_0| \leqq (1/7)\, R_0 \text{ for all } \nu \in \mathscr{B} \end{aligned}$$

where

$$H[\delta, M, R_0] = \frac{1}{2}(1-\lambda^2)(1-\rho^2)\exp(-125\delta)\left\{\log\left[\left(\frac{4M}{3\varepsilon R_0}\right)^2 + e\right]\right\}^{-1}.$$

Now by an inductive argument one concludes that

$$\begin{aligned} H[\delta, M, (1/7)^n R_0] &= \frac{1}{2}(1-\lambda^2)(1-\rho^2)\exp(-125\delta) \\ &\quad \cdot \left\{\log\left[\left(\frac{4M(7)^n}{3\varepsilon R_0}\right)^2 + e\right]\right\}^{-1} \geqq \frac{1}{n} H[\delta, M, 1/7\, R_0] \end{aligned}$$

and

$$z_\nu(x, y) > \frac{1}{n} H[\delta, M, 1/7\, R_0] z_\nu(x_0, y_0) \quad \text{on } 0 \leqq |P - P_0| \leqq (1/7)^{n+1} R_0\ ,$$

$n = 1, 2, 3, \cdots$ for all $\nu \in \mathscr{B}$, thus proving the lemma.

LEMMA 2. *Using the assumptions of Lemma* 1

$$z_\nu(x, y) < \frac{1}{H[\delta, M, 7/8\, R_0]} z_\nu(x_0, y_0)$$

for $0 \leqq |P - P_0| \leqq 1/8\, R_0$ *for all* $\nu \in \mathscr{B}$.

Proof. Follows directly from Lemma 1.

It is of interest to note that if $z(x, y)$ is a positive solution of $A(x, y, p, q)r + 2B(x, y, p, q)s + C(x, y, p, q)t = 0$ in a domain T, then for any compact $U \subset T$ and compact S properly contained in U there is an $H > 0$ depending only on the bound of $z(x, y)$ on U and the distance from S to the boundary of U such that

$$\frac{1}{H} z(x_2, y_2) \leqq z(x_1, y_1) \leqq Hz(x_2, y_2)$$

for any two points (x_1, y_1) and (x_2, y_2) in S.

LEMMA 3. *If* $z_\nu(x, y)$ $\nu \in \mathscr{B}$ *is a solution of* (1) *on the interior of a closed circular disk* K_0 *of radius* $R_0 \leqq 1$ *with center* $P_0(x_0, y_0)$, *then there exists a continuous decreasing function* $g_{P_0}(r)$, $0 \leqq r < R_0$, $g_{P_0}(0) = 1$, *and a continuous increasing function* $f_{P_0}(r)$, $0 \leqq r < R_0$, $f_{P_0}(0) = 1$ *such that*

$$g_{P_0}(r) z_\nu(x_0, y_0) \leqq z_\nu(x, y) \leqq f_{P_0}(r) z_\nu(x_0, y_0)$$

for $0 \leqq |P - P_0| \leqq r$ *where* g *and* f *are independent of* ν.

Proof. Define

$$g_{P_0}(r) \equiv \inf_{\nu \in \mathscr{B}} \inf_{|P - P_0| \leqq r} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)}$$

and

$$f_P(r) \equiv \sup_{\nu \in \mathscr{B}} \sup_{|P - P_0| \leqq r} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} .$$

By Lemma 1, Lemma 2, and an argument similar to that used in Kellogg [4] (page 263) $f_{P_0}(r)$ and $g_{P_0}(r)$ exist for each $0 \leqq r < R_0$. Using standard arguments it is clear that

$$(2) \qquad \lim_{r \to r_0^-} \inf_{\nu \in \mathscr{B}} \inf_{|P - P_0| \leqq r} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} = \inf_{\nu \in \mathscr{B}} \inf_{|P - P_0| \leqq r_0} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)}$$

for $0 < r_0 < R_0$ and

$$(3) \qquad \lim_{r \to r_0^-} \sup_{\nu \in \mathscr{B}} \sup_{|P - P_0| \leqq r} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} = \sup_{\nu \in \mathscr{B}} \sup_{|P - P_0| \leqq r_0} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)}$$

for $0 < r_0 < R_0$. Also,

$$\lim_{r \to 0+} \inf_{\nu \in \mathscr{B}} \inf_{|P-P_0| \leqq r} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} = 1 . \tag{4}$$

This follows by observing that whenever $z_\nu(x, y) > 0$ for $0 \leqq |P - P_0| \leqq R_0$

$$z_\nu(x, y) > H[\delta, M, R_0] z_\nu(x_0, y_0) \quad \text{for } 0 \leqq |P - P_0| \leqq 1/7\, R_0$$

and all $\nu \in \mathscr{B}$. This latter inequality implies

$$\begin{aligned} z_\nu(x, y) &- H[\delta, M, R_0] z_\nu(x_0, y_0) \\ &> H[\delta, M, 1/7\, R_0]\{z_\nu(x_0, y_0) - H[\delta, M, R_0] z_\nu(x_0, y_0)\} \end{aligned}$$

for $0 \leqq |P - P_0| \leqq (1/7)^2 R_0$ and all $\nu \in \mathscr{B}$. Thus, for $0 \leqq |P - P_0| < (1/7)^2 R_0$ and all $\nu \in \mathscr{B}$

$$z_\nu(x, y) > [1 - (1 - H[\delta, M, R_0])(1 - H[\delta, M, 1/7\, R_0])] z_\nu(x_0, y_0)$$

By induction

$$\begin{aligned} z_\nu(x, y) &> \left[1 - \prod_{i=0}^{n} (1 - H[\delta, M, (1/7)^i R_0])\right] z_\nu(x_0, y_0) \\ &> \left[1 - (1 - H[\delta, M, R_0]) \prod_{i=1}^{n} \left(1 - H[\delta, M, 1/7\, R_0] \frac{1}{i}\right)\right] z_\nu(x_0, y_0) \end{aligned} \tag{5}$$

for $0 \leqq |P - P_0| \leqq (1/7)^{n+1} R_0$ and all $\nu \in \mathscr{B}$. Hence,

$$\begin{aligned} 1 - \inf_{\nu \in \mathscr{B}} \inf_{|P-P_0| \leqq (1/7)^{n+1} R_0} & \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} \\ &< \exp\left(-H[\delta, M, R_0] - H[\delta, M, 1/7\, R_0] \sum_{i=1}^{n} \frac{1}{i}\right) . \end{aligned}$$

(4) then follows by the usual argument.

Suppose P is any point in the circle $0 \leqq |P - P_0| \leqq (1/7)^n R_0/[1 + (1/7)^n]$ and let K be the interior of a closed circular disk of radius $1/[1 + (1/7)^n] R_0$ about P. Since $z_\nu(x, y) > 0$ on $0 \leqq |P - P_0| \leqq R_0$ we have $z_\nu(x', y') > 0$ on $0 \leqq |P - P'| \leqq R_0/[1 + (1/7)^n]$ for all $\nu \in \mathscr{B}$. Also

$$\begin{aligned} z_\nu(x', y') - \Big[1 &- (1 - H[\delta, M, 7/8\, R_0])^i \\ &\cdot \prod_{i=1}^{n} \left(1 - H[\delta, M, 1/8\, R_0] \frac{1}{i}\right)\Big] z_\nu(x, y) > 0 \end{aligned}$$

on

$$0 \leqq |P - P'| \leqq (1/7)^{n+1} \frac{1}{1 + (1/7)^n} R_0 \qquad \text{for all } \nu \in \mathscr{B} .$$

Now $P_0(x_0, y_0)$ is such a point $P'(x', y')$; therefore, for all $\nu \in \mathscr{B}$ and $0 \leqq |P - P_0| \leqq (1/7)^{n+1}[1/(1 + (1/7)^n)] R_0$

$$z_\nu(x_0, y_0) > \Big[1 - (1 - H[\delta, M, 7/8\, R_0]) \cdot \prod_{i=1}^{n}\Big(1 - H[\delta, M, 1/8\, R_0]\frac{1}{i}\Big)\Big] z_\nu(x, y)\ ,$$

$$\sup_{\nu\in\mathscr{B}} \sup_{|P-P_0|\leq (1/7)^{n+1}[1/(1+(1/7)^n)]R_0} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} - 1 < \exp\Big(-H[\delta, M, 7/8\, R_0] - H[\delta, M, 1/8\, R_0]\sum_{i=1}^{n}\frac{1}{i}\Big)$$

and we may conclude that

$$\lim_{r\to 0+}\sup_{\nu\in\mathscr{B}} \sup_{|P-P_0|\leq r} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} = 1\ . \tag{6}$$

We will now show that

$$\lim_{r\to r_0^+}\sup_{\nu\in\mathscr{B}} \sup_{|P-P_0|\leq r} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} = \sup_{\nu\in\mathscr{B}} \sup_{|P-P_0|\leq r_0} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} \tag{7}$$

for $0 \leqq r_0 < R_0$.

Suppose the contrary, then since $f_{P_0}(r)$ is increasing $\lim_{r\to r_0^+} f_{P_0}(r) > f_{P_0}(r)$. Hence, there exists an $\varepsilon > 0$ and a decreasing sequence $\{r_n\}$ converging to r_0 such that for all positive integers n $f_{P_0}(r_n) - f_{P_0}(r_0) > \varepsilon$. By the definition of supremum there exists for the above ε and each n a function $z_n(x, y)$ such that

$$\sup_{\nu\in\mathscr{B}} \sup_{|P-P_0|\leq r_n} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} - \sup_{|P-P_0|\leq r_n} \frac{z_n(x, y)}{z_n(x_0, y_0)} \leqq \frac{\varepsilon}{2}$$

and thus,

$$\sup_{|P-P_0|\leq r_n} \frac{z_n(x, y)}{z_n(x_0, y_0)} - \sup_{\nu\in\mathscr{B}} \sup_{|P-P_0|\leq r_0} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} > \frac{\varepsilon}{2}\ .$$

By the Maximum Principle

$$\sup_{|P-P_0|\leq r_n} \frac{z_n(x, y)}{z_n(x_0, y_0)}$$

is assumed at some point $P_n(x_n, y_n)$ on $|P - P_0| = r_n$. Hence, there exists a sequence of points $\{P_n(x_n, y_n)\}$ which contains a convergent subsequence which converges to a point $P_0'(x_0', y_0') \in |P - P_0| = r_0$. Suppose our sequence is such without relabeling. Let

$$\varepsilon_1 = \varepsilon \Big/ \sup_{\nu\in\mathscr{B}} \sup_{|P-P_0|\leq r_0} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)}\ .$$

Therefore,

$$\frac{z_n(x_n, y_n)}{z_n(x_0, y_0)} - \frac{z_n(x_0', y_0')}{z_n(x_0, y_0)} \geqq \frac{z_n(x_n, y_n)}{z_n(x_0, y_0)} - \sup_{\nu \in \mathscr{B}} \sup_{|P-P_0| \leqq r_0} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)}$$
$$> \frac{\varepsilon_1}{2} \sup_{\nu \in \mathscr{B}} \sup_{|P-P_0| \leqq r_0} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} .$$

Let us center our attention on the point $P_0'(x_0', y_0')$. Then, using (6), there exists a $\delta_1 > 0$ such that

$$\sup_{\nu \in \mathscr{B}} \sup_{|P-P_0'| \leqq r} \frac{z_\nu(x, y)}{z_\nu(x_0', y_0')} - 1 \leqq \frac{\varepsilon_1}{2} \qquad \text{if } r < \delta_1 .$$

Also, by (4) there exists a $\delta_2 > 0$ such that if $r < \delta_2$

$$1 - \inf_{\nu \in \mathscr{B}} \inf_{|P-P_0'| \leqq r} \frac{z_\nu(x, y)}{z_\nu(x_0', y_0')} \leqq \frac{\varepsilon_1}{2} .$$

Thus, if $|P - P_0'| \leqq \min[\delta_1, \delta_2]$

$$\left| \frac{z_\nu(x, y)}{z_\nu(x_0', y_0')} - 1 \right| \leqq \frac{\varepsilon_1}{2} \qquad \text{for all } \nu \in \mathscr{B} .$$

It then follows that if $|P_n - P_0'| \leqq \min[\delta_1, \delta_2]$

$$\frac{\varepsilon_1}{2} \sup_{\nu \in \mathscr{B}} \sup_{|P-P_0| \leqq r_0} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} < \frac{z_n(x_n, y_n) - z_n(x_0', y_0')}{z_n(x_0, y_0)}$$
$$< \frac{z_n(x_n, y_n) - z_n(x_0', y_0')}{z_n(x_0', x_0')} \cdot \frac{z_n(x_0', y_0')}{z_n(x_0, y_0)}$$
$$< \frac{z_n(x_n, y_n) - z_n(x_0', y_0')}{z_n(x_0', y_0')} \cdot \sup_{\nu \in \mathscr{B}} \sup_{|P-P_0| \leqq r_0} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)}$$
$$< \frac{\varepsilon_1}{2} \sup_{\nu \in \mathscr{B}} \sup_{|P-P_0| \leqq r_0} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} ,$$

a contradiction. By a similar argument we may conclude

$$(8) \qquad \operatorname*{Lim}_{r \to r_0^+} \inf_{\nu \in \mathscr{B}} \inf_{|P-P_0| \leqq r} \frac{z_\nu(x, y)}{z_\nu(x_0, y_0)} = \inf_{\nu \in \mathscr{B}} \inf_{|P-P_0| \leqq r_0} \frac{z_\nu(x, y)}{z_\nu(y_0, x_0)} ,$$

Hence, by (2), (3), (7), and (8) our lemma is true.

Proof of Theorem 1. Recall that for $\nu \in \mathscr{A}$ $|f_\nu| < M$ on ω and $|z_\nu(x, y)| < M$ on $\bar{\Omega}$. Also, for all $\nu \in \mathscr{A}$, $z_\nu(x, y) + M$ satisfies (1) and $z_\nu(x, y) + M > 0$ on $\bar{\Omega}$.

Let $P_0(x_0, y_0)$ be any point of Ω and assume K is a closed circular disk whose center is $P_0(x_0, y_0)$ and such that $K \subset \Omega$. Hence, by Lemma 3 there exists positive continuous functions $f_{P_0}(r)$ and $g_{P_0}(r)$ (independent of ν) such that $\lim_{r \to 0} f_{P_0}(r) = 1$, $\lim_{r \to 0} g_{P_0}(r) = 1$, and on the interior of K

$$g_{P_0}(r)[z_\nu(x_0, y_0) + M] \leqq z_\nu(x, y) + M \leqq f_{P_0}(r)[z_\nu(x_0, y_0) + M]$$

and

$$-|z_\nu(x_0, y_0) + M|\,|g_{P_0}(r) - 1| \leqq z_\nu(x, y) - z_\nu(x_0, y_0) \leqq |z_\nu(x_0, y_0) + M|\,|f_{P_0}(r) - 1|$$

for all $\nu \in \mathscr{A}$. It then follows that since $\{z_\nu(x, y) : \nu \in \mathscr{A}\}$ is uniformly bounded on $\bar{\Omega}$ that $\{z_\nu(x, y) : \nu \in \mathscr{A}\}$ is equicontinuous on Ω thus proving Theorem 1.

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