# MAXIMUM MODULUS ALGEBRAS AND LOCAL APPROXIMATION IN $C^n$

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1. In [4] W. Rudin established an important result concerning maximum modulus algebras $A$ of continuous complex-valued functions defined on the closure $K$ of a Jordan domain in the complex plane (see also [5]). Rudin's result states, under the assumptions (a) $A$ contains a function $\Psi$ which is schlicht on $K$, and (b) $A$ contains a non-constant function $\phi$ which is analytic in the interior, int $K$, of $K$, that every function in $A$ is analytic in int $K$. In this note we will establish conditions under which assumption (b) alone yields the desired conclusion in a slightly more general setting. We assume that $K$ is a compact set, with interior, of a Riemann surface, but also assume that int $K$ is essentially open in the maximal ideal space $\Sigma_A$ of $A$ ($A$ being regarded as a Banach algebra with the sup norm $||f|| = \sup_{p \in K} |f(p)|$; see [2]). This means that each point of int $K$, excepting a set of points having no limit point in int $K$, has a neighborhood in int $K$ which is open in $\Sigma_A$ under the natural mapping of $K$ into $\Sigma_A$. Under these assumptions it is easy to show, using the Local Maximum Modulus Principle of H. Rossi [3; Theorem 6.1] and Rudin's results, that (b) is sufficient to guarantee that $A$ consists only of analytic functions. Our main purpose, however, is to establish the result by a geometric method, independent of Rudin's work, which is based on an appropriate local approximation in $C^n$. Unfortunately the geometric approach being used here only allows us to make the desired conclusion for twice continuously differentiable functions in $A$ whereas the use of Rubin's results would give a proof valid for any function in $A$. However it is hoped that our method will be of some interest in itself.

The basic idea of the proof is as follows. For simplicity let $K$ be the unit circle $\{z \in C: |z| \leqq 1\}$ in the complex plane, and let $f$ and $g$ be nonconstant functions in the maximum modulus algebra $A$. Suppose that $\Sigma_A = K$. Use $f$ and $g$ to map $K$ into $C^2$ (the space of 2 complex variables) in the obvious way. If $f$ and $g$ are twice continuously differentiable in the neighborhood of a given point in int $K$ then the image of this neighborhood in $C^2$ will be a two (real) dimensional surface possessing a tangent plane at the image $p$ of the point. Let $\pi$ be the two (real) dimensional tangent plane to this surface at $p$. If this plane is nonanalytic (Definition 1) then we can find a polynomial in the coordinates $w_1$ ane $w_2$ of $C^2$ which locally peaks [3] at $p$ when

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restricted to the surface. The results of Rossi and the Arens-Calderon Theorem [1] then show that this will contradict the maximum modulus property. Thus $\pi$ cannot be nonanalytic. This gives a relation between the complex derivatives of $f$ and $g$ which, in particular, implies that both functions are analytic at the pre-image of $p$ if one of them is analytic there. In § 2 the essential geometric lemma is established and in § 3 it is used to prove the main result.

2. Let $F\colon M \to R^n$ be an immersion (a regular map in the sense of Whitney [6]) of a two (real) dimensional twice continuously differentiable manifold $M$ into real Euclidean $n$-space $R^n$. Let $p \in M$ and let $(U, h)$ give local coordinates about $p$, where $U$ is an open set in $M$, and $h$ is a homeomorphism from $U$ onto $D = \{(u, v) \in R^2\colon u^2 + v^2 < 1\}$, with $h(p) = (0, 0)$. If $x_j$ $(j = 1, \cdots, n)$ is a coordinate function in $R^n$ then the functions $\phi_j(u, v) = x_j \circ F \circ h^{-1}(u, v)$ $(j = 1, \cdots, n)$ are differentiable and give a map $\Phi\colon D \to R^n$ defined by $\Phi(u, v) = (\phi_1(u, v), \cdots, \phi_n(u, v))$. Since $F$ is an immersion, the 2 by $n$ matrix

$$\left(\frac{\partial \phi_j}{\partial u}, \frac{\partial \phi_j}{\partial v}\right) = \begin{pmatrix} \dfrac{\partial \phi_1}{\partial u} & \dfrac{\partial \phi_1}{\partial v} \\ \vdots & \vdots \\ \dfrac{\partial \phi_n}{\partial u} & \dfrac{\partial \phi_n}{\partial v} \end{pmatrix}$$

has rank 2 and the mapping $\Phi$ is one-to-one in some disc $V = \{(u, v) \in R^2\colon u^2 + v^2 < r^2 < 1\}$. Further, the set $\Phi(V)$ is a surface element having a tangent plane at $\Phi(0, 0)$).

We can suppose for our purposes that $\Phi(0, 0)$ is the origin 0 in $R^n$. The tangent plane $\pi$ to $\Phi(V)$ at 0 is then given parametrically by

$$x_j = \frac{\partial \phi_j}{\partial u} u + \frac{\partial \phi_j}{\partial v} v \qquad (j = 1, \cdots, n)\,, \tag{2.1}$$

where the derivatives are evaluated at $u = v = 0$. A change of local parameters from $u$ and $v$ to $u' = u'(u, v)$ and $v' = v'(u, v)$ with $\partial(u', v')/\partial(u, v) \neq 0$ (the inverse transformation being given by $u = u(u', v')$ and $v = v(u', v')$ in some neighborhood of $u = v = 0$) yield new functions $\phi'_j(u', v') = \phi_j(u(u', v'), v(u', v'))$ and a new parametrization of the tangent plane, namely,

$$\begin{aligned} x_j &= \frac{\partial \phi'_j}{\partial u'} u' + \frac{\partial \phi'_j}{\partial v'} v' \\ &= \left(\frac{\partial \phi_j}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial \phi_j}{\partial v}\frac{dv}{\partial u'}\right) u' + \left(\frac{\partial \phi_j}{\partial u}\frac{\partial u}{\partial v'} + \frac{\partial \phi_j}{\partial v}\frac{\partial v}{\partial v'}\right) v' \\ &\qquad\qquad (j = 1, \cdots, n)\,. \end{aligned}$$

Note that the rank of the matrix $(\partial\phi_j/\partial u, \partial\phi_j/\partial v)$ is the same as that of $(\partial\phi'_j/\partial u', \partial\phi'_j/\partial v')$ since $\partial(u', v')/\partial(n, v) \neq 0$.

Now $u$ and $v$ parametrize both the surface element $\Phi(V)$ and the tangent plane (given by (2.1)). Let $\eta_j$ and $\eta'_j$ $(j = 1, 2, \cdots n)$ denote the coordinates in $R^n$ of the points $B$ and $B'$ on $\pi$ and $\Phi(V)$, respectively, corresponding to the parameters $u$ and $v$ $(u^2 + v^2 < r^2)$. For sufficiently small $u$ and $v$,

$$\eta'_j = \frac{\partial\phi_j}{\partial u}u + \frac{\partial\phi_j}{\partial v}v + \frac{1}{2}\left(\frac{\partial^2\phi_j}{\partial u^2}u^2 + 2\frac{\partial^2\phi_j}{\partial u\partial v}uv + \frac{\partial^2\phi_i}{\partial v^2}v^2\right)$$

where the first derivatives are evaluated at $u = v = 0$ and the second derivatives are evaluated at $u' = \theta u$, $v' = \theta v$ for some $\theta$ satisfying $0 < \theta < 1$. Since $M$ is twice continuously differentiable, the second derivatives of $\phi_j$ are bounded in absolute value in some sufficiently small neighborhood of $(0, 0)$ and we obtain

$$\sum_1^m (\eta_j - \eta'_j)^2 \leqq K(|u| + |v|)^4$$

and so

$$(2.2) \qquad |\eta_j - \eta'_j| \leqq L(|u| + |v|)^2$$

where $K$ and $L$ are constants depending on these bounds and on $n$, and $u$ and $v$ are sufficiently small. These estimates will be used later.

Suppose now that $n = 2m$. One can define complex coordinates $w_j = x_{2j-1} + ix_{2j}$ making $R^n$ into complex Euclidean space $C^m$. Also the $(u, v)$-plane can be formally complexified by writing $z = u + iv$, $\bar{z} = u - iv$. We then have a mapping $\Psi$: $V \rightarrow C^m$ defined by $\Psi(z, \bar{z}) = (w_1, \cdots, w_m)$ where

$$w_j = \Psi_j(z, \bar{z}) = \phi_{2j-1}\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + i\phi_{2j}\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) \qquad (j = 1, \cdots m)\,.$$

An elementary computation shows that in this formalism the tangent plane $\pi$ to $\Psi(V)$ at the origin 0 is given parametrically by

$$w_j = \frac{\partial\Psi_j}{\partial z}z + \frac{\partial\Psi_j}{\partial\bar{z}}\bar{z} \qquad (j = 1, \cdots, m)$$

where the derivatives are evaluated at $z = \bar{z} = 0$. Furthermore, under a change of local coordinates in the parameter plane from $z$ and $\bar{z}$ to $z' = u' + iv'$ and $\bar{z}' = u' - iv'$, the tangent plane is given parametrically by

$$\begin{aligned} w_j &= \frac{\partial\Psi'_j}{\partial z'}z' + \frac{\partial\Psi'_j}{\partial\bar{z}'}\bar{z}' \\ &= \left(\frac{\partial\Psi_j}{\partial z}\frac{\partial z}{\partial z'} + \frac{\partial\Psi_j}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial z'}\right)z' + \left(\frac{\partial\Psi_j}{\partial z}\frac{\partial z}{\partial\bar{z}'} + \frac{\partial\Psi_j}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial\bar{z}'}\right)\bar{z}' \end{aligned}$$

where $\Psi'_j(z', \bar{z}') = \Psi_j(z(z', \bar{z}'), \bar{z}(z', \bar{z}))$. Since, as a short calculation shows,

$$\frac{\partial(z, \bar{z})}{\partial(z', \bar{z}')} = \frac{\partial(u, v)}{\partial(u', v')} \neq 0 ,$$

the complex rank of $(\partial\Psi_j/\partial z, \partial\Psi_j/\partial\bar{z})$ remains unaffected by a parameter change. We now make the following definition.

**Definition 1.** The two (real) dimensional plane in $C^m$ defined parametrically by $w_j = \alpha_j z + \beta_j \bar{z}$ $(j = 1, 2, \cdots, m)$ is said to be *nonanalytic* if the rank of the 2 by $m$ (complex) matrix $(\alpha_j, \beta_j)$ is 2.

The preceding remarks show that if $\pi$ is nonanalytic in one coordinate parametrization then it remains so under any change of coordinates in the parameter plane. We want to establish the following.

LEMMA. *Suppose the tangent plane $\pi$ to $\Psi(V)$ at the origin in $C^m$ is nonanalytic. Then there is a polynomial in the coordinates $w_j$ whose absolute value takes on a local maximum at the origin when restricted to $\Psi(V)$.*

*Proof.* Since $\pi$ is nonanalytic there exist new coordinates $w'_i = \sum_{j=1}^m \gamma_{ij} w_j$ $(i = 1, \cdots, m)$, where the matrix $(\gamma_{ij})$ is nonsingular, such that in the $w'_i$-coordinates $\pi$ is given parametrically by $w'_1 = z$, $w'_2 = \bar{z}$, and $w'_j = 0$ $(m \geqq j \geqq 3)$. Now let $B$ and $B'$ be points on $\pi$ and $\Psi(V)$, respectively, corresponding to the parameters $u$ and $v$. Let $\eta_j$ and $\eta'_j$ $(j = 1, \cdots, 2n)$ be the real coordinates of $B$ and $B'$ (with $C^m$ regarded as $R^{2n}$) in the new coordinate system. Clearly $\eta_1 = u$, $\eta_2 = v$, $\eta_3 = u$, $\eta_4 = -v$, and $\eta_j = 0$ for $5 \leqq j \leqq 2n$. Let $\eta'_j - \eta_j = \varepsilon_j$ $(j = 1, \cdots, 4)$.

Now consider the function $P(w_i) = 1 - w'_1 w'_2$ (a polynomial in $w_1, \cdots, w_m$). When restricted to $\pi$, $P(w_i)$ is real-valued and has a maximum in absolute value at the origin. We would like to show that $|P(w_i)|$ also has a local maximum at the origin when restricted to a sufficiently small neighborhood of the origin on $\Psi(V)$. This will be true essentially because $\pi$ has a contact of order at least 1 with $\Psi(V)$ at the origin (here we will use the estimates (2.2)).

We have, at the point $B'$,

$$\begin{aligned} |P(B')|^2 &= |1 - (\eta'_1 + i\eta'_2)(\eta'_3 + i\eta'_4)|^2 \\ &= 1 - 2(u^2 + v^2) + (u^2 + v^2) + 2Q(u, v)[u^2 + v^2 - 1] \\ &\quad + [Q(u, v)]^2 + [u(\varepsilon_2 + \varepsilon_4) + v(\varepsilon_3 - \varepsilon_1)]^2 \end{aligned}$$

where

$$Q(u, v) = u(\varepsilon_1 + \varepsilon_3) + v(\varepsilon_2 - \varepsilon_4) + \varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4 .$$

Using inequalities (2.2) for $|\varepsilon_j|$ $(j = 1, \cdots, 4)$ we obtain

$$
\begin{aligned}
|P(B')|^2 &\leqq 1 - 2(u^2 + v^2) + M(|u| + |v|)^3 \\
&\leqq 1 - [1 - M(|u| + |v|)](|u| + |v|)^2
\end{aligned}
$$

for $u$ and $v$ sufficiently small and some constant $M$. If $|u| + |v| < 1/M$ we see that $|P(B')| < 1$ unless $B' = 0$.

3. Let $K$ be a compact subset, with nonempty interior, of a Riemann surfce $M$.

DEFINITION 2. An algebra $A$ of continuous functions on $K$ is said to be a maximum modulus algebra on $K$ if for every $f \in A$ there is a point $p$ on the boundary $\partial K$ of $K$ such that $|f(q)| \leqq |f(p)|$ for all $q \in K$.

As remarked in [4], we can suppose without loss of generality that $A$ is uniformly closed and contains the constants and so is a Banach algebra with identity and norm $||f|| = \sup_{p \in K} |f(p)|$. It is well known that there is a natural continuous mapping $i: K \to \Sigma_A$, where $\Sigma_A$ is the maximal ideal space of $A$ (with the usual Gelfand topology), defined by point evaluation (which is not $1:1$ unless $A$ separates points in $\Sigma_A$).

THEOREM. *Let $A$ be a uniformly closed algebra of continuous functions, containing the constants, on the compact subset $K$ (with nonempty interior) of the Riemann surface $M$. Suppose that there is a set $D$ of points in* int $K$ *having no limit point in* int $K$, *such that each $p \in \operatorname{int} K - D$ has a neighborhood $U$ for which $i(U)$ is open in $\Sigma_A$. Suppose further that $A$ contains one nonconstant analytic function $g = g^1 + ig^2$. Then any function $f = f^1 + if^2$ in $A$ such that $f^1$ and $f^2$ are twice continuously differentiable is analytic in* int $K$.

*Proof.* Let $S$ be the discrete subset of int $K$ on which the differential $dg$ vanishes. For any point $p$ in int $K - S$ there is a neighborhood containing $p$ and contained in int $K - S$ and in which $g$ is one-to-one. Thus for any point $p \in \operatorname{int} K - (D \cup S)$ there exists a neighborhood $U$ containing $p$ which is mapped homeomorphically by $i$ onto an open set $W$ in $\Sigma_A$ and hence local coordinates in $U$ may be transferred to $W$. Define the mapping $F: \Sigma_A \to C^2$ by $F(q') = (f(q'), g(q'))$, $q' \in \Sigma_A$ (where we have used the letters $f$ and $g$ to denote the extension, via the Gelfand representation, of $f$ and $g$, defined on $i(K)$, to $\Sigma_A$). For any point $q'$ in $W$ we have $f(q') = f(i^{-1}(q'))$ and $g(q') = g(i^{-1}(q'))$ so that $F$ can be regarded as a mapping defined on $U$ by $F(q) = (f(q), g(q))$, $q \in U$. $F$ defines an immersion of $W$ since in the local coordinates $z = u + iv$ the matrix

$$
\begin{pmatrix}
f^1_u & f^2_u & g^1_u & g^2_u \\
f^1_v & f^2_v & g^1_v & g^2_v
\end{pmatrix}
$$

(here the subscripts $u$ and $v$ denote partial differentiation) is of rank 2 due to the nonvanishing of the differential $dg = \partial g/\partial z\, dz$—apply the Cauchy-Riemann equations to the matrix

$$\begin{pmatrix} g_u^1 & g_u^2 \\ g_v^1 & g_v^2 \end{pmatrix}.$$

Since $A$ contains the constants we can suppose without loss of generality that $F(p)$ is the origin 0 in $C^2$. We have thus a mapping $F: \Sigma_A \to C^2$ which maps a neighborhood $W$ of $i(p)$ onto a two-dimensional surface element $F(W)$ having a tangent plane $\pi$ at 0.

We now note that $\pi$ cannot be nonanalytic. For if this were the case then by the lemma of § 2 there would be a polynomial in the coordinates $w_1$ and $w_2$ of $C^2$ taking on a local maximum in absolute value at 0 when restricted to $F(W)$. By the Arens-Calderon theorem [**1**; Theorem 3.3] there would then be a function $k \in A$ taking on a local maximum at $i(p)$, and finally, by Rossi's Local Peak-Point Theorem [3, Theorem 4.1] there would be a function $\tilde{k} \in A$ taking on its maximum value exactly at $i(p)$, contradicting the fact that $A$ is a maximum modulus algebra.

Thus the rank of

$$\begin{pmatrix} \dfrac{\partial g}{\partial z} & 0 \\ \dfrac{\partial f}{\partial z} & \dfrac{\partial f}{\partial \bar{z}} \end{pmatrix}$$

(the derivatives being evaluated in the local coordinates at $p$) must be 1 and this implies that $\partial f/\partial \bar{z} = 0$. The same conclusion could be drawn for any $p \in \inf K - (D \cup S)$ and so by the theorem of Riemann on removable singularities, $f$ is analytic in int $K$.

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