SEMIGROUPS AND THEIR SUBSEMIGROUP LATTICES

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1. Introduction. Let S be a semigroup of order at least 2, and $L(S)$ be the system of all subsemigroups of S. Generally $L(S)$, including the empty subset, is a lattice with respect to inclusion. $L(S)$ is called the subsemigroup lattice of S. A semigroup S contains at least one nonempty subsemigroup besides S itself. In the previous paper [4], as the first step towards the investigation of the structure of S with a given type of $L(S)$, we determined all the Γ-semigroups,[1] namely, the semigroups S's in which $L(S)$'s are chains. In the present paper we shall define Γ^*-semigroups as generalization of Γ-semigroups and shall obtain all the types of Γ^*-semigroups except for infinite simple Γ^*-groups.

Since all the semigroups of order 2 are Γ^*-semigroups, we shall treat non-trivial Γ^*-semigroups, namely, those of order $\geqq 3$ in the discussion below. First, in § 2 we shall prove that Γ^*-semigroups of order $\geqq 3$ are unipotent, i.e., having a unique idempotent, and that they are periodic; and hence a Γ^*-semigroup is determined by a group and a Z-semigroup, i.e., a unipotent semigroup with zero. Accordingly, in § 3 we shall determine all the types of Γ^*-Z-semigroups which will have to be of order <5; in § 4 we shall treat solvable Γ^*-groups and prove that finite Γ^*-groups or non-simple Γ^*-groups are solvable; finally in § 5, unipotent Γ^*-semigroups which are neither groups nor Z-semigroups will be discussed. It is interesting that there are no infinite unipotent Γ^*-semigroups except groups.

For convenience, the results from the paper [4] are stated as follows:

LEMMA 1.1. *A semigroup is a Γ-semigroup if and only if it has one of the following types.*[2] *Except for* (1.3) *they are all cyclic semigroups, i.e., semigroups generated by an element d. We show defining relations below.*

(1.1) *Z-semigroups*:

(1.1.1) $\quad d^2 = d^3 \quad$ (*order* 2)

(1.1.2) $\quad d^3 = d^4 \quad$ (*order* 3)

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[1] The author called them Γ-monoids in [4].

[2] As the trivial case, a semigroup of order 1 is also regarded as a Γ-semigroup. This remark will be needed for the definition of a Γ^*-semigroup.

(1.2) *Cyclic groups* $G(p^m)$ *of a prime power order*: $d = d^{p^m+1}$

(1.3) *Quasicyclic groups* [1]: $G(p^\infty)$, *i.e.*,

$$G(p^\infty) = \sum_{k=1}^{\infty} G(p^k)$$

where $Q(p) \subset G(p^2) \subset \cdots \subset G(p^k) \subset \cdots$, *p being a prime.*

(1.4) *Unipotent semigroups of order n, the kernel (the least ideal) of which is a group* $G(p^m)$:

(1.4.1)	*if* $p = 2$	$d^2 = d^{2^m+2}$	(*order* $n = 2^m + 1$)
(1.4.2)	*if* $p \neq 2$		
(1.4.2.1)		$d^2 = d^{p^m+2}$	(*order* $n = p^m + 1$)
(1.4.2.2)		$d^3 = d^{p^m+3}$	(*order* $n = p^m + 2$)

2. Preliminaries.

DEFINITION. A semigroup S is called a Γ^*-semigroup if every subsemigroup different from S is a Γ-semigroup.

S is a Γ^*-semigroup if and only if the subsemigroup lattice $L(S)$ is a lattice satisfying

(2.1) Any subset which cantains the greatest element 1 is a subsemilattice with respect to join, equivalently to

(2.1′) Let x, y be any elements of a lattice. Then

$$x \cup y = x \text{ or } y \text{ or } 1 .$$

Notation. If X and Y are subsets of S, $X \mid Y$ means either $X \subseteqq Y$ or $X \supseteqq Y$; $X \parallel Y$ means that X and Y are incomparable, that is, neither is contained in the other. $((X, Y, \cdots))$ denotes the subsemigroup generated by $X, Y, \cdots$. In particular, $((x))$ denotes the subsemigroup generated by an element x, $((x, y))$ the subsemigroup generated by elements x and y, while $\{x_1, x_2, \cdots\}$ is the set composed of $x_1, x_2, \cdots$.

S is a Γ^*-semigroup if and only if any two subsemigroups A and B satisfy the following condition: $A \parallel B$ implies $S = ((A, B))$. Of course a Γ-semigroup is a Γ^*-semigroup. Since the homomorphic image of a Γ-semigroup is also a Γ-semigroup, we get easily

LEMMA 2.1. *A homomorphic image of a* Γ^**-semigroup is a* Γ^**-semigroup.*

LEMMA 2.2. *A* Γ^**-semigroup is periodic.*

Proof. Suppose there is an element x of infinite order. S contains an infinite cyclic subsemigroup $\{x^i; i = 1, 2, \cdots\}$. Hence we can consider a proper subsemigroup[3] T of S.

$$T = \{x^{2i}; i = 1, 2, \cdots\}$$

which contains two incomparable subsemigroups T_1 and T_2:

$$T_1 = \{x^{4i}; i = 1, 2, \cdots\}\ , \qquad T_2 = \{x^{6i}; i = 1, 2, \cdots\}\ .$$

This contradicts the assumption of S.

By Lemma 2.2, we have seen that a Γ^*-semigroup has at least one idempotent. However, we have

THEOREM 2.1. *A Γ^*-semigroup of order >2 is unipotent.*

Proof. Suppose that a Γ^*-semigroup S of order >2 contains at least two idempotents, say, e, f. First, since e is a right identity of Se, and f is a left identity of fS, we see easily that if $Se = fS$, then $e = f$. Second, we shall say that either both of Se and Sf or both of eS and fS are proper subsemigroups. Suppose either of Se and Sf is equal to S, say, $Se = S$. Then, by the above fact, $fS \subset S$, and so we have to show $eS \subset S$. Let us assume $Se = eS = S$. There is a proper subsemigroup $\{e, f\}$ of order 2 because $ef = fe = f$; but $\{e, f\}$ is not a Γ-semigroup since e and f are both idempotents. This is a contradiction. Therefore $eS \subset S$.

Next, assume that both eS and fS are proper subsemigroups of S. Since eS and fS are Γ-semigroups with left identities, they are groups by Lemma 1.1. We shall prove that $\{e, f\}$ is a proper subsemigroup which is not a Γ-semigroup, and then the contradiction will be derived. For proof, the idempotency of ef and fe is shown as follows:

$$(ef)(ef) = (efe)f = (ef)f = e(ff) = ef$$
$$(fe)(fe) = (fef)e = (fe)e = f(ee) = fe$$

because e and f are two-sided identities of the groups eS and fS respectively. Since $ef \in eS$ and $fe \in fS$, we have

$$ef = e\ , \qquad fe = f$$

whence $\{e, f\}$ is a subsemigroup. We can have the same result, when $Se \subset S$ and $Sf \subset S$. Thus the proof of the theorem has been completed.

LEMMA 2.3. *The index of an element a of a Γ^*-semigroup S cannot exceed* 3.

[3] By "a proper subsemigroup T of S" we mean "a subsemigroup T which is different from S."

Proof. Let a have index greater than 1. Then $((a)) - \{a\}$ is a Γ-semigroup, so $((a^2)) \mid ((a^3))$. Hence there is a positive integer n such that either

$$a^2 = a^{3n} \quad \text{or} \quad a^3 = a^{2n} .$$

This shows that a has index 2 or 3.

3. **Γ^*-Z-Semigroups.** In this section we shall determine the types of Γ^*-Z-semigroups, i.e., unipotent Γ^*-semigroup with zero 0.

Let S be a Γ^*-Z-semigroup with 0. Since S is periodic, every element of S is nilpotent, that is, some power of the element is 0. By Lemma 2.3,

$$x^3 = 0 \quad \text{for all} \quad x \in S .$$

LEMMA 3.1. *$x = xy$ implies $x = 0$; $x = yx$ implies $x = 0$.*

Proof. $x = xy = xy^2 = xy^3 = 0$; the proof of the second part is obtained in a similar way.

LEMMA 3.2. *If $x^2 = 0$, then $xy = yx = 0$ for all y.*

Proof. We may assume $x \neq 0$, let $y \neq 0$. If $((x)) \mid ((xy))$, $xy = 0$ because of Lemma 3.1. If $((x)) \| ((xy))$, then $S = ((x, xy))$ and so $y = xu$ for some u.

$$xy = x^2u = 0 .$$

The proof of $yx = 0$ is similar.

To determine the types of Γ^*-Z-semigroups, we consider the possible three cases:

Case I. $x^2 = 0$ for all $x \neq 0$.

Case II. There exists only one nonzero element x such that $x^3 = 0$, $x^2 \neq 0$.

Case III. There exist at least two nonzero elements x and y such that $x^3 = 0$, $x^2 \neq 0$, $y^3 = 0$, $y^2 \neq 0$.

THEOREM 3.1. *S is a non-trivial Γ^*-Z-semigroup if and only if S is isomorphic or anti-isomorphic to one of the following:*

Case I. *$S = \{0, a, b\}$ where $xy = 0$ for all $x, y \in S$.*

Case II. $S=\{0, a, a^2\}$ *where* $a^3=0$. *This is a* Γ*-semigroup which is isomorphic to* (1.1.2).

Case III. $S=\{0, a, b, c\}$ *where* $a^2=b^2=c$, $a^2x=xa^2=0$ *for all* $x \in S$.

Subcase III_1 $ab=ba=c$
Subcase III_2 $ab=c, ba=0$
Subcase III_3 $ab=ba=0$

Proof.

Case I. Let a and b be distinct nonzero elements of S. Since $((a)) \| ((b))$, $S=((a, b))$. By Lemma 3.2, we have $ab=ba=0$. Hence

$$S=((a, b))=\{0, a, b\} .$$

Case II. Let a be an element with index 3. Suppose that there is $b \in S-((a))$. In the present case we know $b^2=0$. By Lemma 3.2, $ab=ba=0$, whence $A=\{0, a^2, b\}$ is a subsemigroup which does not contain a, and hence A is a Γ-semigroup. On the other hand, since $b \neq a^2$, we have $((a^2)) \| ((b))$. It is impossible in a Γ^*-semigroup S. Therefore $S=((a))$.

Case III. Let a and b be distinct nonzero elements, both of which have index 3. Since $(a^2)^2=(b^2)^2=0$, Lemma 3.2 gives us

$$a^2b=ba^2=b^2a=ab^2=0 \quad \text{and so } a^2b^2=b^2a^2=0 . \tag{3.1}$$

Using (3.1) and Lemma 3.2 repeatedly, since $(aba)^2=aba^2ba=0$, we have

$$(ab)^2=(aba)b=0 \tag{3.2}$$

and hence

$$aba=0 . \tag{3.3}$$

Similarly we get

$$bab=0 . \tag{3.3'}$$

Now we have two subsemigroups $T=((a^2, b^2))$ and $U=((ab, a^2))$:

$$T=((a^2, b^2))=\{0, a^2, b^2\} \not\ni a$$

where we see $a \neq b^2$, otherwise, $a=b^2$ would imply $a^2=0$; also

$$U=((ab, a^2))=\{0, ab, a^2\} \not\ni b .$$

Accordingly both T and U are Γ-semigroups and so

$$((a^2)) \mid ((b^2)) \quad \text{and} \quad ((ab)) \mid ((a^2)) .$$

The first implies (3.4); the second implies (3.5)

$$a^2 = b^2 \tag{3.4}$$

$$ab = a^2 \quad \text{or} \quad 0 . \tag{3.5}$$

Similarly we have

$$ba = a^2 \quad \text{or} \quad 0 . \tag{3.5$'$}$$

Clearly $((a)) \| ((b))$. By (3.1) through (3.5′),

$$S = ((a, b)) = \{0, a, b, a^2\}$$

which consists of exactly four elements. Thus we have obtained the three types for Case III. It is easy to show that the systems thus obtained are Γ^*-Z-semigroups.

4. **Γ^*-groups.** By Lemma 2.2, a group G is a Γ^*-semigroup if and only if it is a Γ^*-group, i.e, every proper gubgroup of G is a Γ-group. By Lemma 1.1, every Γ-group is of type $G(p^k)$, $k \leqq \infty$. In this chapter we determine all solvable Γ^*-groups. We also show that every finite Γ^*-group is solvable. The question whether infinite simple Γ^*-groups can exist remains open.

LEMMA 4.1. *Let G be a non-abelian solvable Γ^*-group which is not also a Γ-group. Then G contains a proper normal subgroup $N \neq 1$ and an element a not in N, such that*

$$N \| ((a)), \quad \textit{so that} \quad G = ((N, a)) \tag{4.1}$$

$$a^q \in N \quad \textit{for a prime number } q . \tag{4.2}$$

Proof. Since G is solvable, it contains a proper normal subgroup N such that G/N is abelian. $N \neq 1$ since G is not abelian. Since N is a proper subgroup of G, it is a Γ-group. Since G is not itself a Γ-group, there exist a and b in G such that $((a)) \| ((b))$, and then we have $G = ((a, b))$. If $N \| ((b))$, then (4.1) holds with b instead of a. To prove (4.1) suppose $N \| ((b))$. If $N \supsetneqq ((b))$, then $N \not\supseteq ((a))$, since N is a Γ-group; and $((a)) \| ((b))$, and $N \not\subseteq ((a))$ since otherwise $((b)) \subsetneqq N \subsetneqq ((a))$. Hence $N \| ((a))$ in this case. If $N \subsetneqq ((b))$, then, since G/N is abelian, $aba^{-1}b^{-1} \in N \subsetneqq ((b))$, so $aba^{-1} \in ((b))b \subsetneqq ((b))$. Since $G = ((a, b))$, we conclude that $N' = ((b))$ is a normal subgroup of G, and (4.1) holds with N' instead of N. Hence N and a exist such that (4.1) holds. Let k be the least positive integer such that $a^k \in N$, and let $k = k'q$ with q a prime. Let $a' = a^{k'}$. Then (4.1) and (4.2) hold with a' instead of a.

THEOREM 4.1. *Let G be a solvable Γ^*-group which is not a Γ-group. Then one of the following holds*:

(4.3) *G is a group of order pq, p and q primes excluding the cyclic group of order p^2.*

(4.4) *G is the quaternion group of order* 8.

Proof. First let us take the case G abelian. If G were directly indecomposable, it would be isomorphic with $G(p^k)$ for some $k \leqq \infty$ (cf. Theorem 10, p. 22, [2]), and so would be a Γ-group. Hence G is directly decomposable: $G = G_1 \times G_2$ where $G_1 \neq 1$, $G_2 \neq 1$. Let a_i be an element of G_i of prime order p_i $(i = 1, 2)$. Then $((a_1)) \| (a_2))$, so $G = ((a_1, a_2)) = ((a_1)) \times ((a_2))$. Thus G has type (4.3).

Let G be non-abelian. By Lemma 4.1, G contains a proper normal subgroup $N \neq 1$, and an element a not in N such that $N \| ((a))$ and $a^q \in N$ for some prime q. Since N is a proper subgroup of G, it is isomorphic with $G(p^k)$ for some prime p and some $k \leqq \infty$. Hence a^q has prime power order p^n, say.

If $q \neq p$, then $a_1 = a^{p^n} \notin N$, and $a_1^q = 1$. If b is any element of N of order p, we have $((a_1)) \| ((b))$ and hence $G = ((a_1, b))$. Since $a_1 N a_1^{-1} \subseteqq N$, and every subgroup of N is characteristic, $a_1((b))a_1^{-1} \subseteqq ((b))$. Hence G is an extension of the cyclic group $((b))$ of order p by the cyclic group $((a_1))$ of order q.

We may now assume $q = p$. Since $N \nsubseteqq ((a))$, there exists b in N such that $b^p = a^p$. Let $c = a^p = b^p$. Since c commutes with a and b, and $G = ((a, b))$, c belongs to the center C of G. If $c = 1$, then, as in the above statements, G is an extension of the cyclic group $((b))$ of order p by the cyclic group $((a))$ of order p. Hence we may assume that the order of c is p^n with $n > 0$.

Since $((b))$ is invariant under a, we have $aba^{-1} = b^r$ for some positive integer $r > 1$. Then

$$c = b^p = ab^pa^{-1} = (aba^{-1})^p = b^{rp} = c^r ,$$

whence $r = 1 + sp^n$ for some integer s. Hence

$$aba^{-1} = b^r = bd \quad \text{or} \quad b^{-1}aba^{-1} = d \neq 1$$

where $d = b^{sp^n} = c^{sp^{n-1}}$ is an element of C of order p. As in the familiar way,

$$(ab^{-1})^p = d^{p(p-1)/2}a^pb^{-p} = d^{p(p-1)/2} .$$

If p is odd, we conclude that $(ab^{-1})^p = 1$. Let $a_1 = ab^{-1}$. Then $a_1^p = 1$ and this case is reduced to the previous case $c = 1$. We are left with the case $p = 2$. Setting $a_1 = ab^{-1}$, we have $a_1^2 = d$. Let b_1 be an element of N such that $b_1^2 = d$. Then $G = ((a_1, b_1))$, and $((b_1))$ is invariant under a_1. Since $b_1^4 = 1$, and G is not abelian, we must have

$$a_1 b_1 a_1^{-1} = b_1^3 .$$

Together with $a_1^4 = b_1^4 = 1$, this shows that G is the quaternion group of order 8. Thus this theorem has been proved.

THEOREM 4.2. *A finite Γ^*-group is solvable.*

Proof. For Γ-groups, the theorem is obvious. Let G be a finite Γ^*-group which is not a Γ-group. If G is of order p^m of a prime power, then this theorem holds, since G has a normal subgroup of order p^{m-1} by the familiar theorem of p-groups. So we may assume that the order of G has at least two distinct prime divisors.

First we shall prove that G has a proper normal subgroup. Let M be a Sylow subgroup of G, and consider the normalizer H of M. If $H = G$, then M is normal; if $M \subseteqq H \subset G$, then H is a Γ-group, a cyclic group. By Burnside's theorem ([8], p. 169), G has a proper normal subgroup N such that $G = NH$, $N \cap H = 1$.

Since N is a proper subgroup, it is a Γ-group, say, $G(p^{\alpha_1})$. Then, suppose the order of the factor group G/N is

$$p^{\alpha_2} q^{\beta} r^{\gamma} \cdots, \quad \alpha_2 \geqq 0, \beta \geqq 1, \quad \gamma \geqq 0, \cdots \tag{4.5}$$

which has a prime divisor $q \neq p$. Since G/N has a subgroup of order q, G has a proper subgroup of order $p^{\alpha_1} q$, which contains two incomparable subgroups, unless

$$\alpha_2 = 0, \beta = 1 . \tag{4.6}$$

Thus we have proved that the index of N is a prime q.

THEOREM 4.3. *A non-simple Γ^*-group is solvable.*

Proof. Let G be a non-simple Γ^*-group and N be a proper normal subgroup of G. We may assume that G/N contains a proper subgroup $\bar{H}$ of prime order p, since G/N is a Γ^*-group and so G/N is periodic. Consider a coset xN which is a generator of $\bar{H}$ and take an element $a \in xN$. Then $H = ((a))$ is a group of order p, and there is a subgroup K of G such that $K/N \cong \bar{H}$. Clearly $K = NH$. On the other hand, since $N \| H$, we have $G = ((N, H)) = NH = K$. Accordingly, $G/N \cong \bar{H}$, which is prime order. Thus the proof has been completed.

Consequently, (4.3) and (4.4) of Theorem 4.1 give us all the types of finite or non-simple Γ^*-groups which are not Γ-groups.

5. Unipotent Γ^*-semigroups.

1. In this chapter we shall discuss unipotent Γ^*-semigroups S's which are neither groups nor Z-semigroups. By Lemma 2.2 and Theorem 2.1 we see that a Γ^*-semigroup S of order >2 is a unipotent inversible semigroup. By "inversible" we mean "for any element a of S there is an element b such that $ab = e$ where e is a unique idempotent." According to [5], [6], a unipotent inversible semigroup which contains properly a group is determined by a group G (or kernel, i.e., least ideal), and a Z-semigroup D (the difference semigroup of S modulo G), and certain mapping of the bases of D into G: $a \to ea$.

First of all we shall prove that the kernel is finite.

LEMMA 5.1. *Let S be a unipotent inversible semigroup with the kernel G of type $G(p^k)$, k being infinite or finite, and let d be an element of S which is not in G such that ed generates $G(p^m)$, $m < k$, and $d^{l-1} \notin G(p^k)$, $d^l \in G(p^k)$. Then there is a subsemigroup H of order $p^{m+1} + l - 1$ of S which contains two incomparable subsemigroups: $G(p^{m+1})$ and $\{d^i; i \geqq 1\}$.*

Proof. Let $a = ed$. As is easily seen (cf. [5]), we have

$$a = ed = de,\ d^i = a^i,\ i \geqq l \tag{5.1}$$

$$xd = dx = xa = ax \quad \text{for every} \quad x \in G\ . \tag{5.2}$$

Especially for $x \in G(p^{m+1})$, $xd = dx \in G(p^{m+1})$. Therefore the set union $H = G(p^{m+1}) \cup \{d^i; l - 1 \geqq i \geqq 1\}$ is a subsemigroup of S; and the two subsemigroups $G(p^{m+1})$ and $\{d^i; i \geqq 1\}$ are incomparable, because $\{d^i; i \geqq l\} \subseteqq G(p^m)$.

THEOREM 5.1. *Let S be a unipotent inversible semigroup which is neither a group nor a Z-semigroup. If S is a Γ^*-semigroup, then S is finite.*

Proof. The proper subgroup G is a Γ-group $G(p^\infty)$ or $G(p^n)$, and the difference semigroup $D = (S\colon G)$ of S modulo G in Rees' sense [3] is a Γ^*-Z-semigroup of order $\leqq 4$ by theorems in § 3. There is an element z_1 outside G such that $z_1^2 \in G$, for example, we may take a nonzero annihilator as z_1 (cf. [6]); and let m be a positive integer such that ez_1 generates a subgroup $G(p^m)$. If S is infinite, then G is of the type $G(p^\infty)$ and so S has a proper subsemigroup of order $p^{m+1} + 1$, which contains two incomparable subsemigroups by Lemma 5.1. This contradicts the definition of Γ^*-semigroups of S. Thus the theorem has been proved.

Hereafter we shall determine the desired semigroups S in each case according as the order of D.

2. The case with D of order 2.

Let $G(p^n)$ denote the kernel of S, and let d be a unique element outside $G(p^n)$. Of course $d^2 \in G(p^n)$. $G(p^k)$ denotes the subgroup generated by $a = ed$. If $k = n$, then, by (5.1), we have

$$S = \{d^i; i \geqq 1\}, \qquad G(p^n) = \{d^i; i \geqq 2\}$$

that is, S is a Γ-semigroup of type (1.4.1) or (1.4.2.1).

If $k < n$, then by Lemma 5.1 there is a subsemigroup $H = G(p^{k+1}) \cup \{d\}$ of order $p^{k+1} + 1$ which contains two incomparable subsemigroups, so that $S = H$ and hence we have $k = n - 1$. In other words, a is a generator of $G(p^{n-1})$; this a determines S and there is a unique S to within isomorphism, independent of choice of generator a (cf. [6]). Conversely, a semigroup S thus obtained is easily seen to be a Γ^*-semigroup. In fact, by (5.1) we see that a proper subsemigroup incomparable to $G(p^n)$ is nothing but

$$G(p^{n-1}) \cup \{d\} = ((d)).$$

3. The case with D of type Case I of order 3.

Let $S = G(p^n) \cup \{d_1, d_2\}$ where $d_1d_2, d_1^2, d_2^2, d_2d_1 \in G(p^n)$. S is determined by the two elements a_1, a_2, i.e.,

$$a_1 = ed_1, \qquad a_2 = ed_2$$

where a_1 and a_2 can be taken independently arbitrarily. The proper subsemigroups $G(p^n) \cup \{d_1\}$ and $G(p^n) \cup \{d_2\}$ are Γ-semigroups of type (1.4.1) or (1.4.2.1). We have already known that a_1 and a_1 are the generators of $G(p^n)$, and

$$G(p^n) \cup \{d_1\} = ((d_1)), \qquad G(p^n) \cup \{d_2\} = ((d_2)).$$

We can easily prove that there are two possible distinct types

$$a_1 = a_2, \qquad a_1 \neq a_2$$

in all cases except for the case $p = 2$ and $n = 1$. They are immediately seen to be Γ^*-semigroups.

4. The case with D of type Case II of order 3.

Let d be a generator of D: $D = \{0, d, d^2\}$, $d^3 = 0$, and let $S =$

$G(p^n) \cup \{d, d^2\}$. We shall prove that $a = ed$ generates $G(p^n)$. Suppose that an element a generates $G(p^k)$, $k < n$. Then, since $ed^2 = (ed)^2$ and $(d^2)^2 \in G(p^n)$, ed^2 generates a subgroup $G(p^m)$, $m \leqq k$, and a subsemigroup $K = G(p^{m+1}) \cup \{d^2\}$ contains two incomparable subsemigroups by Lemma 5.1. K is a proper subsemigroup of S because

$$p^{m+1} + 1 < p^n + 2 .$$

This contradicts the assumption of Γ^*-semigroup of S. Hence it has been proved that $G(p^n)$ is generated by ed. Accordingly we get $G(p^n) = \{d^i; i \geqq 3\}$ by (5.1), whence S is generated by d. In the same way as the Case with D of order 2, we see that arditrary different generators of $G(p^n)$ give some isomorphic S's.

The remaining thing to do is to testify the subsemigroup lattice of such semigroups.

If $p \neq 2$, then ed^2 generates $G(p^n)$, and only a proper subsemigroup between S and $G(p^n)$ is

$$((d^2)) = G(p^n) \cup \{d^2\} \quad \text{by (5.1)}$$

and so S is a Γ-semigroup of type (1.4.2.2).

If $p = 2$, then ed^2 generates $G(2^{n-1})$ and so, by Lemma 5.1, we have a proper subsemigroup

$$G(2^n) \cup \{d^2\}$$

which contain. two incomparable $G(2^n)$ and $((d^2))$. Therefore, S is not a Γ^*-semigroup.

5. The case with D of order 4.

Let $S = G(p^n) \cup \{d_1, d_2, d_3\}$ where $d_1 = d_2^2 = d_3^2$. D has any one of the types of Case III with elements denoted by d_1, d_2, d_3 instead of a, b, c, respectively. Since the proper subsemigroups $G(p^n) \cup \{d_1, d_2\}$ and $G(p^n) \cup \{d_1, d_3\}$ are both Γ-semigroups of type (1.4.2.2), we have by (5.1)

$$G(p^n) \cup \{d_1, d_2\} = ((d_2)) , \qquad G(p^n) \cup \{d_1, d_3\} = ((d_3))$$

where $p \neq 2$, and $a_2 = ed_2$ and $a_3 = ed_3$ are both generators of $G(p^n)$. One the other hand, there are relations between a_2 and a_3 as follows: (We called these relations the primary equations for D in [6], § 3.)

$$a_2^2 = a_3^2 \qquad \text{in Case III}_3 ,$$
$$a_2 = a_3 \qquad \text{in Cases III}_1 \text{ and III}_2 .$$

We see easily that $a_2^2 = a_3^2$ in $G(p^n)$ implies $a_2 = a_3$ because $p \neq 2$. Thus for $G(p^n)$ and each D, there is a unique S to within isomorphism.

As far as the subsemigroups containing $G(p^n)$ are concerned, besides $((d_2))$ and $((d_3))$, there is $((d_1))$ and we have

$$((d_1)) = ((d_2)) \cap ((d_3))$$

because $p \neq 2$. Accordingly it can be seen that S is a Γ^*-semigroup. Thus we have

THEOREM 5.2. *When $G(p^n)$ is given, all the possible unipotent Γ^*-semigroups S whose kernel is $G(p^n)$ and which are not Γ-semigroups are determined as shown below. Let e be the unique idempotent of S, and let $D = (S: G(p^n))$. We remark $G(p^0) = 1$, $G(p^{-1}) = $ empty.*

(5.3.1) *In the case D of order 2, $S = G(p^n) \cup \{d\}$, $n \neq 0$, $ed \in G(p^{n-1}) - G(p^{n-2})$*

(5.3.2) *In the case D of order 3, D is of Case I and $S = G(p^n) \cup \{d_1, d_2\}$, $n \neq 0$*

(5.3.2.1) $\quad ed_1 = ed_2 \in G(p^n) - G(p^{n-1})$

(5.3.2.2) $\quad p^n \neq 2, ed_1 \neq ed_2,$ *and* $ed_1, ed_2 \in G(p^n) - G(p^{n-1})$

(5.3.3) *In the case D of order 4, $S = G(p^n) \cup \{d_1, d_2, d_3\}$, $d_2^2 = d_3^2 = d_1$, $n \neq 0$, $p \neq 2$*

$$\left.\begin{array}{ll} (5.3.3.1) & D \text{ of type Case III}_1 \\ (5.3.3.2) & D \text{ of type Case III}_2 \\ (5.3.3.3) & D \text{ of type Case III}_3 \end{array}\right\} ed_2 = ed_3 \in G(p^n) - G(p^{n-1}) .$$

After all, under the given $G(p^n)$, if $p \neq 2$, then there are six types of S; if $p = 2$ and $n \neq 1$, then three types of S; if $p = 2$ and $n = 1$, then two types of S.

REFERENCES

1. L. Fuchs, *Abelian groups*, Publishing House of the Hungarian Acad. of Sci., Budapest, (1958).
2. I. Kaplansky, *Infinite abelian groups*, University of Michigan Press, (1954).
3. D. Rees, On semigroups, Proc. Cambridge Phil. Soc., **36** (1940), 387-400.
4. T. Tamura, *On a monoid whose submonoids form a chain*, Jour. of Gakugei, Tokushima Univ., **5** (1954), 8-16.
5. ———, *Note on unipotent inversible semigroups*, Kodai Math. Sem. Rep., **3** (1954), 93-95.
6. ———, *The theory of construction of finite semigroups III*, Osaka Math. Jour., **10** (1958), 191-204.
7. ———, *Note on Γ^*-semigroups*, Bull. Amer. Math. Soc., **68** (1962), 505-508.
8. H. Zassenhauss, *The theory of groups*, Chelsea Publ. Comp., New York (2nd edition), (1956).

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