CHAINS AND GRAPHS OF OSTROM PLANES

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1. In 1961, in a letter to D. Hughes, T. G. Ostrom communicated a process that, as developed by Hughes and set forth by A. A. Albert [1], transformed a projective plane of a particular type into another using a coordinatizing ring of the first as a tool. This process may be modified to make more direct use of the algebra to a point where, indeed, it may be employed to create new rings out of old without the mediation of a plane. On the other hand the process may be dualized to alleviate a disadvantage of the essentially involutory nature of the original; from a given initial plane the Ostrom process gives one new plane; if repeated the original plane results. In the process to be discussed below a number of planes result, and, in particular, from a Desarguesian plane of order at least 9, three others, the Hall plane, its dual, and a self-dual plane make a complete set. Recently, Ostrom has published in [5] a development of his original process. We shall refer primarily to [1] as it more directly affects the development of the results to be presented.

2. First we establish some notation: Let π be a finite projective plane coordinatized by a ternary ring R whose additive structure is a group. For purposes of symmetry, we modify the usual notation and denote by $y = x \cdot m \circ b$ the line through the point (m) of L_∞ and the point $(0, -b)$. Let π^* be the dual of π and let it be coordinatized by R^* where R^* is defined by $b = m \cdot x \circ y$ in R^* when $y = x \cdot m \circ b$ in R. We note that, if π (and R) are such that $x \cdot m \circ b = xm - b$ for all x, m, b, then R^* is just the multiplicative mirror image of R and if, further, R is commutative, $R = R^*$.

Second we assume some additional restrictions on R (and thereby on R^*, π, and π^*).

(a) The additive structure of R is that of an abelian group.

(b) R is a vector space of dimension 2 over a field K whose elements commute with all elements of R in the standard binary multiplication in R.

(c) For $a, b \in R$, $\alpha, \delta \in K$,

$$\alpha(\delta a) = (\alpha\delta)a$$
$$(\alpha + \delta)a = \alpha a + \delta a$$
$$\alpha(a + b) = \alpha a + \alpha b$$
$$a \cdot \alpha \circ b = a\alpha - b = \alpha a - b = \alpha \cdot a \circ b\ .$$

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(These conditions are a bit more restrictive than those in [1] because of the need for symmetry.) Note that R^* automatically has all the properties specified for R.

Third, we define a transformation of π (or of π^*), O_T. The points, not on L_∞ may be written as $(x, y) = (x_1t + x_2, y_1t + y_2)$, $t \notin K$, $x_1, x_2, y_1, y_2 \notin K$. This will be abbreviated when the notation will be unambiguous by (x_1x_2, y_1y_2).

Now we introduce the mapping, α: $(x_1x_2, y_1y_2) \xrightarrow{\alpha} (x_1y_1, x_2y_2)$. The new lines of πO_T will be:

(1) All lines of π of the forms $x = a$, $y = mx - b$, $m \in K$, and L_∞;

(2) The sets of points which are the images under α of lines in π with slope $m \notin K$. (The designations of the individual points on L_∞ remain to be defined.) The new complex, πO_T, which will be shown to be a projective plane, is denoted by π_T ($\pi^* O_T = \pi_T^*$). It will be coordinatized by a system based on the axes and the unit line as given and the following determination of slopes: Take the line $y = x(m_1t + m_2)$ in π, $m_1 \neq 0$. Find on it a point of the form $(0m_1', 1m_2')$; indeed,

$$t + m_2' = m_1'(m_1t + m_2) = m_1'm_1t + m_1'm_2 .$$

Hence $m_1' = m_1^{-1}$, $m_2' = m_1^{-1}m_2$ and the point exists and is unique. The algebraic manipulations are justified by the restrictions placed on R. Under α this point passes to $(01, m_1'm_2')$ and we label the intersection of the transformed line and L_∞ by m'. The remaining points on L are designated by the appropriate slope in K. We finish by defining the new lines $y = x \cdot m' \circ b'$ by finding the parallel class m' and the negative intercept b'. (The fact that there is indeed a unique intercept is almost obvious and, further, it will be dealt with in the general considerations of intersections in the proof that π_T is a projective plane.)

Fourth, we define a dual transformation of π^* (or π). This begins with a mapping δ of the lines: $(m_1m_2, b_1b_2) \xrightarrow{\delta} (m_1b_1, m_2b_2)$. The new points will be defined by considering the set of lines through a point (x_1x_2, y_1y_2) of π, $x_1 \neq 0$. This set transformed by δ defines the new point. Also we will have points (x, y), $x \in K$ and the points on L_∞. It remains to indicate the mappings of the lines of the form $x = a$. We follow the dual of the former situation. Take a point in π^* of the form $(x_1x_2, 00)$, $x_1 \neq 0$. Find a line through it with slope in K, say x_1', and negative intercept $t + x_2'$. The line through the transformed point and the special point (∞) will be designated $x = x_1't + x_2'$. Other lines $x = a$ are designated to conform with the points whose coordinates are in K on the axis.

Finally, we find the coordinates of the still unlocated points by referring to the corresponding lines $x = x_0$, $y = y_0$. The resulting complex we will call π_S^*; the corresponding transformation is O_S.

3. The next step is to show that π_T is a projective plane coordinatized by a ternary ring R_T. When this is done, the equivalent statements for π_T^*, π_S, and π_S^* will be obvious by the duality of the constructions.

The counts are clearly correct. Further, a line of the (new) form $y = x \cdot m \circ b$, $m \notin K$, intersects another line of that form in precisely the image point of the intersection of the two precedent lines in π.

As for the intersection of such a line with one of the form $y = x\mu - b_1$, consider that, for a fixed μ, each point not on L_∞ lies on such a line so that $y = x \cdot m \circ b$ intersects the n lines $y = x\mu - b_i$ in n points. No intersection is on L_∞. It remains to be shown that no two are on any one line. If this happened we may see that the predecessor line would have a duplicate intersection with a line of slope in K (or undefined slope). If

$$\begin{aligned} y_1 &= x_1\mu - b_1 \\ y_2 &= x_2\mu - b_1 \\ y_2 - y_1 &= (x_2 - x_1)\mu \,. \end{aligned}$$

Now, if $(x_1, y_1) = (\xi_1\xi_2, \eta_1\eta_2)$, $(x_2, y_2) = (\xi_3\xi_4, \eta_3\eta_4)$, we have

$$\eta_3 t + \eta_4 - \eta_1 t - \eta_2 = [\xi_3 t + \xi_4 - \xi_1 t - \xi_2]\mu$$

so that

$$(\xi_3\xi_4, \eta_3\eta_4) = [\xi_3 t + \xi_4, (\eta_1 + \xi_3\mu - \xi_1\mu)t + \eta_2 + \xi_4\mu - \xi_2\mu] \,.$$

Hence the two precursor points have the forms:

$$(\xi_1\eta_1, \xi_2\eta_2) = (x_1', y_1')$$

and

$$[\xi_3 t + \eta_1 + \xi_3\mu - \xi_1\mu, \xi_4 t + \eta_2 + \xi_4\mu - \xi_2\mu] = (x_2', y_2') \,.$$

Thus the differences $y_2' - y_1' = (\xi_4 - \xi_2)t + (\xi_4 - \xi_2)\mu$ and

$$x_2' - x_1' = (\xi_3 - \xi_1)t + (\xi_3 - \xi_1)\mu$$

are related by the expression:

$$(\xi_4 - \xi_2)(x_2' - x_1') = (\xi_3 - \xi_1)(y_2' - y_1') \,.$$

The intersections with lines of the form $x = a$ can be similarly treated. Finally, mutual intersections of unchanged lines are unchanged.

(In the calculations carried out in this part of the proof, those done in the new plane before the final establishment of the properties of its coordinatizing ring are based on the definitions of the special lines in terms of R and on the restrictions on R.)

We must further show that every pair of points is on at least one line. The case where one of the pair is on L_∞ was dealt with in the original construction. If the difference of the x coordinates of the two points is zero or if the y difference is a multiple of the x difference by an element of K, it is possible to locate the points on one of the special (unchanged) lines. Otherwise, look at the precursors of the pair involved; they lie on a line whose image under α is a line containing the specified pair.

4. The above proofs and those immediately to follow on the properties of the new coordinatizing ring are included to indicate the greater directness possible with the present definition of the Ostrom transformation which, on the surface, differs greatly from that given by Albert. It would be enough to prove that O_T was simply the combination of the transformation described by Albert with a coordinatization of the result differing in a comparatively minor way from that given by him. When this difference had been shown inconsequential, all the results in his exposition would follow. However, we believe that the direct proofs are interesting both comparatively and in their own rights.

The elements and operations of K are again to be found in the coordinatizing ring R_T of π_T since the lines of the Desarguesian subplane coordinatized by K are unchanged. If we look at the algebraic implications of the special (unchanged) lines, we see that for $\alpha, \delta \in K$,

$$\begin{aligned} a \cdot \alpha \circ b &= a\alpha - b \\ (a\alpha)\delta &= a(\alpha\delta) \\ a(\alpha + \delta) &= a\alpha + a\delta \\ (a + b)\alpha &= a\alpha + b\alpha \end{aligned}$$

form the restrictions on R and the inheritance of slopes in K. To establish the remaining special properties of elements of K, it is necessary to show that $a\alpha = \alpha a$ in R_T. Without loss of generality, $a = a_1 t + a_2$, $a_1 \neq 0$. On the line $y = xa$, consider the point $(\alpha, y) = (0\alpha, y_1 y_2)$. Its predecessor, $(0y_1, \alpha y_2)$ lies on a line in π with slope $a_1^{-1}t + a_1^{-1}a_2$. That is, since the line passes through the origin,

$$\begin{gathered} \alpha t + y_2 = y_1(a_1^{-1}t + a_1^{-1}a_2) = y_1 a_1^{-1} t + y_1 a_1^{-1} a_2 \,. \\ \alpha = y_1 a_1^{-1};\ y_2 = y_1 a_1^{-1} a_2 \,, \quad \text{multiplications in } K \,. \\ y_1 = a_1 \alpha;\ y_2 = a_2 \alpha \,. \end{gathered}$$

In the latter stages of the above proof, use was made of the fact that the elements of R_T are again a right vector space of dimension two over K. This follows from the preservation of the form of the elements in R_T, the properties mentioned above, and the preservation of the lines with slope 1 among the inherited lines.

THEOREM 1. *Under the notations and restrictions of* § 2, *the transformations* O_T *and* O_S *carry a projective plane with restricted coordinatizing ring* R *into another (not necessarily distinct) projective plane with coordinatizing ring* R_T *or* R_S *having precisely the same properties as specified for* R.

THEOREM 2. *Right distributivity of* R *implies right distributivity of* R_T; *left distributivity of* R *implies left distributivity of* R_S.

Proof. We prove the first half, leaving the remainder to follow by duality. Consider

$$(a + b)m = (a_1t + a_2 + b_1t + b_2)(m_1t + m_2)$$

where, without loss of generality $m_1 \neq 0$. The precursor of the point (c_1c_2, y_1y_2) is (c_1y_1, c_2y_2) where $c_1 = a_1 + b_1$, $c_2 = a_2 + b_2$. That is, $c_2t + y_2 = (c_1t + y_1)m'$ where m' is the precursor of m. Similarly, from

$$am = y_3t + y_4,\ bm = y_5t + y_6,\ a_2t + y_4 = (a_1t + y_3)m' ;$$
$$b_2t + y_6 = (b_1t + y_5)m' .$$

Expanding all three expressions by the right distributivity of R, and substituting the two latter in the first:

$$(y_1 - y_3 - y_5)m' = (y_2 - y_4 - y_6) .$$

But the right hand side is in K, hence so is the left. Therefore both sides are zero. This completes the proof.

THEOREM 3. *If* R *admits the following weak right distributivity*: $(a + \alpha)b = ab + \alpha b$ *for* α *in* K, *and if the line in* π, $y = x \cdot m \circ b$, *has the form* $y = xm - b$, *then the successor line in* π_T, $y = x \cdot m' \circ b'$, *also has the form* $y = xm' - b'$. *The equivalent statement can be made for points in* π_S, *if* R *admits weak left distributivity.*

Proof. Again it suffices to prove the case for π_T. For a line $y = xm - b$, where, without loss of generality, we may assume $m = m_1t + m_2$, $m_1 \neq 0$, we find not only $m' = m_1^{-1}t + m_1^{-1}m_2$, but also b'. If

$$b = b_1t + b_2,\ b' = b_1't + b_2' ,$$

we solve:

$$-b_2' = -b_1'(m_1t + m_2) - (b_1t + b_2) ,$$

and find

$$b_1' = -m_1^{-1}b_1;\ b_2' = b_2 - m_1^{-1}m_2b_1 .$$

Now denote by $X_1t + X_2$, the value $(x_1t + x_2)(m_1t + m_2)$. Then the point on the line:

$$(x_1t + x_2, (X_1 - b_1)t + X_2 - b_2)$$

becomes

$$(x_1t + X_1 - b_1, x_2t + X_2 - b_2)$$

and we must determine if this satisfies $y = xm' - b'$. From the point (x_1x_2, X_1X_2) on the line $y = xm$, we know that the point (x_1X_1, x_2X_2) is on $y = xm'$ or $x_2t + X_2 = (x_1t + X_1)m'$. Then, in the present case,

$$\begin{aligned} xm' - b' &= (x_1t + X_1 - b_1)m' - b' \\ &= (x_1t + X_1)m' - b_1(m_1^{-1}t + m_1^{-1}m_2) - (-m_1^{-1}b_1t + b_2 - m_1^{-1}m_2b_1) \\ &= x_2t + X_2 - b_2 , \end{aligned}$$

which was to be shown. We conclude this section by stating the duality relations which follow from the construction procedures.

THEOREM 4. *Using, as before, the asterisk to denote duality*:

$$(\pi_T)^* = \pi_S^*, (\pi_S)^* = \pi_T^* .$$

5. In this section we will consider the planes related to the Desarguesian plane by the transformations O_T and O_S. It is shown in [1] that the transform of a Desarguesian plane by O_T is a Hall plane. The proof will be given in the present notation. Theorem 4 then gives the result that the transform by O_S is the dual of the Hall plane.

THEOREM 5. *The transform of a Desarguesian plane by O_T is a Hall plane.*

Proof. For R to be the coordinatizing ring of a Desarguesian plane, R must be a quadratic field over K. Let the equation satisfied by t be $t^2 = rt + s$. We desire to investigate R_T and to show it is a Hall system. In [2] Hall gives three conditions in his Theorem 20.4.7. The first two amount to requiring that the ring be a right vector space over K; this, and more, is guaranteed by Theorem 2. The third may be quoted verbatim: (substituting t for u): For $z = a + tb$ where, $a, b \in K, b \neq 0$, and $w = c + zd, c, d \in K$, put

$$wz = ds + z(c + dr) = ac + adr + ds + t(bc + bdr) .$$

To better conform with our notation and procedures, let $z = mt + n$,

$w = xt + y$ so that $a = n$, $b = m$, $c = y - nm^{-1}x$, $d = m^{-1}x$. The product is then

$$wz = ny - n^2m^{-1} + nm^{-1}xr + m^{-1}xs + (my - nx + xr)t \ .$$

Now, if in R_T, $wz = (xt + y)(mt + n) = et + f$, then in R:

$$(xt + e)(m^{-1}t + m^{-1}n) = yt + f$$

$$xm^{-1}(rt + s) + (em^{-1} + xm^{-1}n)t + em^{-1}n = yt + f$$

$$(xm^{-1}r + em^{-1} + xm^{-1}n)t + xm^{-1}s + em^{-1}n = yt + f \ .$$

$$e = my - xr - xn$$

$$f = xm^{-1}s + em^{-1}n = xm^{-1}s + ny - m^{-1}nxr - m^{-1}n^2x \ .$$

If the two expressions are examined, they are found to be identical except for the signs on terms containing r. That is, R_T is the Hall system which results from employing the polynomial $x^2 + rx - s$ in place of $x^2 - rx - s$.

It remains to determine what happens when we perform O_S on the Hall plane or O_T on its dual. Except in the case where K is $GF(2)$, the Hall plane is distinct from the Desarguesian plane and from its dual. Then, because of the involutory nature of the transformation, the O_S transform of π_T cannot be the original. Also, it cannot be π_S since then the repeated transformation of the Desarguesian plane by O_S would yield π_T rather than the original. Thus, either π_T is invariant under O_S or a new plane results. We shall show that the latter is the case and that the result is a self-dual plane which then results also from the effect of O_T on the dual Hall plane.

THEOREM 6. *The result of performing first O_S, then O_T on the Desarguesian plane is the same as performing first O_T, then O_S and is a self-dual plane distinct* (*except in the case of order* 4) *from the original plane.*

Proof. We will prove that the incidence sets are the same. For convenience, we will denote an incidence of the point (x, y) with the line of slope m and negative intercept b (the line (m, b)) by $[x, y, m, b]$, or in our expanded notation by $[x_1, x_2, y_1, y_2, m_1, m_2, b_1, b_2]$. There are four cases, only one of which offers any interest of difficulty.

Case I. $x_1 = m_1 = 0$. Neither O_S nor O_T change this incidence which refers to an unchanged line under O_T and an unchanged point of O_S.

Case II. $x_1 = 0 \neq m_1$. For O_T, this involves a changed line; for O_S an unchanged point.

$$
\begin{aligned}
&[0, x_2, y_1, y_2, m_1, m_2, b_1, b_2] \\
\xrightarrow{O_T}\ &[0, y_1, x_2, y_2, m_1^{-1}, m_1^{-1}m_2, -m_1^{-1}b_1, b_2 - m_1^{-1}m_2b_1] \\
\xrightarrow{O_S}\ &[0, y_1, x_2, y_2, m_1^{-1}, m_1^{-1}m_2, -m_1^{-1}b_1, b_2 - m_1^{-1}m_2b_1] .
\end{aligned}
$$

In the other order

$$
\begin{aligned}
&[0, x_2, y_1, y_2, m_1, m_2, b_1, b_2] \\
\xrightarrow{O_S}\ &[0, x_2, y_1, y_2, m_1, m_2, b_1, b_2] \\
\xrightarrow{O_T}\ &[0, y_1, x_2, y_2, m_1^{-1}, m_1^{-1}m_2, -m_1^{-1}b_1, b_2 - m_1^{-1}m_2b_1] .
\end{aligned}
$$

In the calculation of the new negative intercepts we use the fact that both the Hall plane and its dual are linear (Theorem 3) and the property that this sufficed for the calculation in the proof of Theorem 4. The distributivity hypothesis was not used at this point.

Case III. $m_1 = 0 \neq x_1$. The calculation is precisely the same as in Case II. For completeness we write the final incidence:

$$[x_1^{-1}, x_1^{-1}x_2, -x_1^{-1}y_1, y_2 - x_1^{-1}x_2y_1, 0, b_1, m_2, b_2] .$$

Case IV. $m_1x_1 \neq 0$. Here we interpose a lemma: It is always possible to choose t so that, if $[x_1, x_2, y_1, y_2, m_1, m_2, b_1, b_2]$ is an incidence, so is: $[x_1, -x_2, -y_1, y_2, m_1, -m_2, -b_1, b_2]$, the plane being Desarguesian. To prove the lemma, we note that if the characteristic is 2, there is nothing to prove and that if the characteristic in not 2, K contains a nonsquare so that we may assume that the equation satisfied by t is $t^2 = \alpha$, α in K. Then we know that

$$y_1t + y_2 = (x_1t + x_2)(m_1t + m_2) - (b_1t + b_2) ,$$

or

$$y_1 = x_1m_2 + x_2m_1 - b_1;\ y_2 = x_1m_1\alpha + x_2m_2 - b_2 .$$

Further,

$$
\begin{aligned}
(x_1t - x_2)(+m_1t - m_2) - (-b_1t + b_2) &= (-x_1m_2 - x_2m_1 + b_1)t \\
&\quad + (x_1m_1\alpha + x_2m_2 - b_2) = -y_1t + y_2 .
\end{aligned}
$$

We make the obvious one-one correspondence between the two sets of indices for this case and carry out the now familiar calculations on the original incidence for O_T followed by O_S, and on the incidence with the four minus sign for O_S followed by O_T. The end result is the derived incidence:

$$[x_1^{-1}, x_1^{-1}y_1, -x_1^{-1}x_2, y_2 - x_1^{-1}x_2y_1, m_1^{-1}, -m_1^{-1}b_1, m_1^{-1}m_2, b_2 - m_1^{-1}m_2b_1] .$$

Thus the theorem is proved and the structure of the new self-dual plane explicitly revealed by defining its incidence matrix in terms of that of the Desarguesian plane. This is the best we could hope to de since the plane is clearly not linear.

The question naturally arises: What is the relationship of these planes to the Hughes planes? For order 9, they are identical. Indeed the ring R_{ST} is precisely the ring given from the construction of Hughes in [4] and reproduced explicitly in [3]. For higher orders it will follow from as yet unpublished work of Ostfom that they are distinct. It is to be noted that these planes do not depend on an odd order and also, there is always only one of these planes for any order while there are two distinct Hughes planes for $11^2, 23^2, 29^2, 59^2$.

6. What is the general situation for planes with rings of the prescribed type? If we take such a plane and its dual and proceed to form the successive O_T and O_S transformations we get parallel structures or graphs which have no more than two joins from each point and which are, hence, rings or chains. If a self-dual plane occurs the parallel structures merge. The structures will be closed loops unless a plane is left invariant under the transformation. The general possible structures of these graphs are not known. It is possible to formulate a number of possible hypotheses such as that each graph is a ring or lozenge having two self-dual planes but direct calculation seems impossible except for such simple cases as the Desarguesian case considered in the previous section.

What part has the restriction to finite orders played? The sole explicit use was in the first proof where a finitistic counting argument was used to prove that a changed line intersected an unchanged line. This argument can be replaced by other conditions such as actually calculating the intersection where the ring is simple enough. Thus, of course, this method of deriving the Hall plane extends to the infinite case.

The situation for the self-dual plane is a bit more complicated. Since the unique solution is calcuable in fields of all even powers of all primes, we could, perhaps appeal to local-global theorems but the direct proof is more satisfying. We will carry out the proof for the intersection of lines of the form $y = x\delta - d$, δ in K, with lines $y = x \cdot m \circ b$, m not in K, in a field quadratic over K where $t^2 = \alpha$. The other possible cases will be seen to be essentially similar. Under the conditions we seek an incidence in the new plane of the form:

$$[x_1, x_2, x_1\delta - d_1, x_2\delta - d_2, m_1, m_2, b_1, b_2] .$$

In carrying this incidence back to the Desarguesian plane, two possibilities arise depending on a possible solution with $x_1 = 0$. If such a solution exists, the corresponding incidence in the Desarguesian plane is:

$$[0,\ -d_1,\ x_2,\ x_2\delta - d_2,\ m_1^{-1},\ m_1^{-1}m_2,\ -m_1^{-1}b_1,\ b_2 - m_1^{-1}m_2b_1]\ .$$

If not, we have:

$$[x_1^{-1},\ \delta - d_1x_1^{-1},\ -x_1^{-1}x_2,\ x_1^{-1}x_2d_1 - d_2m_1^{-1},\ -m_1^{-1}b_1,\ m_1^{-1}m_2,\ b_2 - m_1^{-1}m_2b_1]\ .$$

The first case leads to the equations in the field:

$$m_1x_2 = b_1 - d_1;\ \delta m_1x_2 = m_1d_2 - m_2d_1 - m_1b_2 + m_2b_1\ ,$$

with a consequent solubility condition:

$$m_1(d_2 - b_2) + (\delta - m_2)(d_1 - b_1) = 0\ .$$

The second case to the equations:

$$\begin{aligned}(\delta - m_2)x_1 + m_1x_2 &= b_1 + d_1\\ (b_2m_1 + b_1\delta - m_2b_1 - m_1d_2)x_1 + m_1d_1x_2 &= \alpha + b_1d_1\ .\end{aligned}$$

The determinant of this system is just m_1 times the expression which must be zero in the first case and the two possibilities are complementary. Incidentally, although the uniqueness did not depend on finitistic arguments, it is clearly revealed in the computation. If $b_1 + d_1 = 0$, $\alpha + b_1d_1 = \alpha - b_1^2 \neq 0$ since α cannot be a square. Thus we have the theorem:

THEOREM 6′. *Theorem 6 is true as stated, without recourse to the initial restriction to finite orders, for any system beginning with a Desarguesian plane coordinatized by a quadratic field (i.e. a simple quadratic extension of a subfield).*

It is an open question whether the finiteness assumption is necessary at all. No other proof of the intersection property is known; the proof in [1] is essentially similar to the one given here.

The type of purely algebraic proof used in Theorems 2, 3 and 5 illustrates the possibility of divorcing the transformation from the plane and considering it only as a 'twist' employed on a given ring.

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