ON FUNDAMENTAL PROPERTIES OF A BANACH SPACE WITH A CONE

T. ANDÔ

1. Introduction. Normed vector lattices have been investigated from various angles (see [1] Chap. 15 and [7] Chap. 6). On the contrary, it seems that there remain several problems unsolved in the theory of general normed spaces with a cone since the pioneer works of Riesz and Krein, though recently Namioka [8], Schaefer [9] and others made many efforts in analysing and extending the results of Riesz and Krein. In this paper we shall discuss two among them. Let E be a Banach space with a closed cone K (for the terminologies see § 2);

(A) What condition on the dual E^* is necessary and sufficient for that $E = K - K$?

(B) What condition on the dual is necessary and sufficient for the interpolation property of E?

Grosberg and Krein [3] dealt with (A) in a reversed form;

(A′) What condition on E is necessary and sufficient for that $E^* = K^* - K^*$ where K^* is the dual cone?

Schaefer ([9], Th. 1.6) obtained a complete answer to (A′) within a scope of locally convex spaces. A result of Riesz gives a half of an answer to (B), while Krein [6] obtained a complete answer only under the assumption that the cone has an inner point.

The purpose of this paper is to give answers to both (A) and (B) in natural settings. Our starting assumptions consist of the completeness of E and of the closedness of the cone K.

After several comments on order properties in § 2, Lemmas in § 3 present algebraic forms to both the property named normality by Krein [5] and that named (BZ)-property by Schaefer [9], supported by Banach's open mapping theorem. Then Theorem 1 will produce an answer to (A) via these Lemmas. § 4 is devoted to an answer to (B) under the condition that E is an ordered Banach space. It should be remarked that our main theorems are also valid for (F) spaces, that is, metrisable complete locally convex spaces.

2. Definitions and consequences. Let E[1] be a real normed space and let K be a cone, that is, a subset of E with the following properties:

(1) $K + K \subset K$,

(2) $\alpha K \subset K$ for all $\alpha \geqq 0$, and

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[1] Elements of E are denoted by x, y, a, $\cdots$, e, and those of the dual E^* by f, g, h. Scalars are denoted by Greek letters. θ is reserved for the zero element.

(3) $K \cap (-K) = \{\theta\}$. Then the natural partial ordering $\geqq$ is associated with the cone K, i.e. $a \geqq b$ in case $a - b \in K$. A subset of the form $\{x; a \leqq x \leqq b\}$ will be called an *interval*. The dual E^* of E is also partially ordered by the *dual cone* $K^* \stackrel{\text{def}}{=} \{f \in E^*; f(x) \geqq 0$ for all $x \in K\}$, though K^* does not always satisfy the condition (3).[2]

The cone K is said to *generate* E or to be a *generating cone* in case every element in E can be written as difference of two in K, that is, $E = K - K$. E is said to have the *interpolation property* with respect to $\geqq$ in case $a, b \geqq c, d$ implies the existence of x such that $a, b \geqq x \geqq c, d$. This property is equivalent to the following one named the *decomposition property*: whenever $a, b, x \in K$ and $a + b \geqq x$, there exist $c, d \in K$ such that $x = c + d$, $a \geqq c$ and $b \geqq d$. When for any pair $a, b \in E$ there exists the *supremum* $a \vee b$, E is called a *vector lattice*. A vector lattice has the interpolation property and its cone is generating.

There are several notions connected with the so-called order topology. E is said to be *(o)-complete* in case any upward directed subset with an upper bound (with respect to $\geqq$) has the supremum. When the directed subset in question is restricted to that consisting of countable members, E is said to be *σ-(o)-complete*. As a less restrictive completeness, E is said to be *quasi-(o)-complete* in case any sequence $\{a_i\}$, such that $\theta \leqq a_1 \leqq a_2 \leqq \cdots \leqq a$ and $a_{i+j} - a_i \leqq \varepsilon_i a$ with $\varepsilon_i \downarrow 0$, has the supremum. In many cases (o)-completeness can be derived from σ-(o)-completeness. It is clear that if E with the generating cone is (o)-complete and has the interpolation property, it is a vector lattice (cf. [9] Th. 13.2).

Usually a complete normed vector lattice is called a *Banach lattice* in case its norm satisfies the following condition: $|a| \leqq |b|$ implies $\|a\| \leqq \|b\|$ where $|a| \stackrel{\text{def}}{=} a \vee (-a)$. The cone in a Banach lattice is obviously closed. In general, order topology is connected with the norm topology through the closedness of the cone in the following way: if $a_i \leqq a$ $i = 1, 2, \cdots$ and $\lim_{i\to\infty} a_i = x$ then $x \leqq a$, in particular, if $a_1 \leqq a_2 \leqq \cdots$ and $\lim_{i\to\infty} a_i = a$ then a is the supremum of $\{a_i\}$. Thus a Banach lattice is quasi-(o)-complete. In this connection a quasi-(o)-complete Banach space with a closed generating cone will be called an *ordered Banach space*.

3. Generating cone.

In this section E is a Banach space with a closed cone K. First on the ground of Klee's theorem [4] it will be proved that the generating property is equivalent to the stronger one named strict *(BZ)-property* in Schaefer [9] ((3) in Lemma 1 below).

LEMMA 1. *The following conditions are mutually equivalent, where α, β and ρ are positive constants and U denotes the unit ball of E:*

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[2] K^* satisfies the condition (3), if and only if $K - K$ is dense in E.

(1) *K generates E,*
(2) $(K \cap U - K \cap U)^- \supset \alpha U$ *where* $(\cdot)^-$ *denotes the closure,*
(3) $(K \cap U - K \cap U) \supset \beta U$,
(4) *any* $x \in E$ *admits a decomposition* $x = a - b$ *such that* $a, b \in K$ *and* $\| a \|, \| b \| \leqq \rho \| x \|$.

Proof. (1) $\Rightarrow$ (2) follows from the second category of E, because $E = K - K = \bigcup_{n=1}^{\infty} n(K \cap U - K \cap U)^-$. In order to see (2) $\Rightarrow$ (3), let $V \overset{\text{def}}{=} K \cap U - K \cap U$ and let F be the subspace generated by V. Then on the basis of completeness of K, Klee ([4] and [8] Th. 5.5) shows that F is complete under the norm defined by $\| x \|_V = \inf \{| \lambda |; x \in \lambda V\}$. Then (2) shows that under the natural injection of F into E the closure of the image of the unit ball V is a neighborhood of the origin in E. A modification of Banach's open mapping theorem (see [2] Chap. I, § 3) yields (3). (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are trivial.

In the next place quasi-(o)-completeness will be connected with the property named *normality* in Krein [5] ((3) in Lemma 2 below).

LEMMA 2. *The following conditions are mutually equivalent, where* ρ *is a positive constant*:
(1) *E is quasi-(o)-complete,*
(2) *every interval is bounded in norm,*
(3) $a \leqq x \leqq b$ *implies* $\| x \| \leqq \rho \cdot \max (\| a \|, \| b \|)$,
(4) $(U + K) \cap (U - K) \subset \rho U$.

Proof. In order to see (1) $\Rightarrow$ (2), for each $a \in K$ let

$$V_a \overset{\text{def}}{=} \{x: -a \leqq x \leqq a\}$$

and let F_a be the subspace generated by V_a. F_a is complete under the norm defined by $\| x \|_a = \inf \{\lambda: -\lambda a \leqq x \leqq \lambda a\}$. In fact, if

$$\| x_{i+1} - x_i \|_a < 1/2^i \qquad (i = 1, 2, \cdots),$$

by the definition of the norm $\theta \leqq y_i \leqq a/2^{i-1}$ where $y_i = x_{i+1} - x_i + a/2^i$. Then quasi-($o$)-completeness implies the existence of the supremum y of the sequence $\{\sum_{i=1}^{n} y_i\}_n$. Put $x = y + x_1 - a$, then $x - x_i$ is the supremum of the sequence $\{x_n - x_i - a/2^{n-1}\}_{n \geqq i}$ hence $x - x_i \leqq a/2^{i-1}$, and similarly $x - x_i \geqq - a/2^{i-1}$. This means that

$$\| x - x_i \|_a \leqq 1/2^{i-1} \qquad (i = 1, 2, \cdots),$$

hence $\lim_{i \to \infty} x_i = x$. Since K is closed, as remarked in § 2, the natural injection of F_a into E is a closed linear mapping, hence on account of Banach's closed graph theorem (see [2] Chap. I, § 3) it is bounded, i.e. V_a is bounded in E. Now every interval is readily proved to be bounded.

(3) follows from (2) via a standard argument (see [8] p. 32). (3) $\Rightarrow$ (4) is trivial. (4) $\Rightarrow$ (1) follows from the closedness of an interval and the completeness of K.

Before going into the first theorem, let us recall the definition of polar sets. The *polar set* A^0 of $A \subset E$ (resp. $\subset E^*$) is defined by $A^0 = \{f \in E^*; f(x) \leqq 1 \text{ for all } x \in A\}$ (resp. $= \{x \in E; f(x) \leqq 1 \text{ for all } f \in A\}$). For example, U^0 is the unit ball of E^* and $K^0 = -K^*$. The bipolar theorem (see [2] Chap. IV, § 1) asserts that (1) $\Gamma(A, B)^0 = A^0 \cap B^0$ where $\Gamma(A, B)$ denotes the convex hull of $A \cup B$, and (2) if $A \ni \theta$ and $B \ni \theta$ are closed convex sets in E (resp. weakly[3], i.e. $\sigma(E^*, E)$, closed convex sets in E^*) $(A \cap B)^0 = \Gamma(A^0, B^0)^{w-}$ (resp. $= \Gamma(A^0, B^0)^-$) where $(\cdot)^{w-}$ denotes the weak closure, and (3) if A contains θ and is a closed convex set in E (resp. weakly closed convex set in E^*), $A^{00} = A$. By the way, remark that the weak compactness of U^0 and the weak closedness of imply that both $U^0 + K^*$ and $U^0 - K^*$ are weakly closed.

THEOREM 1. (1) *K generates E if and only if E^* is quasi-(o)-complete.*

(2) *K^* generates E^* if and only if E is quasi-(o)-complete.*

Proof. (1) First remark the formula: $A + B \supset \Gamma(A, B) \supset \frac{1}{2}A + \frac{1}{2}B$ for any convex sets $A \ni \theta$ and $B \ni \theta$. Now the following chain of equivalences is valid, where α, β, γ and ρ are positive constants:

K generates E

$\Longleftrightarrow$ $(U \cap K - U \cap K)^- \supset \alpha U$ by Lemma 1

$\Longleftrightarrow$ $\Gamma(U \cap K, -U \cap K)^- \supset \beta U$ by the above remark

$\Longleftrightarrow$ $\Gamma(U^0, -K^*)^{w-} \cap \Gamma(U^0, K^*)^{w-} \subset \gamma U^0$ by the bipolar theorem

$\Longleftrightarrow$ $(U^0 - K^*)^{w-} \cap (U^0 + K^*)^{w-} \subset \rho U^0$ by the above remark

$\Longleftrightarrow$ $(U^0 - K^*) \cap (U^0 + K^*) \subset \rho U^0$ by the weak closedness of $U^0 \pm K^*$

$\Longleftrightarrow$ E^* is quasi-(o)-complete by Lemma 2. A proof of (2) is similar and is omitted.

The "only if" part of (1) is essentially known (see [8] p. 46), while (2) is a restatement of Grosberg-Krein's theorem [3] in terms of order properties[4].

If E^* is quasi-(o)-complete, in view of Lemma 2 every interval of E^* is bounded in norm and weakly closed, hence weakly compact. Therefore it is readily shown that all the three notions of completeness are the same thing on E^*.

4. Interpolation property. In this section E is an ordered Banach space. Then Theorem 1 guarantees that E^* is also an ordered Banach

[3] The weak topology always refers to the topology $\sigma(E^*, E)$.

[4] Grosberg-Krein's proof differs essentially from ours.

space. A result of Riesz can be stated as follows (see [8] Th. 6.1): if E is an ordered Banach space with the interpolation property, the dual has the same property, hence by the remark at the end of § 3 it is a vector lattice. In this section the converse will be proved.

LEMMA 3. *Let E be an ordered Banach space. Then the interpolation property can be derived from the following less restrictive one: for any $\varepsilon > 0$ and $a_i \geqq b_j$ in E $(i, j = 1, 2, \cdots, n)$ there exist $x \in E$ and $y \in K$ such that $a_i \geqq x - y$ and $x \geqq b_i (i = 1, 2, \cdots, n)$ and $\| y \| \leqq \varepsilon$.*

Proof. Let $a, b \geqq c, d$. We can successively find $x_i \in E$ and $y_i \in K$ (x_0 and y_0 being disregarded) such that $a, b, x_{i-1} \geqq x_i - y_i$ and $x_i \geqq c, d, x_{i-1} - y_{i-1}$ and $\| y_i \| \leqq 1/2^i$. Then $-y_{i-1} \leqq x_i - x_{i-1} \leqq y_i$, hence by Lemma 2 $\| x_i - x_{i-1} \| \leqq \rho/2^i$ $(i = 1, 2, \cdots)$. The completeness of E implies that $\lim_{i\to\infty} x_i = x$ exists. Since $\lim_{i\to\infty} y_i = \theta$ and K is closed, we can conclude that $a, b \geqq x \geqq c, d$.

Before going into the second theorem, in order to simplify the notations, for each $A \subset E$ (resp. $\subset E^*$) define $A^\# \stackrel{\text{def}}{=} \{f \in K^*; f(x) \geqq 1$ for all $x \in A\}$ (resp. $\stackrel{\text{def}}{=} \{x \in K: f(x) \geqq 1$ for all $f \in A\}$). Since K is closed convex, on account of the separation theorem (see [2] Chap. II § 3), for $a \in K$ $\{x; x \geqq a\} = a + K = \{a\}^{\#\#}$.

THEOREM 2. *An ordered Banach space E has the interpolation property, if (and only if) the dual E^* has the same property.*

Proof. Suppose that E^* has the interpolation property. It suffices to prove the less restrictive form of the interpolation property for E in Lemma 3. Let $a_i \geqq b_j$ $(i, j = 1, 2, \cdots, n)$. All b_j may be assumed to be in K because K generates E. For any $\varepsilon > 0$ and $\gamma > 0$

$$A \stackrel{\text{def}}{=} (1 + \varepsilon)\Gamma(\{b_i\}^\#; i = 1, 2, \cdots, n)$$

is disjoint from

$$B \stackrel{\text{def}}{=} \Gamma((a_i - K)^0 \cap \gamma U^0; i = 1, 2, \cdots, n) .$$

Otherwise, since E^* is an ordered Banach space with the interpolation property, in view of Riesz result stated above the second dual E^{**} has the same property, therefore there exists $X \in E^{**}$ such that $a_i \geqq X \geqq b_i$ $(i = 1, 2, \cdots, n)$ where E is imbedded into E^{**} in the natural, linear-order preserving way, then $X(f) \leqq 1$ and $X(f) \geqq 1 + \varepsilon$ for $f \in A \cap B$, because, for example, f can be represented as $f = \sum_{i=1}^n \alpha_i g_i$ such that $g_i \in \{b_i\}^\#$ and $\alpha_i \geqq 0$, $\sum_{i=1}^n \alpha_i = 1 + \varepsilon$, hence

$$X(f) = \sum_{i=1}^n \alpha_i X(g_i) \geqq \sum_{i=1}^n \alpha_i g_i(b_i) \geqq \sum_{i=1}^n \alpha_i = 1 + \varepsilon ,$$

This contradiction proves the expected disjointness. Next we shall prove that A is weakly closed and B is weakly compact. Take, for example, the former and suppose $n = 2$ for the simplicity sake. On account of Banach's theorem (see [2] Chap. IV, § 2) it suffices to prove that

$$\Gamma(\{b_i\}^{\#}; i = 1, 2) \cap \rho U^0$$

is weakly closed for all $\rho > 0$. Suppose that a net $\{\alpha_\lambda f_\lambda + (1 - \alpha_\lambda) g_\lambda\}_\lambda$ converges weakly to some h in E^* where $f_\lambda \in \{b_1\}^{\#}$, $g_\lambda \in \{b_2\}^{\#}$, $0 \leqq \alpha_\lambda \leqq 1$ and $\| \alpha_\lambda f_\lambda + (1 - \alpha_\lambda) g_\lambda \| \leqq \rho$. Since E^* is quasi-(0)-complete, by Lemma 2 $\| \alpha_\lambda f_\lambda \|$ and $\| (1 - \alpha_\lambda) g_\lambda \|$ are uniformly bounded. We may assume that $\{\alpha_\lambda\}_\lambda$ converges to some α. If $0 < \alpha < 1$, $\| f_\lambda \|$ and $\| g_\lambda \|$ are uniformly bounded, hence we may even assume that $\{f_\lambda\}_\lambda$ and $\{g_\lambda\}_\lambda$ converge weakly to some f and to some g respectively because of the weak compactness of U^0. Since both $\{b_1\}^{\#}$ and $\{b_2\}^{\#}$ are weakly closed, it follows that $h = \alpha f + (1 - \alpha) g$ is in $\Gamma(\{b_1\}^{\#}, \{b_2\}^{\#})$. If $\alpha = 1$, say, we may assume that $\{f_\lambda\}_\lambda$ converges weakly to some f in $\{b_1\}^{\#}$, therefore $h \geqq f$, hence $h \in \{b_1\}^{\#}$ by the definition of $\{b_1\}^{\#}$. Thus the proof of the weak closedness is over.

Now since A is convex, weakly closed and is disjoint from the convex weakly compact set B, by the separation theorem (see [2] Chap. II, § 3) there exists $c \in E$ such that $f(c) \geqq 1 > g(c)$ for all $f \in A$ and $g \in B$. From the remark preceding the theorem and by the bipolar theorem it follows that $(1 + \varepsilon)c \geqq b_i (i = 1, 2, \cdots, n)$ and $c \in \bigcap_1^n (a_i - K + 1/\gamma\, U)^-$. Therefore there exist $\{c_i\}_1^n$ such that $c + c_i \leqq a_i$ and $\| c_i \| \leqq 2/\gamma$ $(i = 1, 2, \cdots, n)$. Since the cone of E is generating, by Lemma 1 each c_i admits a decomposition $c_i = d_i - e_i$ with $d_i, e_i \in K$ such that

$$\| d_i \|, \| e_i \| \leqq \rho_1 \| c_i \|$$

where ρ_1 is a positive constant. Finally let $x = (1 + \varepsilon)c$ and

$$y = \varepsilon c + \sum_{i=1}^n e_i ,$$

then $x - y \leqq a_i$ and $x \geqq b_i \, (i = 1, 2, \cdots n)$ and, for some $\rho_2 > 0$,

$$\| y \| \leqq \varepsilon \| c \| + \sum_{i=1}^n \| e_i \| \leqq \varepsilon(\rho_2 \| a_1 \| + 4/\gamma) + 2\rho_1 n/\gamma .$$

Since $\varepsilon > 0$ and $\gamma > 0$ are arbitrary, the expected conclusion has been obtained.

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HOKKAIDO UNIVERSITY
AND
INDIANA UNIVERSITY