ALGEBRAIC PROPERTIES OF THE COMPOSITION OF SOLUTIONS OF PARTIAL DIFFERNTIAL EQUATIONS

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1. Introduction. Let $Lu = 0$ be a homogeneous linear partial differential equation with constant coefficients in two independent variables x and y. In [1] Lewy introduced a class of compositions each of which associates with any two solutions u and v of the differential equation a function w which depends bilinearly on u and v. The main result of [1] is that w is a solution of the same differential equation. In this paper we first give a simpler proof of this result and then investigate when the composition is commutative and associative. We prove that for every differential equation at least one of the compositions is commutative. For associativity our results are limited to second order equations. We show that when the differential equation is parabolic, one of the compositions is both commutative and associative. For elliptic and hyperbolic equations none of the compositions are associative, but by a suitable modification an operation which is both commutative and associative is obtained.

2. The compositions. The solutions u and v will be assumed to be defined in a domain $\mathscr{D}$ of the x, y-plane with the origin O as an interior point. Let x_P, y_P and x_Q, y_Q be the coordinates of the points P and Q, respectively; and let $P - Q$ be the point with coordinates $x_P - x_Q, y_P - y_Q$. We use ξ and η to denote operators which map functions into functions such that

$$(2.1) \qquad \xi f(P) = \frac{\partial}{\partial x_P} f(P), \quad \eta f(P) = \frac{\partial}{\partial y_P} f(P) \,.$$

Note that this implies that

$$\xi f(Q) = \frac{\partial}{\partial x_Q} f(Q), \quad \eta f(Q) = \frac{\partial}{\partial y_Q} f(Q) \,,$$

$$\xi f(P - Q) = \frac{\partial}{\partial x_P} f(P - Q) = -\frac{\partial}{\partial x_Q} f(P - Q) \,.$$

All derivatives occurring in formulas will be assumed to be continuous so that the operators ξ and η commute. When a path of integration from O to P is considered, it will always be assumed that for each

point Q of the path both Q and $P - Q$ are in the domain $\mathscr{D}$.

If L is the operator defined by the equation

(2.2) $Lu(P) = \Sigma a_{kl} \xi^k \eta^l u(P)$

where the sum is over nonnegative integers k and l with $k + l \leqq \rho$, then by Green's formula if u and v have continuous derivatives of order $\leqq \rho$

(2.3) $\iint_{\mathscr{G}} [u(P - Q)Lv(Q) - v(Q)Lu(P - Q)]dx_Q\, dy_Q$

$$= \int_{\partial \mathscr{G}} B_1(u(P - Q),\, v(Q))dx_Q + B_2(u(P - Q),\, v(Q))dy_Q$$

where the B_i are certain bilinear forms in u and derivatives of order $\leqq \rho - 1$ evaluated at $P - Q$ on the one hand and v and derivatives of order $\leqq \rho - 1$ evaluated at Q on the other. In general, these bilinear forms are not uniquely determined by L.

Using an arbitrary choice of the B_i for which (2.3) holds, Lewy defines the composition $u \circ v$ of two solutions u and v of

(2.4) $Lu(P) = 0$

by setting

(2.5) $u \circ v(P) = \int_O^P [B_1(u(P - Q),\, v(Q))dx_Q + B_2(u(P - Q),\, v(Q))dy_Q]\,.$

By (2.3) the integral is independent of the path from O to P. In this way we obtain, in general, not a single composition but a class of compositions.

Let $\mathscr{D}'$ be the domain consisting of all points P in $\mathscr{D}$ which can be reached by paths from O such that for each point Q on the path both Q and $P - Q$ are in $\mathscr{D}$. (In particular, if $\mathscr{D}$ is convex, $\mathscr{D}'$ is identical with $\mathscr{D}$.) The following theorem is the main result of [1].

THEOREM 1. *If u and v are solutions of (2.4) with continuous derivatives of order $2\rho - 1$ in $\mathscr{D}$ and if $u \circ v$ is an arbitrary composition given by (2.5), then $L(u \circ v) = 0$ in the domain $\mathscr{D}'$.*

3. **Proof of Theorem 1.** We begin by establishing that if the theorem holds for some one of the compositions given by (2.5) it holds for all of them. If B_1, B_2 are the bilinear forms associated with one composition and B_1', B_2' the forms associated with another composition, then by (2.3) for all sufficiently differentiable functions $u(P - Q)$ and $v(Q)$ (whether solutions of (2.4) or not)

(3.1) $[B_1(u, v) - B_1'(u, v)]dx_Q + [B_2(u, v) - B_2'(u, v)]dy_Q$

must be an exact differential of some function F. Let us prove that this function F is a bilinear form with constant coefficients in $u, \xi u, \eta u, \cdots$ and $v, \xi v, \eta v, \cdots$. By repeated use of the equation

$$\xi^k \eta^l u(P - Q)\xi^m \eta^n v(Q) = \frac{\partial}{\partial x_Q} [\xi^k \eta^l u(P - Q)\xi^{m-1}\eta^n v(Q)]$$
$$+ \xi^{k+1}\eta^l u(P - Q)\xi^{m-1}\eta^n v(Q)$$

we can remove differentiations of v with respect to x in the expression multiplying dx_Q in (3.1). Thus we find (3.1) is equal to

(3.2) $\frac{\partial}{\partial x_Q}(F_0(u, v))dx_Q + \frac{\partial}{\partial y_Q}(F_0(u, v))dy_Q + F_1(u, v)dx_Q$
$$+ F_2(u, v)dy_Q$$

where the F_i are forms bilinear in $u, \xi u, \eta u, \cdots$ and $v, \xi v, \eta v, \cdots$ and where all derivatives of v in F_1 are with respect to y. Also, we must have $(\partial/\partial y_Q)F_1(u, v) = (\partial/\partial x_Q)F_2(u, v)$. If F_2 did not vanish identically, then $(\partial/\partial x_Q)F_2(u, v)$ when written as a form in $u, \xi u, \cdots$ and $v, \xi v, \cdots$ would contain a term in which differentiation of v with respect to x occurs. But, in $(\partial/\partial y_Q)F_1(u, v)$ all derivatives of v are with respect to y. Consequently F_2 and F_1 vanish identically, and $F = F_0(u, v)$.

We observe that all derivatives of u and v occurring in F_0 must be of order $\leq \rho - 2$. The difference between the two compositions $F_0(u(O), v(P)) - F_0(u(P), v(O))$ is a linear combination of $u(P), \xi u(P), \cdots, v(P), \xi v(P), \cdots$ with constant coefficients. Now, if u and v are solutions of (2.4) with continuous derivatives of order $\leq 2\rho - 2$, then because L has constant coefficients, all derivatives of u and v of order $\leq \rho - 2$ are solutions of (2.4). Thus the difference between the two compositions is a solution.

We now define a particular composition, which we shall call the symmetric composition, and verify that it is a solution of the differential equation. Consider the exterior differential form

$$[u(P - Q)\xi^k \eta^l v(Q) - v(Q)\xi^k \eta^l u(P - Q)]dx_Q \, dy_Q \, .$$

Regarding P as fixed and differentiating with respect to Q, we find that this form is equal to

$$d\left\{\left[\sum_{i=0}^{k-1} \xi^i v(Q)\xi^{k-1-i}\eta^l u(P - Q)\right]dy_Q\right.$$
$$\left. - \left[\sum_{j=0}^{l-1} \xi^k \eta^j v(Q)\eta^{l-1-j}u(P - Q)\right]dx_Q\right\}$$

and also is equal to

$$d\Big\{\Big[\sum_{i=0}^{k-1}\eta^i\xi^i v(Q)\xi^{k-1-i}u(P-Q)\Big]dy_Q$$

$$-\Big[\sum_{j=0}^{l-1}\eta^j v(Q)\eta^{l-1-j}\xi^k u(P-Q)\Big]dx_Q\Big\}\ .$$

With

$$V_{kl}(u,\,v) = \int_0^P\Big(\Big[\sum_{i=0}^{k-1}\xi^i v(Q)\xi^{k-1-i}\eta^l u(P-Q)\Big]dy_Q$$

$$-\Big[\sum_{j=0}^{l-1}\xi^k\eta^j v(Q)\eta^{l-1-j}u(P-Q)\Big]dx_Q\Big)\ ,$$

(3.3)

$$V^*_{kl}(u,\,v) = \int_0^P\Big(\Big[\sum_{i=0}^{k-1}\eta^l\xi^i v(Q)\xi^{k-1-i}u(P-Q)\Big]dy_Q$$

$$-\Big[\sum_{j=0}^{l-1}\eta^j v(Q)\eta^{l-1-j}\xi^k u(P-Q)\Big]dx_Q\Big)$$

we set

(3.4)
$$w(P) = \sum_{k,l}a_{kl}V_{kl}(u,\,v)\ ,$$

$$w^*(P) = \sum_{k,l}a_{kl}V^*_{kl}(u,\,v)\ .$$

Both $w(P)$ and $w^*(P)$ are compositions of u and v of Lewy's type. The proof of Theorem 1 given in [1] was made by verifying that $L(w) = 0$. It turns out, however, to be more convenient to prove that $L(u \circ v) = 0$ where $u \circ v$ is the *symmetric composition* given by the formula

(3.5)
$$u \circ v(P) = \frac{1}{2}(w(P) + w^*(P))\ .$$

The line integrals V_{kl} and V^*_{kl} are, in general, not independent of path. However, in computing the partial derivatives of $u \circ v$ it is convenient to be able to operate on the terms V_{kl} and V^*_{kl} separately. For this reason we extend the operators ξ and η. If

$$I = f_0(P) + \int_0^P(f_1(P,\,Q)dx_Q + f_2(P,\,Q)dy_Q)\ ,$$

then we set

(3.6)
$$\xi I = \frac{\partial f_0}{\partial x}(P) + f_1(P,\,P) + \int_0^P\Big(\frac{\partial}{\partial x_P}f_1(P,\,Q)dx_Q + \frac{\partial}{\partial x_P}f_2(P,\,Q)dy_Q\Big)\ ,$$

$$\eta I = \frac{\partial f_0}{\partial y}(P) + f_2(P,\,P) + \int_0^P\Big(\frac{\partial}{\partial y_P}f_1(P,\,Q)dx_Q + \frac{\partial}{\partial y_P}f_2(P,\,Q)dy_Q\Big)\ .$$

The consistency of these definitions follows from the usual rules for differentiating an integral which is independent of the path. We

note, however, that some care is required in using these extended operators because the extended operators ξ and η may not commute. We have

$$(3.7) \quad \begin{aligned} V_{kl}(u, v) &= V_{0l}(u, \xi^k v) + V_{k0}(\eta^l u, v) , \\ V_{kl}^*(u, v) &= V_{0l}(\xi^k u, v) + V_{k0}(u, \eta^l v) . \end{aligned}$$

Setting

$$(3.8) \quad S_{kn}(u, v) = \sum_{i=0}^{k-1} \sum_{j=0}^{n-1} \xi^i \eta^j v(P) \xi^{k-1-i} \eta^{n-1-j} u(O)$$

and applying (3.6), we obtain

$$\begin{aligned} \xi V_{0l}(u, v) &= V_{0l}(\xi u, v) - S_{1l}(u, v) \\ \eta V_{k0}(u, v) &= V_{k0}(\eta u, v) + S_{k1}(u, v) \\ \xi V_{k0}(u, v) &= V_{k0}(\xi u, v) \\ \eta V_{0l}(u, v) &= V_{0l}(\eta u, v) . \end{aligned}$$

Moreover, since

$$\begin{aligned} \xi S_{kn}(u, v) + S_{1n}(\xi^k u, v) &= S_{k+1,n}(u, v) , \\ \eta S_{kn}(u, v) + S_{k1}(\eta^n u, v) &= S_{k,n+1}(u, v) , \end{aligned}$$

we obtain

$$(3.9) \quad \begin{aligned} \xi^m V_{0l}(u, v) &= V_{0l}(\xi^m u, v) - S_{ml}(u, v) , \\ \eta^n V_{k0}(u, v) &= V_{k0}(\eta^n u, v) + S_{kn}(u, v) , \\ \xi^m V_{k0}(u, v) &= V_{k0}(\xi^m u, v) , \\ \eta^n V_{0l}(u, v) &= V_{0l}(\eta^n u, v) . \end{aligned}$$

Hence by (3.7)

$$\begin{aligned} \eta^n \xi^m V_{kl}(u, v) &= V_{0l}(\xi^m \eta^n u, \xi^k v) + V_{k0}(\xi^m \eta^n \eta^l u, v) \\ &\quad + S_{kn}(\xi^m \eta^l u, v) - S_{ml}(u, \xi^k \eta^n v) \\ \xi^m \eta^n V_{kl}^*(u, v) &= V_{0l}(\xi^m \eta^n \xi^k u, v) + V_{k0}(\xi^m \eta^n u, \eta^l v) \\ &\quad - S_{ml}(\xi^k \eta^n u, v) + S_{kn}(u, \xi^m \eta_l v) . \end{aligned}$$

By (2.2) and (3.4) we have

$$\begin{aligned} Lw(P) &= \sum_{m,n} a_{mn} \eta^n \xi^m w(P) \\ &= \sum_{m,n} \sum_{k,l} a_{mn} a_{kl} \eta^n \xi^m V_{kl}(u, v) . \end{aligned}$$

If $Lu = 0$ and $Lv = 0$, then

$$\sum_{k,l} a_{kl} [V_{0l}(Lu, \xi^k v) + V_{k0}(L\eta^l u, v)] = 0 .$$

Therefore,

$$(3.10) \qquad Lw(P) = \sum_{m,n} \sum_{k,l} a_{mn} a_{kl} [S_{kn}(\xi^m \eta^l u, v) - S_{ml}(u, \xi^k \eta^n v)] \ .$$

Similarly

$$(3.11) \qquad Lw^*(P) = \sum_{m,n} \sum_{k,l} a_{mn} a_{kl} \xi^m \eta^n V_{kl}^*(u, v)$$

$$= \sum_{m,n} \sum_{k,l} a_{mn} a_{kl} [-S_{ml}(\xi^k \eta^n u, v) + S_{kn}(u, \xi^m \eta^l v)] \ .$$

The value of this sum is unchanged if we interchange k and m and interchange l and n throughout. Doing this, we obtain

$$(3.12) \qquad\qquad Lw^*(P) = -Lw(P) \ .$$

Thus for the symmetric composition

$$L(u \circ v) = \frac{1}{2}(Lw + Lw^*) = 0 \ ;$$

and, as we have seen, this implies that $L(u \circ v) = 0$ for any of the compositions. This completes the proof of Theorem 1.

If we assume that there is a straight line segment joining O to P which lies entirely in $\mathscr{D}$, then the change of variable $Q' = P - Q$ takes the line segment from O to P into the segment from P to O. If this path of integration is used, it is easily verified that

$$(3.13) \qquad\qquad V_{kl}(v, u) = V_{kl}^*(u, v)$$

and hence for the symmetric composition $u \circ v(P) = v \circ u(P)$, i.e., the symmetric composition is commutative.

4. A covariant composition. In this section we introduced for operators of all orders a special composition of Lewy's type which we shall call the covariant composition. For operators of order ≤ 2 it coincides with the symmetric composition of § 3.

If u is a solution of the differential equation

$$(4.1) \qquad\qquad Lu = \Sigma a_{kl} \frac{\partial^{k+l}}{\partial x^k \partial y^l} u = 0$$

and the linear transformation

$$(4.2) \qquad \begin{aligned} x' &= \alpha x + \beta y \\ y' &= \gamma x + \delta y \ , \qquad \Delta = \alpha\delta - \beta\gamma \neq 0 \end{aligned}$$

is made, then u considered as a function of x' and y' satisfies a new differential equation of the same order

$$(4.3) \qquad\qquad L'u = \Sigma a'_{kl}\frac{\partial^{k+l}}{\partial x'^k \partial y'^l}u = 0 \ .$$

For the transformed operator L' we can then obtain the Lewy compositions. We now show how to single out a composition such that if $\Delta = 1$ the same function $w = u \circ v$ is obtained whether the composition considered is with respect to L or with respect to L'.

We begin by introducing certain exterior differential forms Ω_{nj} which depend bilinearly on u and v. Let A and B be arbitrary real constants (indeterminates). Then Ω_{nj} is defined by equating coefficients in the equation

$$(4.4) \quad \sum_{j=0}^{n-1}(A\xi + B\eta)^j v(Q)(A\xi + B\eta)^{n-1-j}u(P - Q)(Ady_Q - Bdx_Q)$$

$$= \sum_{j=0}^{n}\binom{n}{j}A^j B^{n-j}\Omega_{nj}(u, v) \ .$$

We define the *covariant composition* of two solutions u and v of (4.1) by the equation

$$(4.5) \qquad\qquad u \circ v(P) = \int_O^P \Sigma\, a_{kl}\Omega_{k+l,k}(u(P - Q), v(Q)) \ .$$

Before stating our main result for the covariant composition, let us verify that it is indeed a composition of the type considered by Lewy. Regarding P as fixed and differentiating with respect to Q, we obtain

$$d\left\{\sum_{j=0}^{n}\binom{n}{j}A^j B^{n-j}\Omega_{nj}\right\} = \sum_{j=0}^{n-1}\{(A\xi + B\eta)^{j+1}v(Q)(A\xi + B\eta)^{n-1-j}u(P - Q)$$

$$- (A\xi + B\eta)^j v(Q)(A\xi + B\eta)^{n-j}u(P - Q)\}dx_Q dy_Q$$

$$= \{(A\xi + B\eta)^n v(Q)u(P - Q) - v(Q)(A\xi + B\eta)^n u(P - Q)\}dx_Q dy_Q \ .$$

Hence

$$(4.6) \quad d\Omega_{nj} = \{\xi^j\eta^{n-j}v(Q)u(P - Q) - v(Q)\xi^j\eta^{n-j}u(P - Q)\}dx_Q dy_Q \ ,$$

and consequently (4.5) defines a composition.

THEOREM 2. *If the linear change of variables* (4.2) *is made, then the covariant composition with respect to the transformed operator L' is equal to Δ times the covariant composition with respect to L.*

Proof. Starting with $u \circ v(P)$ as given by (4.5) we make the change of variables (4.2) and express $u \circ v(P)$ in terms of the new variables x' and y'. With $\xi' = \partial/\partial x'$ and $\eta' = \partial/\partial y'$ we have

$$\xi = \alpha\xi' + \gamma\eta' \,, \qquad dy = \frac{1}{\Delta}(\alpha dy' - \gamma dx') \,,$$

(4.7)

$$\eta = \beta\xi' + \delta\eta' \,, \qquad -dx = \frac{1}{\Delta}(\beta dy' - \delta dx') \,.$$

We define $\Omega'_{nj}(u, v)$ by the following equation which corresponds to (4.4):

(4.8) $$\sum_{j=0}^{n-1}(A\xi' + B\eta')^j v(Q)(A\xi' + B\eta')^{n-1-j}u(P - Q)(Ady'_Q - Bdx'_Q)$$

$$= \sum_{j=0}^{n}\binom{n}{j}A^j B^{n-j}\Omega'_{nj} \,.$$

We now express the left hand side of (4.4) in terms of ξ', η', dx'_Q, dy'_Q. Since

$$A\xi + B\eta = (A\alpha + B\beta)\xi' + (A\gamma + B\delta)\eta'$$

$$Ady_Q - Bdx_Q = \frac{1}{\Delta}\{(A\alpha + B\beta)dy'_Q - (A\gamma + B\delta)dx'_Q\} \,,$$

the left hand side of (4.4) is equal to $1/\Delta$ times the expression obtained by replacing A by $A\alpha + B\beta$, B by $A\gamma + B\delta$, ξ by ξ', η by η', dx_Q by dx'_Q, dy_Q by dy'_Q. Therefore

(4.9) $$\sum_{j=0}^{n}\binom{n}{j}A^j B^{n-j}\Omega_{nj} = \frac{1}{\Delta}\sum_{j=0}^{n}\binom{n}{j}(A\alpha + B\beta)^j(A\gamma + B\delta)^{n-j}\Omega'_{nj} \,.$$

Next we introduce a linear operator T which takes differential forms into differential forms and satisfies the equation

(4.10) $$T\Omega'_{nj} = \Omega'_{n,j+1} \qquad (j = 0, 1, 2, \cdots, n - 1) \,.$$

From (4.9) we obtain

$$\sum_{j=0}^{n}\binom{n}{j}A^j B^{n-j}\Omega_{nj} = \frac{1}{\Delta}\sum_{j=0}^{n}\binom{n}{j}(A\alpha T + B\beta T)^j(A\gamma + B\delta)^{n-j}\Omega'_{n0}$$

$$= \frac{1}{\Delta}(A\alpha T + B\beta T + A\gamma + B\delta)^n\Omega'_{n0}$$

$$= \frac{1}{\Delta}\sum_{j=0}^{n}\binom{n}{j}A^j B^{n-j}(\alpha T + \gamma)^j(\beta T + \delta)^{n-j}\Omega'_{n0} \,.$$

Therefore

(4.11) $$\Omega_{nj} = \frac{1}{\Delta}(\alpha T + \gamma)^j(\beta T + \delta)^{n-j}\Omega'_{n0} \,.$$

The covariant composition of u and v with respect to the operator L' is given by the integral

(4.12)
$$\int_O^P \Sigma a'_{kl}\Omega'_{k+l,k}(u(P-Q),\,v(Q))\;.$$

By (4.7) a term $a_{j,n-j}\xi^{n-j}$ in the operator L transforms into

$$a_{j,n-j}(\alpha\xi'+\gamma\eta')^j(\beta\xi'+\delta\eta')^{n-j}\;.$$

Hence

(4.13)
$$\Sigma a'_{kl}\Omega'_{k+l,k} = \sum_{\substack{0\leq n\leq\rho \\ 0\leq j\leq n}} a_{j,n-j}\{(\alpha T+\gamma)^j(\beta T+\delta)^{n-j}\Omega'_{n0}\}\;.$$

Comparing (4.13) with (4.11), we see that the composition with respect to L' is $\varDelta$ times the composition with respect to L; this completes the proof of the theorem.

REMARK. If the straight line segment from O to P lies in the domain $\mathscr{D}$, then one can easily show that $u\circ v(P) = v\circ u(P)$, i.e. that the covariant composition is commutative. The proof is similar to that given in §3 for the commutativity of the symmetric composition.

5. **Associativity.** In this section we consider an arbitrary second-order operator with constant coefficients,

(5.1)
$$Lu = a_{20}\frac{\partial^2 u}{\partial x^2} + 2a_{11}\frac{\partial^2 u}{\partial x\partial y} + a_{02}\frac{\partial^2 u}{\partial y^2} + a_{10}\frac{\partial u}{\partial x}$$
$$+ a_{01}\frac{\partial u}{\partial y} + a_{00}\;.$$

Throughout, the composition considered will be the covariant composition of §4; and the path of integration will be the straight line segment from O to P. As we have seen, the covariant composition is commutative. We now investigate when it is associative.

THEOREM 3. *For the second-order operator* (5.1) *and the covariant composition*

(5.2) $u\circ(v\circ w) - (u\circ v)\circ w = Dv(O)\big(u(O)w(P) - u(P)w(O)\big)$

where

$$D = a_{11}^2 - a_{20}a_{02}\;.$$

REMARK. From the theorem it follows that the covariant composition is associative if and only if the operator L is parabolic. Note, however, that if the convariant composition is restricted to the subset of solutions of $Lu = 0$ which vanish at O, then it is always associative.

Proof. It is convenient to define two operations $\underset{x}{*}$ and $\underset{y}{*}$ by the equations

(5.3)
$$u \underset{y}{*} v = \int_0^P u(P - Q)v(Q)dy_Q .$$
$$u \underset{x}{*} v = -\int_0^P u(P - Q)v(Q)dx_Q .$$

where the integrals are taken along the straight line segment from O to P. It is well known that these convolution operations are commutative and associative, i.e.,

(5.4)
$$u \underset{x}{*} v = v \underset{x}{*} u , \qquad\qquad u \underset{y}{*} v = v \underset{y}{*} u ,$$
$$u \underset{x}{*} (v \underset{x}{*} w) = (u \underset{x}{*} v) \underset{x}{*} w , \qquad u \underset{y}{*} (v \underset{x}{*} w) = (u \underset{y}{*} v) \underset{x}{*} w ,$$
$$u \underset{x}{*} (v \underset{y}{*} w) = (u \underset{x}{*} v) \underset{y}{*} w , \qquad u \underset{y}{*} (v \underset{y}{*} w) = (u \underset{y}{*} v) \underset{y}{*} w .$$

By a linear transformation of the independent variables the operator L can be brought into the canonical form

(5.5)
$$\frac{\partial^2}{\partial x^2} + \varepsilon \frac{\partial^2}{\partial y^2} + a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c$$

where $\varepsilon = 1$ for elliptic operators, $\varepsilon = 0$ for parabolic operators, and $\varepsilon = -1$ for hyperbolic operators. In the following computations we shall assume that L has this form and that $Lu = 0$, $Lv = 0$, and $Lw = 0$. We have

$$u \circ v = v \underset{y}{*} \xi u + \xi v \underset{y}{*} u + \varepsilon v \underset{x}{*} \eta u + \varepsilon \eta v \underset{x}{*} u + av \underset{y}{*} u + bv \underset{x}{*} u ,$$
$$v \circ w = \xi w \underset{y}{*} v + w \underset{y}{*} \xi v + \varepsilon \eta w \underset{x}{*} v + \varepsilon w \underset{x}{*} \eta v + aw \underset{y}{*} v + bw \underset{x}{*} v ,$$
$$\xi(v \circ w) = \xi v \circ w - \varepsilon \eta w(P)v(O) - \varepsilon w(P)\eta v(O) - bw(P)v(O) ,$$
$$\eta(v \circ w) = \eta v \circ w + \xi w(P)v(O) + w(P)\xi v(O) + aw(P)v(O) .$$

Consequently, $u \circ (v \circ w)$ is equal to

$$\Phi(u, v, w) - \varepsilon v(O)\eta w \underset{y}{*} u - \varepsilon \eta v(O)w \underset{y}{*} u - bv(O)w \underset{y}{*} u + \varepsilon v(O)\xi w \underset{x}{*} u$$
$$+ \varepsilon \xi v(O)w \underset{x}{*} u + av(O)w \underset{x}{*} u$$

where $\Phi(u, v, w)$ is equal to the sum of the following 36 terms:

$$\xi w *_y v *_y \xi u \qquad w *_y \xi v *_y \xi u \qquad \varepsilon \eta w *_x v *_y \xi u$$

$$\xi w *_y \xi v *_y u \qquad w *_y \xi^2 v *_y u \qquad \varepsilon \eta w *_x \xi v *_y u$$

$$\varepsilon \xi w *_y v *_x \eta u \qquad \varepsilon w *_y \xi v *_x \eta u \qquad \varepsilon^2 \eta w *_x v *_x \eta u$$

$$\varepsilon \xi w *_y \eta v *_x u \qquad \varepsilon w *_y \xi \eta v *_x u \qquad \varepsilon^2 \eta w *_x \eta v *_x u$$

$$a \xi w *_y v *_y u \qquad a w *_y \xi v *_y u \qquad \varepsilon a \eta w *_x v *_y u$$

$$b \xi w *_y v *_x u \qquad b w *_y \xi v *_x u \qquad \varepsilon b \eta w *_x v *_x u$$

$$\varepsilon w *_x \eta v *_y \xi u \qquad a w *_y v *_y \xi u \qquad b w *_x v *_y \xi u$$

$$\varepsilon w *_x \xi \eta v *_y u \qquad a w *_y \xi v *_y u \qquad b w *_x \xi v *_y u$$

$$\varepsilon^2 w *_x \eta v *_x \eta u \qquad \varepsilon a w *_y v *_x \eta u \qquad \varepsilon b w *_x v *_x \eta u$$

$$\varepsilon^2 w *_x \eta^2 v *_x u \qquad \varepsilon a w *_y \eta v *_x u \qquad \varepsilon b w *_x \eta v *_x u$$

$$\varepsilon a w *_x \eta v *_y u \qquad a^2 w *_y v *_y u \qquad a b w *_x v *_y u$$

$$\varepsilon b w *_x \eta v *_x u \qquad a b w *_y v *_x u \qquad b^2 w *_x v *_x u \ .$$

If these 36 terms are arranged as a 6×6 matrix with the first six lines above as the first three columns of the matrix and the last six lines as the last three columns, then terms on the main diagonal are unchanged if u and w are interchanged, while a term off the diagonal is changed into the one symmetric with respect to the diagonal. Hence $\Phi(u, v, w) = \Phi(w, v, u)$. From this it follows easily that

$$u \circ (v \circ w) - (u \circ v) \circ w = \varepsilon v(O)\{\eta u *_y w - \xi u *_x w - \eta w *_y u + \xi w *_x u\}$$

$$= \varepsilon v(O)\left\{ \int_O^P w(P - Q) du(Q) \right.$$

$$\left. - \int_O^P u(P - Q) dw(Q) \right\}$$

$$= \varepsilon v(O)\{w(O)u(P) - w(P)u(O)\} \ .$$

This proves the theorem if L is in the canonical form (5.5). The general case is then easily obtained by using Theorem 2.

Although for $D \neq 0$ the covariant composition is not associative, there is a modification of this operation which is both commutative and associative. Using $\circ$ to denote the covariant composition, we let

(5.6) $$u \cdot v = u \circ v + c_1 u(P)v(O) + c_2 u(O)v(P) \ .$$

where c_1 and c_2 are constants.

THEOREM 4. *The operation* $\cdot$ *defined by* (5.6) *is commutative if and only if* $c_1 = c_2$. *It is associative if and only if* $c_1 = c_2$ *and* $c_1^2 = D$.

Proof. The first statement is obvious. We have

$$v \cdot w = v \circ w + c_1 v(P) w(O) + c_2 v(O) w(P) \; ,$$

and hence, using the fact that $v \circ w(O) = 0$,

$$
\begin{aligned}
u \cdot (v \cdot w) &= u \circ (v \circ w) + c_1 w(O) u \circ v + c_2 v(O) u \circ w \\
&\quad + c_1(c_1 + c_2) u(P) v(O) w(O) + c_2 u(O) v \circ w \\
&\quad + c_1 c_2 u(O) v(P) w(O) + c_2^2 u(O) v(O) w(P) \; , \\
(u \cdot v) \cdot w &= (u \circ v) \circ w + c_1 v(O) u \circ w + c_2 u(O) v \circ w \\
&\quad + c_1 w(O) u \circ v + c_1^2 u(P) v(O) w(O) \\
&\quad + c_1 c_2 u(O) v(P) w(O) + c_2(c_1 + c_2) u(O) v(O) w(P) \; .
\end{aligned}
$$

Using Theorem 3 we obtain

$$
\begin{aligned}
u \cdot (v \cdot w) - (u \cdot v) \cdot w &= (c_2 - c_1) v(O) u \circ w \\
&\quad + (D - c_1 c_2) v(O)[u(O) w(P) - u(P) w(O)]
\end{aligned}
$$

from which the second statement of the theorem follows immediately.

Theorem 4 can be used to answer the question of which compositions for second order operators are commutative and associative. The operation $\cdot$ is a composition if and only if $c_1 = -c_2$. In § 3 it was shown that for second order operators an arbitrary composition has the form

$$u \circ v + c\big(u(O) v(P) - u(P) v(O)\big)$$

where $u \circ v$ is the convariant composition. Thus by Theorem 4 the covariant composition is the only compostion which is commutative. Also, it follows from Theorem 4 that a composition is associative if and only if $D = 0$ and $c = 0$. Thus for parabolic operators only the covariant composition is associative, while for elliptic and hyperbolic operators none of the compositions are associative.

It is interesting to note that for real hyperbolic equations the two modifications of the composition which are commutative and associative, when applied to two real solutions yield real solutions, but for real elliptic equations no such modification exists. In the case of elliptic equations one must consider complex-valued solutions to obtain an associative operation.

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