A PROBLEM OF LEAST AREA

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McShanes's solution of the Plateau problem made use of a representation theorem to reduce this problem to that of minimizing the Dirichlet integral, a non-parametric problem. The parametric integral considered by Cesari, Sigalov and Morrey [4, 16, 15] can be interpreted as an area integral for an appropriately generalized area. If this area satisfies certain conditions there exists a Dirichlet-type integral and a representation theorem so that the McShane procedure applies. We shall restrict ourselves to such integrands since further information concerning the non-parametric problem is required to handle the general case. Results of [13] ensure the existence of a solution minimizing the Dirichlet integral and, if the integrand is sufficiently smooth, then the solution also has differentiability properties. The representation theorem is used to show that the solution which minimizes the Dirichlet integral also minimizes the parametric integral.

We use Theorem 5.2 and the representation theorem to correct an error in [21].

It seems probable that Fleming's results [9] can be combined with those of this paper to extend to surfaces of other topological types the results derived here for surfaces of the type of the disc.

It is desirable that the theory be broad enough to handle the problem of least area in m, the space of bounded sequences [1], hence an independent proof that the generalized area, given by the integral if the representation is good enough, is lower semi-continuous.

Since the Lebesgue area of a surface is obtained by taking the limit of the areas of a suitable sequence of approximating polyhedra, there is no loss in generality in supposing that all of the Banach spaces considered are separable, except m. If B is such a space, then we can suppose that B is a subspace of m [17].

1. Let M be a metric space, $C(J, M)$ be the space of continuous functions on a Jordan region J into M and $x \in C(J, M)$. Then there exists a monotone-light factorization $x = \lambda_x \mu_x$ such that $\mu_x(J) \subset m$, $L(\mu_x) = L(x)$, where L is Lebesgue area and λ_x is a contraction [17, 18]. If $y \in C(J, M)$ then $\| \mu_x - \mu_y \| \leq 2 \| x - y \|$. If K is a Jordan region contained in J, then $\| \mu_{x|K}(p) - \mu_{x|K}(q) \| \geq \| \mu_x(p) - \mu_x(q) \|$ whenever $p, q \in K$. Furthermore, there exists a nonnegative valued function λ on the subsets of m such that $\lambda(A) \leq \lambda(B)$ whenever $A \subset B$ and $\lambda(\mu_x(J)) = L(x)$. If $M = m$ and x is monotone then $\lambda(x(J)) = L(x)$

[20]. It is not hard to verify that the last equality holds if J is an admissible set [3]. In addition, if A is contained in a cyclic element C of $\mu_x(J)$ then $\lambda(\operatorname{Int}_C A) = \lambda(A)$ where $\operatorname{Int}_C A$ is the interior of A relative to C. Finally, $\lambda(\mu_x(J)) = \Sigma \lambda(C)$ where the summation is taken over the cyclic elements of $\mu_x(J)$.

A Jordan region is the homeomorph of a Jordan region in the plane. The letter i will frequently be a natural number or an index but may also indicate the identity map on m. Thus if $\mu_x(J) = K$ is a Jordan region then μ_x and $i \,|\, K$ are Fréchet equivalent and $L(x) = L(\mu_x) = \lambda(K) = L(i, K)$ where $L(i, K) = L(i \,|\, K)$. If a Jordan region A is cut into two Jordan regions B and C by means of a rectifiable arc, then $\lambda(A) = \lambda(B) + \lambda(C)$.

A subset of $\mu_x(J)$ is open if it is open relative to $\mu_x(J)$. If f is a function on A to B and $p \in B$ then $f^{\vee}(p) = \{q \in A \,|\, f(q) = p\}$.

Let $\mathscr{J}(x)$, $\mathscr{K}(x)$, $\mathscr{R}(x)$, $\mathscr{G}(x)$ and $\mathscr{D}(x)$ be, respectively, the collection of all Jordan regions in Domain x, all Jordan regions in Range μ_x, that subset of $\mathscr{K}(x)$ whose boundaries, relative to Range μ_x, are rectifiable, the open subsets of Range μ_x whose boundaries are boundaries of elements of $\mathscr{R}(x)$, and the inverses, under μ_x, of the elements of $\mathscr{G}(x)$. Thus the elements of $\mathscr{G}(x)$ and $\mathscr{D}(x)$ are open, connected and simply connected.

Now let $Q = \text{Domain } x$ and $\mathscr{Q} = \mu_x(Q)$, and $L(x)$ be finite.

LEMMA 1.1. *If $J \in \mathscr{J}(x)$ then $L(\mu_{x|J}) = L(\mu_x, J)$.*

Proof. By Kolmogoroff's principle, $L(x, J) \leqq L(\mu_x, J) \leqq L(\mu_{x|J})$. We have already noted the equality of $L(x, J)$ and $L(\mu_{x|J})$.

LEMMA 1.2. *If $G \in \mathscr{G}(x)$ and $H = \mu_x^{\vee}(G)$ then $L(x, H) = \lambda(G)$.*

Proof. If $J \in \mathscr{J}(x)$, $J \subset H$, then $L(x, J) = L(\mu_{x|J}) = L(\mu_x, J) = \lambda(\mu_x(J)) \leqq \lambda(G)$. Thus $L(x, H) \leqq \lambda(G)$ since H can be invaded by Jordan regions [3]. Now let $G' = \mathscr{Q} - G - \partial G$ and $H' = \mu_x^{\vee}(G')$. By invading, with multiply connected Jordan regions if necessary, we obtain, as above, that $L(x, H') \leqq \lambda(G')$. Since $x \,|\, \partial H$ is rectifiable (thus $\lambda_x(\partial G)$ is also rectifiable) $L(x) = L(x, H) + L(x, H') \leqq \lambda(G) + \lambda(G') = \lambda(\mathscr{Q}) = L(\mu_x) = L(x)$ and the equality must hold throughout. The lemma follows.

LEMMA 1.3. *Let R_1 and R_2 be Jordan regions with $R_1 \subset \operatorname{Int} R_2$. Suppose that x is light on $A = R_2 - \operatorname{Int} R_1$ into M. Let $\mathscr{F}$ be the set of all continuous f on $[0, 1]$ into A such that $f(0) \in \partial R_1$ and $f(1) \in \partial R_2$. Let $\mathscr{H}$ be the set of all continuous h on $[0, 1]$ into A such that $h(0) = h(1)$ and $h(a) \neq h(b)$ unless $(a, b) = (0, 1)$ or $(1, 0)$,*

*and R_1 is contained in the interior of the Jordan curve determined
by h. Let $\alpha = \inf_{f \in \mathcal{F}}$ length $\mu_x f$ and $\beta = \inf_{h \in \mathcal{H}}$ length $\mu_x h$. Then
α and β are minima and $\beta \leq L(x)/\alpha + \alpha$.*

Proof. Let $f_n \in \mathcal{F}$ with length $\mu_x f_n < \alpha + 1/n$. By Hilbert's
theorem concerning curves uniformly bounded in length there exists
f' continuous on $[0, 1]$ to which a subsequence of $\mu_x f_n$ converge in
the sense of Fréchet. Since x is light, $\mu_x^\vee$ is a homeomorphism. It
is easy to see that $f = \mu_x^\vee f' \in \mathcal{F}$ and length $\mu_x f = \alpha$.

Let $\gamma = f([0, 1])$ and $q \in \operatorname{Int} A - \gamma$. It is obvious that γ is a simple
arc. There exist simple arcs γ_n joining ∂R_1 to ∂R_2 and Jordan regions
J_n with $\partial J_n \subset \partial R_1 \cup \partial R_2 \cup \gamma \cup \gamma_n$ such that $q \in J_n$ and J_n invades A.
By [19], there exists g_n continuous on $[0, 1]$ into J_n such that $g_n(0) \in \gamma$
and $g_n(1) \in \gamma_n$ such that length $\mu_x g_n \leq L(x, J_n)/\alpha \leq L(x)/\alpha$. By appeal-
ing to Hilbert's theorem again we obtain a function g' continuous on
$[0, 1]$ into A such that $g'(0) \in \mu_x(\gamma)$, $g'(1) \in \mu_x(\gamma)$ and R_1 is contained in
the Jordan region R bounded by $g'([0, 1])$ and the piece of γ between
$g'(0)$ and $g'(1)$. The lemma follows.

If $x \in C(J, m)$ then $x^{ik} \in C(J, E_2)$ is defined by $x^{ik}(p) = (x^i(p), x^k(p))$
for each $p \in J$.

LEMMA 1.4. *Let $p \in \mathcal{Q}$ and C be a cyclic element of $\mathcal{Q}$ contain-
ing p. Then for each $\varepsilon > 0$ there exists $R \in \mathcal{R}(x)$ such that diam $R
< \varepsilon$ and $L(i, R) < \varepsilon$.*

Proof. By [19] there exists, for each $\delta > 0$, $R \in \mathcal{R}(x)$ with $p \in \operatorname{Int} R$
and diam $R < \delta$. By [17] there exists a number T such that
$L(y) > L(\mu_x) - \varepsilon/2$ where $y^j = (\mu_x)^j$ for $j \leq T$ and $y^j = 0$ for $j > T$.
Let N_{ik} be the essential multiplicity function, [8], of $(\mu_x)^{ik}$. Since
$\iint N_{ik}$ is finite there exists $\delta_{ik} > 0$ such that $\iint_E N_{ik} < \varepsilon/T^2$ whenever
$|E| < \delta_{ik}$. Let $\delta = \min \delta_{ik}$, $1 \leq i, k \leq T$. Now take diam$^2 R < \delta/4$.
Since diam $(\lambda_x)^{ik}(R) \leq 2$ diam R, $L(i, R) \leq \sum_{1 \leq i < j \leq T} \iint N_{ik} + \varepsilon/2 < \varepsilon$, where
the integral is taken over $(\lambda_x)^{ik}(R)$.

By '$\sigma \prec \mathcal{T}^-(x)$' we mean that σ is a finite family of non-over-
lapping elements of $\mathcal{T}$, where $\mathcal{T}$ is $\mathcal{F}$, $\mathcal{H}$, or $\mathcal{R}$; by '$\sigma \prec \mathcal{S}(x)$'
we mean that σ is a finite family of pairwise disjoint elements of $\mathcal{S}$,
where $\mathcal{S}$ is $\mathcal{G}$ or $\mathcal{D}$.

THEOREM 1.1. *There exists $\sigma_n \prec \mathcal{D}(x)$ such that $\max_{D \in \sigma_n} \operatorname{diam} x(D) \to 0$
and $L(x) = \lim_n \sum_{D \in \sigma_n} L(x, D)$.*

Proof. Let C_i be the cyclic elements of $\mathcal{Q}$. There exists T_n
such that $\sum_{i > T_n} \lambda(C_i) < 1/n$ and $\max_{i > T_n} \operatorname{diam} C_i < 1/n$. Those points

common to C_i and C_j, $i \neq j$, $i, j \leq T_n$, can each be placed in an element of $\mathscr{R}$ the sum of whose areas is less than $1/n$. The part of C_i, $i \leq T_n$, not in any of these regions can be cut up into arbitrarily small regions each of which has a rectifiable boundary, relative to C_i, by means of the intrinsic inequality. The theorem follows from the cyclic additivity of L and the additivity of L relative to a rectifiable cut.

If U is an open subset of Q let $\hat{U} = \mu_x^{\vee}(\text{Int }\mu_x(U))$ [23].

LEMMA 1.5. *If U is an open subset of Q then $L(x, U) = \lambda(\mu_x(U))$.*

Proof. Let J be a Jordan region (possibly multiply connected) contained in U. Then $\mu_x(J) \subset \mu_x(U)$ and $L(x, J) \leq \lambda(\mu_x(U))$. Hence $L(x, U) \leq \lambda(\mu_x(U))$. Let $\varepsilon > 0$. The intrinsic inequality can be used to produce $\sigma \prec \mathscr{G}(x)$, $G' = \bigcup_{G \in \sigma} G \subset \mu_x(\hat{U})$, such that $\lambda(G') > \lambda(\mu_x(\hat{U})) - \varepsilon$. Let $H = \mu_x^{\vee}(G')$. Then $L(x, H) = \lambda(G')$, by Lemma 1.2, and $H \subset U$. Thus $L(x, U) \geq L(x, H) = \lambda(G') > \lambda(\mu_x(\hat{U})) - \varepsilon$.

LEMMA 1.6. *If U is an open subset of Q then $L(x, \hat{U}) = L(x, U)$.*

Proof. $L(x, \hat{U}) = \lambda(\mu_x(\hat{U})) = \lambda(\mu_x(U)) = L(x, U)$.

LEMMA 1.7. *If $U \subset V \subset Q$, U and V open, then*

$$L(x^{ik}, V) - L(x^{ik}, U) \leq L(x, V) - L(x, U) .$$

Proof. Let $\varepsilon > 0$. By Theorem 1.1 there exists $\sigma \prec \mathscr{D}(x)$, $\bigcup_{D \in \sigma} D \subset U$, and $\sum_{D \in \sigma} L(x, D) > L(x, U) - \varepsilon$. Thus

$$L(x^{ik}, V) - L(x^{ik}, U) \leq L(x^{ik}, V) - \sum_{D \in \sigma} L(x^{ik}, D)$$

$$= L(x^{ik}, V - \bigcup_{D \in \sigma} D) \leq L(x, V - \bigcup_{D \in \sigma} D)$$

$$\leq L(x, V) - \sum_{D \in \sigma} L(x, D) < L(x, V) - L(x, U) + \varepsilon .$$

2. If B is a Banach space over the reals, then B_1 is the set of all elements of B having norm one and B^* is the space of continuous linear functionals over B. For the purposes of this paper there is no loss in generality in supposing that B is separable. In this case there exist $f_n \in B_1^*$, $n = 1, 2, \cdots$, such that $\|a\| = \sup_n [a, f_n]$. By identifying $a \in B$ with $\{[a, f_n]\} \in m$ we can suppose that B is a subspace of m [17]. Let m' be the space of bounded functions β on $N \times N$, N is the set of positive integers, such that $\beta(m, n) + \beta(n, m) = 0$. If $a, b \in m$ then we define $a \wedge b \in m'$ by $a \wedge b(m, n) = a^m b^n - a^n b^m$, where $a = \{a^i\}$ and $b = \{b^i\}$. Thus we suppose that the exterior product of B with

itself, $B \wedge B$, is contained in m'. If $\beta \in m'$ then we put $\| \beta \| = \sup |\beta(m, n)|$ and obtain $\| a \wedge b \| \leqq 2 \| a \| \| b \| \leqq \| a \|^2 + \| b \|^2$. If $a \wedge b = 0$ then a and b are linearly dependent, and conversely. We will usually write β^{ik} for $\beta(i, k)$.

If f is continuous on a Jordan region J into the plane then $O(f, \partial J, p)$ is the topological index of p relative to $f(\partial J)$.

LEMMA 2.1. *If U is an open connected subset of Q and if J_n is a sequence of (possibly multiply-connected) Jordan regions invading U then $\lim \iint O(x^{ik}, \partial J_n)$ exists. The limit is independent of the sequence $\{J_n\}$.*

Proof. Let $\varepsilon > 0$ and take K so large that $L(x, J_r) > L(x, U) - \varepsilon/2$ for $r > K$. If $n > m > K$ then

$$\left| \iint O(x^{ik}, \partial J_n) - \iint O(x^{ik}, \partial J_m) \right| - \varepsilon/2 < \left| \iint O(x^{ik}, \partial(J_n - \mathrm{Int}\, J_m)) \right|$$

$$\leqq \iint | O(x^{ik}, \partial(J_n - \mathrm{Int}\, J_m)) | \leqq L(x^{ik}, J_n - \mathrm{Int}\, J_m)$$

$$\leqq L(x, J_n - \mathrm{Int}\, J_m) \leqq L(x, J_n) - L(x, J_m) < \varepsilon \ .$$

Let $\alpha_n^{ik} = \iint O(x^{ik}, \partial J_n)$. Then $\{\alpha_n^{ik}\}$ is a Cauchy sequence for each (i, k). Let $\alpha^{ik} = \lim \alpha_n^{ik}$. Clearly $\alpha_n \in m'$, $\alpha \in m'$, and $\alpha_n \to \alpha$ in m'. The last statement is evident.

If U is an open connected subset of Q then we define $\langle x, U \rangle = \alpha$. We may write $\langle x, J \rangle$ for $\langle x, \mathrm{Int}\, J \rangle$ when J is a Jordan region. Thus we have just shown that $\| \langle x, U \rangle - \langle x, J \rangle \| \leqq L(x, U) - L(x, J)$ if $J \subset U$. If U is not connected we put $\langle x, U \rangle = \Sigma \langle x, W \rangle$ where the sum is taken over the components W of U.

LEMMA 2.2. *If $U \subset V$ are open subsets of Q then*

$$\| \langle x, V \rangle - \langle x, U \rangle \| \leqq L(x, V) - L(x, U) \ .$$

LEMMA 2.3. *If U is an open subset of Q and if $D \subset U$, $D \in \mathscr{D}(x)$, then $\langle x, U \rangle = \langle x, D \rangle + \langle x, U - \mathrm{Clos}\, D \rangle$.*

The proof depends upon the fact that the image of ∂D under x is rectifiable.

The monotone map μ_x induces an orientation of the cyclic elements of $\mathscr{C}$[24]. We assume from now on that $\mathscr{C}$ is a subset of m together with this induced orientation.

LEMMA 2.4. *Let r_C be the monotone retraction of $\mathscr{C}$ onto one of its cyclic elements C. If J is a Jordan region contained in Q then*

$O(x^{ik}, \partial J) = \Sigma O((\lambda_x)^{ik} r_c, \mu_x(\partial J))$, where the sum is taken over the cyclic elements of $\mathscr{Q}$.

Proof. Let $\delta \subset \mu_x(\partial J)$ and suppose that r_c is constant on δ and that $r_c = i$ on $\mu_x(\partial J) - \delta$. Let E be the oriented plane containing $x^{ik}(Q)$, $P \in E$, and L be a half-line in E terminating at P. If $p \in \partial J$ and f is continuous on J into E, let $w(f, p)$ be the angle between the half-line determined by $f(p)$ and P with L. Evidently the change in $w(x^{ik})$ around ∂J is equal to the sum of the changes of $w((\lambda_x)^{ik} r_c)$ on $\mu_x(\partial J)$. Thus $\langle x, J \rangle = \Sigma \langle \lambda_x r_c, \mu_x(J) \rangle$.

LEMMA 2.5. $\langle x, U \rangle = \langle \lambda_x, \mu_x(\hat{U}) \rangle$.

Proof. We can suppose that U is connected. Let $\varepsilon > 0$. There exists a Jordan region $J \subset U$ with $L(x, U) < L(x, J) + \varepsilon$. If $K = \text{Int} J$ then

$$\sum_c [L(\lambda_x r_c, \mu_x(\hat{U})) - L(\lambda_x r_c, \mu_x(\hat{K}))]$$
$$= L(\lambda_x, \mu_x(\hat{U})) - L(\lambda_x, \mu_x(\hat{K})) = L(x, U) - L(x, J) < \varepsilon.$$

Hence

$$\|\langle x, U \rangle - \langle \lambda_x, \mu_x(\hat{U}) \rangle\| \leqq \|\langle x, U \rangle - \langle x, J \rangle\|$$
$$+ \|\langle x, J \rangle - \Sigma \langle \lambda_x r_c, \mu_x(J) \rangle\| + \|\Sigma \langle \lambda_x r_c, \mu_x(\hat{U}) \rangle - \Sigma \langle \lambda_x r_c, \mu_x(J) \rangle\|$$
$$+ \|\Sigma \langle \lambda_x r_c, \mu_s(U) \rangle - \langle \lambda_x, \mu_x(\hat{U}) \rangle\| < \varepsilon + 0 + \varepsilon + 0 = 2\varepsilon.$$

LEMMA 2.6. *Let $\mu_x(U)$ contain a cyclic element C of the type of a sphere. Then $\langle \lambda_x r_c, \mu_x(U) \rangle = 0$.*

Proof. Since U is open and $\mu_x^v(C) \subset U$ there exists a Jordan region J with $\mu_x^v(C) \subset J \subset U$. Hence, since r_c is the identity on C, $\langle \lambda_x r_c, \mu_x(J) \rangle = \langle \lambda_x, r_c \mu_x(J) \rangle = 0$.

LEMMA 2.7. *If x is constant on ∂Q then $\langle x, Q \rangle = 0$, furthermore $\langle x, Q \rangle = \langle \lambda_x r_{c_0}, \mu_x(Q) \rangle$ if C_0 is the cyclic element containing $\mu_x(\partial Q)$. Finally, $\langle x, U \rangle = \Sigma \langle \lambda_x r_c, \mu_x(U) \rangle$ where the sum extends only over those cyclic elements of $\mathscr{Q}$ not contained in $\mu_x(U)$ plus, possibly, C_0.*

LEMMA 2.8. *Let C and C' be two Fréchet curves in E_2, of the type of the circle, each of whose lengths is less than M. Then*

$$\left| \iint_{E_2} [O(C, p) - O(C', p)] dp \right| \leqq 2M \|C, C'\|$$

where $O(C, p)$ is the topological index of p relative to C and $\|C, C'\|$

is the Fréchet distance between C and C'.

Proof. Choose $\varepsilon > 0$. There exist functions f, g, f' and g' on $[0, 1]$, all continuous and of bounded variation such that $\|f - f'\| + \|g - g'\| < \|C, C'\| + \varepsilon$ and

$$\left| \iint [O(C, p) - O(C', p)] dp \right| = \left| \int_0^1 f dg - \int_0^1 f' dg' \right|$$

$$= \left| \int_0^1 (f - f') dg + \int_0^1 (g - g') df' \right| < 2M\{\|C, C'\| + \varepsilon\}.$$

We require an additional property of μ_x. If $p, q \in Q$ then $\|\mu_x(p) - \mu_x(q)\| = \inf \{\text{length } xg \mid g \text{ is continuous on } [0, 1] \text{ into } Q \text{ with } g(0) = p \text{ and } g(1) = q\}$.

THEOREM 2.1. *Suppose that $x_n \to x$ in $C(Q, m)$ and that there exists a number M such that $L(x_n) < M$ for all n. Then for each simply connected Jordan region $J_0 \subset Q$ there exists a finite collection σ_n of non-overlapping simply-connected Jordan regions in J_0 such that*

$$\lim \sum_{J \in \sigma_n} \langle x_n, J \rangle = \langle x, J_0 \rangle.$$

Proof. If the theorem were false there would exist $\varepsilon_0 > 0$ such that $\limsup |\sum_{J \in \sigma_n} \langle x_n, J \rangle - \langle x, J_0 \rangle| > \varepsilon_0$ for each admissible $\{\sigma_n\}$. By extracting a suitable subsequence we can suppose that the limit exists.

Let C_k be the cyclic elements of $\mathscr{Q}$ and r_k be the monotone retraction of $\mathscr{Q}$ onto C_k. There exists a number T such that $\sum_{k > T} \lambda(C_k) < \varepsilon_0/4$. For each $k \leqq T$ let $K_j(k)$ be the Jordan regions in C_k whose boundaries, relative to C_k are subsets of $r_k \mu_x(\partial J_0)$. There exists a number s_k such that $\sum_{j > s_k} \lambda(K_j(k)) < \varepsilon_0/(3T)$. Let $s = \max s_k$ and $\eta = \varepsilon_0/(4Ts)$.

Let us fix $k \leqq T$, $j \leqq s_k$ and write K for $K_j(k)$. There exist Jordan regions $K_i \subset K$, $i = 1, 2, 3$, with $K_1 \subset K_2 - \partial K_2 \subset K_2 \subset K_3 - \partial K_3 \subset K_3 \subset K - \partial K$, where boundaries are taken relative to C_k, such that $\lambda(K_1) > \lambda(K) - \eta$. Let $\psi = \inf \text{diam } A$ where A is a continuum in $K_3 - K_2$ which separates ∂K_3 from ∂K_2. Evidently $\psi > 0$. Let $\xi(j, k) = (1/6) \min \{1, \text{dist} (\partial K_1, \partial K_2), \text{dist} (\partial K_2, \partial K_3), \text{dist} (\partial K_3, \partial K), \psi\}$, $\xi = \min \xi(j, k)$, $N = M/\xi + 1$, and $\zeta = \min \{\xi, \eta/(4N)\}$. We consider only such n for which $\|x_n - x\| < \zeta$.

Let $T_i = (r_k \mu_x)^{\vee}(K_i - \partial K_i)$. Then T_i is an open two-cell and there exist Jordan regions S_2 and S_3 such that $T_2 \subset S_2 \subset S_3 \subset T_3$ and dist $(\mu_x(\partial S_2), \mu_x(\partial S_3)) > 5\xi$. Let $B_n^i = \mu_{x_n}(\partial S_i)$. Then dist $(B_n^2, B_n^3) > 5\xi - 4\zeta \geqq \xi$.

If p is a cutpoint of $\mathscr{Q}_n = \mu_{x_n}(Q)$ on B_n^3 and if $C'(n)$ and $C''(n)$ are two cyclic elements of $\mathscr{Q}_n$ separated by p then only one is not

separated from B_n^2 by p. Hence there is a single cyclic element $C(n)$ of $\mathcal{C}_n$ such that no cut point of $\mathcal{C}_n$ on B_n^3 separates $C(n)$ from B_n^2. Let $r(n)$ be the monotone retraction of $\mathcal{C}_n$ onto $C(n)$.

If $B_n^2 \cap C(n) = 0$ there is a cutpoint p between $C(n)$ and B_n^2. Hence $p \notin B_n^3$ and, consequently, p separates B_n^2 from B_n^3. This implies that there exists a continuum $A \subset K_3 - K_2$ which separates ∂K_3 from ∂K_2 and whose diameter is less than 4ξ, but this is a contradiction. Hence $B_n^2 \cap C(n) \neq 0$ and, therefore, dist $(r(n)B_n^3, r(n)B_n^2) \geq \text{dist}\,(B_n^3, B_n^2) > \xi$. Thus there exist Jordan regions $R_n^2 \subset R_n^3 \subset C(n)$ such that $r(n)B_n^2 \subset R_n^2$, $r(n)B_n^3 \cap R_n^3 = 0$ and dist $(\partial R_n^2, \partial R_n^3) > \xi$. Since $\lambda(R_n^3) < \lambda(C(n)) < M$ there exists a Jordan region $R(n)$ with $R_n^2 \subset R(n) \subset R_n^3$ and length $\partial R(n) < N$.

The curves $\{\partial R(n)\}$ lie in a compact set and are uniformly bounded in length by N; thus there exist representations $\gamma_n \in R(n)$ on $[0, 1]$ such that γ_n converges uniformly to a continuous function γ on $[0, 1]$ onto $\mathcal{C}$. From now on suppose that n is so big that $\|\gamma_n - \gamma\| < \zeta$. Let $t \in [0, 1]$, $p = \gamma(t)$ and $p_n = \gamma_n(t)$. There exists $q_n \in S_3 - S_2$ such that $p_n = \mu_{x_n}(q_n)$. Thus $\|p - \mu_x(q_n)\| \leq \|p - p_n\| + \|\mu_{x_n}(q_n) - \mu_x(q_n)\| \leq \zeta + 2\zeta = 3\zeta$. First suppose that $\mu_x(q_n) \in C_k$. Then $\mu_x(q_n) \in K_3 - K_2$. Whether $p \in C_k$ or not, $\|r_k(p) - \mu_x(q_n)\| \leq \|p - \mu_x(q_n)\| < 3\zeta$. Now suppose that $\mu_x(q_n) \notin C_k$. If $r_k \mu_x(q_n)$ separates p from C_k then $r_k \mu_x(q_n) = r_k(p)$; otherwise $\|r_k(p) - r_k \mu_x(q_n)\| < \|p - \mu_x(q_n)\| < 3\zeta$. Thus if $p = \gamma(t)$ and $q \in q(t) = r_k \mu_x \mu_{x_n}^{\vee} \gamma_n(t)$ then $\|p - q\| < 3\zeta$. Let $A(n) = \cup q(t)$. Then $A(n) \subset K_3 - K_2$, $\gamma([0, 1]) \subset K - K_1$, and $A(n)$ separates ∂K_3 from ∂K_2. If $r_k \gamma$ could be shrunk to a point in $K_3 - K_2$ then γ could be in $\mathcal{C} - K_2$. Now $B_n = \bigcup_t \mu_{x_n} \mu_x^{\vee} \gamma(t)$ would be shrunk to a set of arbitrary small diameter in $C(n) - R_n^2$, for n large enough, and this is impossible since diam $R_n^2 \geq \text{diam}\, K_2 - 2\|x - x_n\|$.

Let G_j be the components of $K - \gamma([0, 1])$ which do not contain ∂K. Then

$$\iint O(\pi^{ik}\gamma, [0, 1]) = \iint O(\pi^{ik}, \gamma([0, 1])) = \Sigma \langle \lambda_x, G_j \rangle^{ik}$$

and

$$\iint O(\pi^{ik}\gamma_n, [0, 1]) = \langle \lambda_{x_n}, R(n) \rangle^{ik} \,,$$

where $\pi^{ik}(a) = (a^i, a^k)$. By Lemma 2.8,

$$\left| \iint O(\pi^{ik}\gamma_n, [0, 1]) - O(\pi^{ik}\gamma, [0, 1]) \right| < 4N\zeta \,.$$

Let the subscripts of the G's be so chosen that $G_1 \supset K_1$. Then

$$\| \langle \lambda_x, K \rangle - \langle \lambda_x, G_1 \rangle \| < \lambda(K) - \lambda(G_1) \leq \lambda(K) - \lambda(K_1) < \eta \,.$$

Hence $||\langle\lambda_{x_n}, R(n)\rangle - \langle\lambda_x, K\rangle|| < 4N\zeta + \eta < 2\eta$.

Next, $\langle\lambda_{x_n}, R(n)\rangle = \langle\lambda_{x_n}, r(n)\check{~}R(n)\rangle = \langle x_n, D_n\rangle$ where $D_n = \mu_{x_n}\check{~}[\text{Int } r(n)\check{~}R(n)]$. There exists a simply connected Jordan region $J_n \subset D_n$ such that $||\langle x_n, D_n\rangle - \langle x_n, J_n\rangle|| < \eta$. Thus $||\langle\lambda_x, K\rangle - \langle x_n, J_n\rangle|| < 3\eta$.

Let $r'(n)$, $R'(n)$, $J'(n)$, K' and K'_3 correspond to $K_{j'}(k')$ as $r(n)$, $R(n)$, $J(n)$, K and K_3 correspond to $K_j(k)$ and suppose either $k \neq k'$ or $j \neq j'$. If $R(n) \cap R'(n) \neq 0$ there exists $p \in R(n) \cap R'(n)$, $q \in K_3$ and $q' \in K'_3$ such that $||p - q|| < 2\zeta_n$ and $||p - q'|| < 2\zeta_n$. This gives $4\zeta \geqq 4\zeta_n > ||q - q'|| > \text{dist}\{q, \partial K\} \geqq \text{dist}\{\partial K_3, \partial K\} > 5\xi > 4\zeta$. Hence $R(n) \cap R'(n) = 0$, $r(n)R(n) \cap r'(n)R'(n) = 0$ and, finally, $J(n) \cap J'(n) = 0$.

Now let σ_n be the collection of all such J_n, one for each $K_j(k)$. By some arithmetic we get

$$\left|\left|\sum_{J \in \sigma_n}\langle x_n, J\rangle - \langle x, J_0\rangle\right|\right| < \varepsilon_0 .$$

COROLLARY. *If* $x_n \to x$, $L(x_n) < M$ *and* $\sigma \prec \mathscr{F}(x)$ *then there exists* $\sigma_n \prec \mathscr{F}(x_n) = \mathscr{F}(x)$ *such that* $\lim \sum_{J \in \sigma_n}\langle x_n, J\rangle = \sum_{J \in \sigma}\langle x, J\rangle$.

3. If $a \in m$ let $\pi_n(a) = {}_na = \{{}_na^i\}$ where ${}_na^i = a^i$ or 0 according as $i \leqq n$ or $i > n$. If $\alpha \in m'$ let ${}_n\alpha^{ik} = \alpha^{ik}$ or 0 according as $i, k \leqq n$ or either i or $k > n$. Let ${}_nm = \pi_n(m)$ and ${}_nm' = {}_nm \wedge {}_nm$. We recall that we can suppose that ${}_nm' \subset m'$.

Let ψ be a nonnegative valued function on ${}_nm'$, for some natural number n, with the following properties:

(i) ψ is continuous,

(ii) ψ is positively homogeneous of degree one,

(iii) ψ is convex, and

(iv) there exist $K \geqq k > 0$ such that $k||\alpha|| \leqq \psi(\alpha) \leqq K||\alpha||$ for all $\alpha \in {}_nm'$. Let F'_n be the collection of such functions ψ.

Let $\mathscr{A}'$ be the set of all ψ defined on m' with the property that $\psi_n = \psi|({}_nm') \in F'_n$ and $\psi_n \leqq \psi_{n+1}$, $\psi_n \to \psi$.

Let $\psi \in \mathscr{A}'$. If there exist $S \subset m^* \times m^*$ such that $\psi(a \wedge b) = \sup\{[a \wedge b, f \wedge g]|(f, g) \in S\}$, where $[a \wedge b, f \wedge g] = f(a)g(b) - f(b)g(a)$, then ψ is *simple*. If $\psi(a \wedge b) = [a \wedge b, f \wedge g]$ for some $(f, g) \in S$ then we write $(f, g) \in S(a \wedge b)$.

We now norm $m \times m'$ and $m \times m'^*$ by $||(a, \alpha)|| = ||a|| + ||\alpha||$ and $||(a, \zeta)|| = ||a|| + ||\zeta||$ where $a \in m$, $\alpha \in m'$, $\zeta \in m'^*$ and $||~~||$ is the appropriate norm.

Suppose that A is a bounded closed subset of ${}_nm$ for some n and ψ is a real-valued continuous function on $A \times {}_nm'$ with $\psi_n \in F'_n$, where $\psi_n(\alpha) = \psi(a, \alpha)$ for each $a \in A$ and $\alpha \in {}_nm'$, with k and K of (iv) independent of a.

LEMMA 3.1. *Let ψ be as in the last paragraph. Then ψ has an extension ψ_0 on ${}_nm \times {}_nm'$ which satisfies all of the conditions imposed on ψ with A replaced by ${}_nm$.*

Proof. Let h be defined on $A \times {}_nm_1'^*$ by

$$\frac{1}{h(a, \zeta)} = \max_{0 \neq \alpha \in {}_nm'} \frac{[\alpha, \zeta]}{\psi(a, \alpha)} \, .$$

It is easy to see that $k \leq h(a, \zeta) \leq K$ and that h is uniformly continuous. The McShane Extension Theorem [15], applied to h, yields an extension k defined on ${}_nm \times {}_nm_1'^*$ which has the same bounds and modulus of continuity as h. Let $\psi_0(a, \alpha) = \max \{k(a, \zeta)[\alpha, \zeta] \mid \zeta \in m_1'^*\}$.

Let F_n be the collection of all such functions ψ_0. Let $\mathscr{A}$ be the set of all ψ defined on $m \times m'$ such that $\psi_a \in \mathscr{A}'$, there exists $K \geq k > 0$ with $k \| \alpha \| \leq \psi(a, \alpha) \leq K \| \alpha \|$ for all $a \in m$, $\alpha \in m'$, $\psi_n = \psi \mid ({}_nm \times {}_nm') \in F_n$ and $\psi_n \leq \psi_{n+1}$, $\psi_n \to \psi$.

Let $\psi \in \mathscr{A}$, α_p and α in m' and suppose that $\alpha_p^{ik} \to \alpha^{ik}$ for all (i, k). Then ${}_n\alpha_p \to {}_n\alpha$ for all n. Hence $\psi_n({}_na, {}_n\alpha) = \lim \psi_n({}_na, {}_n\alpha_p) \leq \liminf \psi(a, \alpha_p)$. Thus $\psi(a, \alpha) \leq \liminf \psi(a, \alpha_p)$.

Suppose that $\psi_n \in F_n$ with $k \| {}_n\alpha \| \leq \psi_n({}_na, {}_n\alpha) \leq K \| {}_n\alpha \|$ for all (a, α). Let $\psi_p({}_pa, {}_p\alpha) = \max (\psi_n({}_na, {}_n\alpha), k \| {}_p\alpha \|)$ for $p > n$. Then $\psi_p \in F_p$ for such p and $\psi = \lim \psi_p$ is in $\mathscr{A}$. Furthermore, ψ is an extension of ψ_n and is simple if ψ_n is.

If $x \in C(Q, m)$ we define, for each $\psi \in \mathscr{A}$,

$$P(\psi, x) = \lim_{\|\sigma\| \to 0} \sum_{J \in \sigma} \min_{p \in J} \psi(x(p), \langle x, J \rangle), \qquad \sigma < \mathscr{J}(x) \, .$$

Suppose that $\psi \in \mathscr{A}$. It is easy to verify that $P(\psi, x) = \lim P(\psi_n, x)$.

In [17] the definition for the Peano area of x was equivalent to

$$P(x) = \sup_{\sigma < \mathscr{J}(x)} \sum_{J \in \sigma} \sup_{i \, k} \int\!\!\int | 0(x^{ik}, \partial J) | \, .$$

If Range x is contained in a Euclidean space (which we can suppose is contained in m) then $P(x)$ is the usual Peano area of x. Let $\psi_0(a, \alpha) = \| \alpha \|$. Evidently $\psi_0 \in \mathscr{A}$.

LEMMA 3.2. $P(\psi_0, x) = P(x)$.

An inequality in one direction is obvious. The difficult inequality follows from a result of Cesari $[U = V, 3]$.

LEMMA 3.3. $P(\psi, x) = \lim\limits_{\|\sigma\| \to 0} \sum_{T \in \sigma} \inf_{\mu_x(p) \in T} \psi(x(p), \langle \lambda_x, T \rangle)$ *where* $\sigma < \mathscr{K}(x)$, $\mathscr{R}(x)$, *or* $\mathscr{G}(x)$, *and*

$$P(\psi, x) = \lim_{||\sigma|| \to 0} \sum_{D \in \alpha} \inf_{p \in D} \psi(x(p), \langle x, D \rangle), \qquad \sigma \prec \mathscr{D}(x) \,.$$

The proof makes use only of the definitions and the results connecting $\langle x, J \rangle$ with $\langle \lambda_x, \mu_x(J) \rangle$.

LEMMA 3.4. *If x and y are Fréchet equivalent then $P(\psi, x) = P(\psi, y)$. Furthermore, $P(\psi, \cdot)$ is cyclically additive.*

We understand that the space of Fréchet surfaces is given the Fréchet metric.

If $\mathscr{S}$ is a Fréchet surface then $P(\psi, \mathscr{S}) = P(\psi, x)$ for any $x \in \mathscr{S}$.

THEOREM 3.1. *$P(\psi, \cdot)$ is lower semi-continuous both on $C(Q, m)$ and on the space of Fréchet surfaces.*

The results of the Corollary to Theorem 2.1 enable us to use the standard arguments.

THEOREM 3.2. *If x_n^i converges uniformly to x^i for each i, then $P(\psi, x) \leqq \liminf P(\psi, x_n)$.*

There is nothing to show if $\psi \in F_n$ for some n. Otherwise we make use of the fact that the limit of an increasing sequence of lower semi-continuous functions is again lower semi-continuous.

4. Our next step is to show that the expected formula holds for $P(\psi, x)$ whenever x is smooth enough. What follows is drawn from [4; §§ 28, 30, 32, 12.10 and 12.12].

Let $x \in C(Q, m)$. If $L(x^{ik}) < \infty$ then the interval function whose value on R is $\langle x, R \rangle^{ik}$ is differentiable and its derivative $\mathscr{J}(x^{ik})$ is the *generalized Jacobian* of $x^{ik} = (x^i, x^k)$. If x^i and x^k have ordinary first derivatives almost everwhere in Int Q then $\mathscr{J}(x^{ik}) = x_u^i x_v^k - x_v^i x_u^k$ almost everwhere. We define the *generalized Jacobian* of x on m' by $[\mathscr{J}(x)]^{ik} = \mathscr{J}(x^{ik})$. Since $P(\psi, x)$ is superadditive, it has a derivative almost everywhere. By Jensen's inequality,

$$\iint_q \psi(x, \mathscr{J}(x)) \geqq \min_{r \in q} \psi(x(r), \iint_q \mathscr{J}(x))$$

for each square q contained in Int Q. It follows that $P'(\psi, x, p) \geqq \psi(x(p), \mathscr{J}(x, p))$ almost everywhere. Thus $P(\psi, x) \geqq \iint \psi(x, \mathscr{J}(x))$. We wish to show that the equality holds for x sufficiently smooth. The proper requirement would be absolute continuity in the sense of Cesari, but we shall content ourselves with showing that the equality holds if x is a D-map [21].

THEOREM 4.1. *If x is a D-map on a Jordan region Q and $\psi \in \mathscr{A}$ then $P(\psi, x) = \iint_{\mathrm{Int}\,Q} \psi(x, \mathscr{J}(x))$.*

Proof. Since x is a D-map, $P(\psi, x \mid \cdot)$ is an absolutely continuous interval function. Hence, by the preceding paragraph, we need only show that $P(\psi, x) \leq \iint \psi(x, \mathscr{J}(x))$. Furthermore, if $\varepsilon > 0$, then there exists a finite collection σ of non-overlapping squares contained in Int Q such that

$$P(\psi, x) - \varepsilon < \sum_{q \in \sigma} \min_{p \in q} \psi\left(x(p), \iint_q \mathscr{J}(x)\right)$$

$$\leq \sum_{q \in \sigma} \iint_q \psi(x, \mathscr{J}(x)) \leq \iint_{\mathrm{Int}\,Q} \psi(x, \mathscr{J}(x))$$

where we have used the fact that x is a D-map to obtain σ and the equality of $\langle x, q \rangle$ and $\iint_q \mathscr{J}(x)$.

If $\mathscr{P}$ is a polyhedron then there exists a quasilinear representation x of $\mathscr{P}$ on the unit square and $P(\psi, \mathscr{P}) = \iint \psi(x, \mathscr{J}(x))$. Since $\iint \|\mathscr{J}(x)\|$ is often referred to as the elementary area of $\mathscr{P}$, we shall regard $\iint \psi(x, \mathscr{J}(x))$ as the elementary area of $\mathscr{P}$ corresponding to the (nongeometric) area ψ. Let us write $\mathscr{E}(\psi, \mathscr{P})$ for this quantity. That the elementary area is lower semi-continuous with respect to Fréchet convergence follows from Theorems 3.1 and 4.1.

Now let $\mathscr{S}$ be a Fréchet surface of the type of the two-cell. We define a Lebesgue area for $\mathscr{S}$ as follows:

$$L(\psi, \mathscr{S}) = \liminf_{\mathscr{P} \to \mathscr{S}} \mathscr{E}(\psi, \mathscr{P})$$

where the convergence is with respect to the Fréchet metric. That $L(\psi, \cdot)$ is lower semi-continuous with respect to Fréchet convergence, and therefore with respect to uniform convergence, follows in the usual manner.

If $k\|\alpha\| \leq \psi(a, \alpha) \leq K\|\alpha\|$ and $x \in C(Q, m)$ then it is easy to see that $kL(x) \leq L(\psi, x) \leq KL(x)$.

5. Let $\mathscr{D}''$ be the collection of all continuous nonnegative valued functions $\bar{\psi}$ on $m \times m$ which are positively homogeneous of degree two, convex, and for which there exists $K \geq k > 0$ such that $k[\|a\|^2 + \|b\|^2] \leq \bar{\psi}(a, b) \leq K[\|a\|^2 + \|b\|^2]$ for all $(a, b) \in m \times m$.

Let $\bar{\psi} \in \mathscr{D}''$, $c \wedge d \neq 0$, and π be the plane determined by c and d.

Lemma 5.1. Let $a \neq 0$, $a \in \pi$, $\pi_a = \{p \in \pi \mid (a \wedge p)/(c \wedge d) > 0\}$ and

$f(p) = \bar{\psi}(a, p)/\|a \wedge p\|$ on π_a. Then the set $E(a)$ of relative minima of f is closed and convex, and f is constant on $E(a)$.

Proof. Suppose that p and q are in $E(a)$ and that $\|a \wedge q\| - \|a \wedge p\| = k > 0$. Let $\varphi(w) = \bar{\psi}(a, wp + (1 - w)q)$ and $A(w) = \|a \wedge q\| - kw$. Since φ is convex and $\varphi(w)/A(w)$ is $(-1/k)$ times the slope of the line joining $(\|a \wedge q\|/k, 0)$ to $(w, \varphi(w))$, it follows that $(0, \varphi(0))$, $(1, \varphi(1))$, and $(\|a \wedge q\|/k, 0)$ are collinear and φ/A is constant on $[0, 1]$. If $k = 0$ the convex function φ has horizontal tangents at $(0, \varphi(0))$ and at $(1, \varphi(1))$. Hence φ, and therefore φ/A, are constant on $[0, 1]$.

Now define $g(a)$ to be the value assumed by f on $E(a)$. If k and K are related to $\bar{\psi}$ in the usual way then $k \leqq g(a) \leqq 2K$.

Lemma 5.2. g *is continuous* (*on* π).

Proof. Let $0 \neq a_n \to a \neq 0$ and $g(a_n) = \bar{\psi}(a_n, p_n)/\|a_n \wedge p_n\|$ where a_n, a and p_n are all in π. We can suppose that $p_n \to p_0$. Since $g(a_n) \leqq 2K$ for all n, $\liminf \|a_n \wedge p_n\| > 0$. Hence $g(a) \leqq \liminf g(a_n)$. On the other hand, suppose that $g(a) = \bar{\psi}(a, p)/\|a \wedge p\|$. Now $g(a_n) \leqq \bar{\psi}(a_n, p)/\|a_n \wedge p\|$, for sufficiently large n, and so $\limsup g(a_n) \leqq g(a)$.

Since g is positively homogeneous of degree zero we can define $K(c \wedge d) = \max_{0 \neq a \in \pi} g(a)$.

Now we use $\bar{\psi}$ to generate an area $A\bar{\psi}$ by the formulas

$$A\bar{\psi}(a \wedge b) = K(a \wedge b)\|a \wedge b\|$$

and

$$A\bar{\psi}(\alpha) = \inf_{\Sigma a_n \wedge b_n = \alpha} \sum A\bar{\psi}(a_n \wedge b_n) .$$

LEMMA 5.3. *If* $A\bar{\psi}(a \wedge b) = \bar{\psi}(a, b)$ *then* $\|a\|/L \leqq \|b\| \leqq \|a\|L$ *where* $L = (K + \sqrt{K^2 - k^2})/k$.

Proof. Let $\|b\| = t^2\|a\|$ where $t^2 > L$. Then $\bar{\psi}(ta, b/t) \leqq K(t^2\|a\|^2 + t^{-2}\|b\|^2) = 2Kt^2\|a\|^2 < k(1 + t^4)\|a\|^2 \leqq \bar{\psi}(a, b)$ which is a contradiction.

If $a \wedge b \neq 0$ then by an application of the Hahn-Banach Theorem, [17], there exist $f, g \in m_1^*$ such that $[a \wedge b, f \wedge g] = \|a \wedge b\|$ and $\max(|f(r)|, |g(r)|) \leqq \|r\| \leqq |f(r)| + |g(r)|$ whenever r is linearly dependent upon a and b. Hence $(f^2(r) + g^2(r))/2 \leqq \|r\|^2 \leqq 2(f^2(r) + g^2(r))$ for all such r and $(\|a\|^2 + \|b\|^2)/4 \leqq \|r\|^2 + \|s\|^2 \leqq 4(\|a\|^2 + \|b\|^2)$ if $r = a\cos\theta - b\sin\theta$ and $s = a\sin\theta + b\cos\theta$.

Let $\mathscr{D}' = \{\bar{\psi} \in \mathscr{D}'' \mid \bar{\psi}(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta) = \bar{\psi}(a, b)$ for all θ, $\bar{\psi} = \lim \bar{\psi}_n(a, b)$ and $\bar{\psi}_n \leqq \bar{\psi}_{n+1}$ where $\bar{\psi}_n(a, b) = \bar{\psi}(_n a, _n b)\}$.

Let $\mathscr{D}$ be the collection of all continuous functions on $m \times m \times m$ for which there exist constants $K \geqq k > 0$ such that $k(\| b \|^2 + \| c \|^2) \leqq \bar{\psi}(a, b, c) \leqq K(\| b \|^2 + \| c \|^2)$ for all $a, b, c \in m$ and such that $\bar{\psi}_a \in \mathscr{D}'$, where $\bar{\psi}_a(b, c) = \bar{\psi}(a, b, c)$. If $\bar{\psi} \in \mathscr{D}$ then $\bar{\psi}_n(a, b, c) = (\bar{\psi}_a)_n(b, c)$. We may write $\bar{\psi}_n$ for $\bar{\psi} \,|\, (_nm \times _nm \times _nm)$. Let $\mathscr{D}_n = \{\bar{\psi} \in \mathscr{D} \,|\, \bar{\psi} = \bar{\psi}_n\}$.

LEMMA 5.4. *If $\bar{\psi} \in \mathscr{D}'$ then $A\bar{\psi}(a \wedge b) = \min \{\bar{\psi}(c, d) \,|\, c \wedge d = a \wedge b\}$.*

The proof is straight-forward.

If $\bar{\psi} \in \mathscr{D}''$ then let $N = 2(\bar{\psi})^{1/2}$.

LEMMA 5.5. *Let $\bar{\psi} \in \mathscr{D}'$ and $(a, b) \in m \times m$. If there exist $(f, g) \in m^* \times m^*$ such that (f, g) is a supporting linear functional to (the convex function) N at (a, b) and if $f(a) = g(b) = N(a, b)/2$, $f(b) = g(a) = 0$, then $A\bar{\psi}(\lambda a + \mu b, \rho a + \sigma b) = [(\lambda a + \mu b) \wedge (\rho a + \sigma b), f \wedge g]$ whenever $\lambda\sigma - \mu\rho \geqq 0$.*

Proof. $N(a, \rho a + \sigma b) \geqq (1 + \sigma)N(a, b)/2$ implies that $\bar{\psi}(a, \rho a + \sigma b) \geqq (1 + \sigma)^2 \bar{\psi}(a, b)/4 \geqq \sigma\bar{\psi}(a, b)$. It follows from Lemma 5.4 that $A\bar{\psi}(a \wedge b) = \bar{\psi}(a, b)$.

LEMMA 5.6. *Let $\bar{\psi} \in \mathscr{D}'$ and suppose that $A\bar{\psi}(a \wedge b) = \bar{\psi}(a, b)$. Let $(f', g') \in \pi^* \times \sigma^*$ satisfy $f'(a) = g'(b) = N(a, b)/2$, $f'(b) = g'(a) = 0$, where π is the plane determined by a and b. Then (f', g') is a supporting linear functional to $N \,|\, (\pi \times \pi)$ at (a, b).*

Proof. We write N for $N \,|\, (\pi \times \pi)$. Let $\lambda, \mu, \rho, \sigma$ be real numbers, $a' = \lambda a + \mu b$ and $b' = \rho a + \sigma b$. By hypothesis, $\lambda\sigma - \mu\rho > 1$ implies $N(a', b') > N(a, b)$. We must show that $N(a', b') \geqq (\lambda + \sigma)N(a, b)/2$ for all $\lambda, \mu, \rho, \sigma$. However, since N is convex, we can suppose that $|\lambda - 1|, |\mu|, |\rho|$ and $|\sigma - 1|$ are all less than $1/2$, which implies that $\lambda\sigma - \mu\rho > 0$. Let $\lambda_n = (n + \lambda)/(n + 1)$, $\mu_n = \mu/(n + 1)$, $\rho_n = \rho/(n + 1)$, $\sigma_n = (n + \sigma)/(n + 1)$, $a_n = \lambda_n a + \mu_n b$, $b_n = \rho_n a + \sigma_n b$ and $t < (\lambda + \sigma)/2$. Then $(a_n, b_n) = (n(a, b) + (a', b'))/(n + 1)$ and, for sufficiently large n, $\lambda_n\sigma_n - \mu_n\rho_n = (n^2 + n(\lambda + \sigma) + \lambda\sigma - \mu\rho)/(n + 1)^2 > (n + t)^2/(n + 1)^2$. For such n, $N(a_n, b_n) > (n + t)N(a, b)/(n + 1)$. Thus $nN(a, b) + N(a', b') > (n + t)N(a, b)$ and $N(a', b') > tN(a, b)$. Hence $N(a', b') \geqq (\lambda + \sigma)N(a, b)/2$.

LEMMA 5.7. *Let $\bar{\psi} \in \mathscr{D}'$ and suppose that $A\bar{\psi}(a \wedge b) = \bar{\psi}(a, b)$. Then there exist $(f, g) \in m^* \times m^*$ such that (f, g) is a supporting linear functional to N at (a, b) and $f(a) = g(b) = N(a, b)/2$, $f(b) = g(a) = 0$.*

This lemma follows from the preceding and the Hahn-Banach Theorem.

If $\bar{\psi} \in \mathscr{D}'$ let $S(\bar{\psi}) = \{(f, g) \mid (f, g)$ supports N for some (a, b) and $f(a) = g(b) = N(a, b)/2$, $f(b) = g(a) = 0\}$.

THEOREM 5.1. *If* $\bar{\psi} \in \mathscr{D}'$ *then*

$$A\bar{\psi}(a \wedge b) = \sup \{[a \wedge b, f \wedge g] \mid (f, g) \in S(\bar{\psi})\} .$$

Proof. Suppose that $A\bar{\psi}(a \wedge b) = \bar{\psi}(a, b)$. Then $A\bar{\psi}(a \wedge b) = N^2(a, b)/4 = [a \wedge b, f \wedge g]$ for some $(f, g) \in S(\bar{\psi})$, by Lemma 5.7. Now let $(c, d) \in m \times m$. If we choose θ so that $f(c) \sin \theta + f(d) \cos \theta = 0$ and let $c' = c \cos \theta - d \sin \theta$, $d' = c \sin \theta + d \cos \theta$, then we obtain
$$[c \wedge d, f \wedge g] = [c' \wedge d', f \wedge g] = f(c')g(d') \leqq (f(c') + g(d'))^2/4$$
$$\leqq N^2(c', d')/4 = \bar{\psi}(c', d') = \bar{\psi}(c, d) = A\bar{\psi}(c \wedge d) .$$

LEMMA 5.8. *Let* $\bar{\psi}_0 \in \mathscr{D}''$ *and* $\bar{\psi}(a, b) = \max_\theta \bar{\psi}_0(a \cos \theta - b \sin \theta, a \sin \theta + b \cos \theta)$. *Then* $A\bar{\psi}_0 = A\bar{\psi}$.

Proof. Let $a' \wedge b' \neq 0$. There exist a_1 and b_1 with $a_1 \wedge b_1 = a' \wedge b'$ such that $A\bar{\psi}(a_1 \wedge b_1) = \bar{\psi}(a_1, b_1)$ and there exists θ such that $\bar{\psi}(a_1, b_1) = \bar{\psi}_0(a, b)$ where $a = a_1 \cos \theta - b_1 \sin \theta$ and $b = a_1 \sin \theta + b_1 \cos \theta$. Let π be the plane determined by a' and b' and $N_0 = 2(\bar{\psi}_0)^{1/2}$. If both $N_0 \mid (\pi \times \pi)$ and $N \mid (\pi \times \pi)$ have unique supporting linear functionals at (a, b) they must coincide since $N \geqq N_0$ and $N(a, b) = N_0(a, b)$. Thus there exist f' and g' on π such that (f', g') is the supporting linear functional and $f'(a) = g'(b) = N(a, b)/2$, $f'(b) = g'(a) = 0$. Thus $N_0(a, \rho a + \sigma b) \geqq (f'(a) + \sigma g'(b))^2 = (1 + \sigma)^2 N^2(a, b)/4$ whenever $\sigma > 0$. Hence $A\bar{\psi}_0(a \wedge b) = \bar{\psi}_0(a, b) = \bar{\psi}(a, b) = A\bar{\psi}(a \wedge b)$. If either $N_0 \mid (\pi \times \pi)$ or $N \mid (\pi \times \pi)$ does not have a unique supporting linear functional at (a, b), let $\varepsilon > 0$ and choose N_0' on $\pi \times \pi$ to be strictly convex, of class C'', positively homogeneous of degree one and such that $N_0 \leqq N_0' < (1 + \varepsilon)^2 N_0$. Let $\bar{\psi}_0' = N_0'^2/4$ and $\bar{\psi}'$ be defined for $\bar{\psi}_0'$ as $\bar{\psi}$ was for $\bar{\psi}_0$. Then $A\bar{\psi} \leqq A\bar{\psi}' = A\bar{\psi}_0' < (1 + \varepsilon)A\bar{\psi}_0$.

THEOREM 5.2. *Let* M *be a norm on* m *and* $\bar{\psi}(a, b) = (M^2(a) + M^2(b))/2$. *If* $M^*(f) = \sup \{f(a) \mid M(a) = 1\}$ *for all* $f \in m^*$ *then* $A\bar{\psi} = \psi$ *where* $\psi(a \wedge b) = \sup \{[a \wedge b, f \wedge g] \mid M^*(f) = M^*(g) = 1\}$.

Proof. Suppose that $a \wedge b \neq 0$. Then, by the argument of [17] where $M = \| \ \|$, there exist $c, d \in m$ and $f, g \in m^*$ such that $M^*(f) = M^*(g) = 1$, $c \wedge d = a \wedge b$, $f(c) = M(c) = M(d) = g(d)$, $f(d) = g(c) = 0$, and $\psi(a \wedge b) = [a \wedge b, f \wedge g]$. It follows that $(f, g) \in S(\bar{\psi})$ and so $A\bar{\psi} \geqq \psi$. On the other hand, if $(f, g) \in S(\bar{\psi})$ then there exists a and

b with $a \wedge b \neq 0$ such that (f, g) supports N at (a, b) and $f(a) = g(b) = N(a, b)/2$, $f(b) = g(a) = 0$. Since $\bar{\psi}(ta, b/t)$ has a relative minimum at one, we see that $M(a) = M(b)$. Now let $c \in m$, and choose $t > 0$ so that $M(tb) = M(c)$. We have $f(c) + M(c) = f(c) + g(tb) \leq 2((M^2(c) + M^2(tb)/2)^{1/2} = 2M(c)$. Thus $M^*(f) \leq 1$ and, similarly, $M^*(g) \leq 1$ and so $\psi \geq A\bar{\psi}$.

6. If x is a D-map on J then $\{x_u^i\}$ is defined almost everywhere in Int J and is an element of m. We define $x_u = \{x_u^i\}$ and $x_v = \{x_v^i\}$ [21]. Let $D(x) = \iint(\|x_u\|^2 + \|x_v\|^2)$, $I(\psi, x) = \iint \psi(x, x_u \wedge x_v)$, and $I(\bar{\psi}, x) = \iint \bar{\psi}(x, x_u, x_v)$ where all of the integrals are taken over Int J.

Suppose that $\{x_n\}$ is a sequence of continuous functions on the unit circle $\mathscr{C}$. Then $\{x_n\}$ satisfies the *three point condition* if there is a $\delta > 0$ and $w_{in} \in \partial \mathscr{C}$, $i = 1, 2, 3$, such that $\|w_{in} - w_{jn}\| > \delta$, $\|x_n(w_{in}) - x_n(w_{jn})\| > \delta$ whenever $i \neq j$, $i, j = 1, 2, 3$.

If $\bar{\psi} \in \mathscr{D}_n$, if x_k and x are D-maps on $\mathscr{C}$ and if x_k^i converges uniformly to x^i for each i, then $I(\bar{\psi}, x) \leq \liminf I(\bar{\psi}, x_k)$ [13]. Hence $I(\bar{\psi}, x) \leq \liminf I(\bar{\psi}, x_k)$ for all $\bar{\psi} \in \mathscr{D}$.

If x is continuous on a Jordan region J into a Banach space B and if $\mathscr{S}$ is the Fréchet surface determined by x then $\partial \mathscr{S}$ is the Fréchet curve represented by $x \mid \partial J$.

The proof of the following lemma is modeled after a proof in [2].

LEMMA 6.1. *Let $\bar{H} \in \mathscr{D}$ where $B = R_n$ and suppose that $\bar{H}$ is of class C'' and strictly convex in its last two arguments. Let $\mathscr{P}$ be an open non-degenerate polyhedron in B. Let p_i and q_i, $i = 1, 2, 3$, be distinct points of ∂C and $\partial \mathscr{P}$, respectively. Then there exists a D-map x^* on $\mathscr{C}$ which represents $\mathscr{P}$ such that $x^*(p_i) = q_i$ and $I(\bar{H}, x^*) = I(A\bar{H}, x^*)$.*

Proof. We mention, first, two properties of $\bar{H}$. If y is a D-map and T is a conformal transformation, then $I(\bar{H}, y) = I(\bar{H}, Ty)$. Also, $\sum_i y_v^i \bar{H}_{,i} (y, y_u, y_v) = \sum_i y_u^i \bar{H}_{,i} i(y, y_u, y_v)$ where $\bar{H}_{,i} = (\partial/\partial y_u^i)\bar{H}$ and $\bar{H}_{,i} = (\partial/\partial y_v^i)\bar{H}$.

Let K be the nonempty class of all representations of $\mathscr{P}$ on $\mathscr{C}$ which are D-maps. Let $\bar{I} = \inf_{x \in K} I(\bar{H}, x)$ and let $\{x_n\}$ be a minimizing sequence in K. Since $I(\bar{H}, x_n) = I(\bar{H}, Tx_n)$ for T as above, we can suppose that $x_n(p_i) = q_i$. By Theorems 5 and 6 [21], the sequence $\{x_n\}$ is equicontinuous. By deleting some terms, if necessary, we can suppose that x_n converges uniformly to x^* where $x^* \in \mathscr{P}$ and x^* is a D-map [21; Th. 3]. Thus $x^* \in K$ and $I(\bar{H}, x^*) = \bar{I}$.

Now let φ and ψ be Lipschitzian with constant M on $\mathscr{C}$. The transformation $U(u, v) = (\alpha, \beta)$, $\alpha = u + \varepsilon\varphi(u, v)$, $\beta = v + \varepsilon\psi(u, v)$, together with its inverse, is Lipschitzian if $|\varepsilon| < 1/(3M)$. Set $x(\alpha, \beta) = x^*(u, v)$. Then x is a D-map. Let T be the conformal transformation of domain x onto $\mathscr{C}$ which takes $U(p_i)$ onto p_i and put $X = xT\check{}$. Then $X \in K$. Put

$$J(\varepsilon) = I(\bar{H}, X) = \iint_{\mathscr{C}} D^{-1}\bar{H}(x^*, x_u^*\beta_v - x_v^*\alpha_u, -x_u^*\alpha_v + x_v^*\alpha_u)dudv$$

where $D = \partial(\alpha, \beta)/\partial(u, v)$. Then

$$0 = 2J'(0) =$$

$$\iint_{\mathscr{C}} \{((G^* - E^*)\varphi_u - 2F^*\varphi_v) + ((E^* - G^*)\psi_v - 2F^*\psi_u)\}dudv$$

where

$$E^* = \sum_i x_u^{i*}\bar{H}_{i,}(x^*, x_u^*, x_v^*) \,,$$

$$F^* = \sum_i x_u^{i*}\bar{H}_{,i}(x^*, x_u^*, x_v^*) = \sum_i x_v^{i*}\bar{H}_{i,}(x^*, x_u^*, x_v^*)$$

and

$$G^* = \sum_i x_v^{i*}\bar{H}_{,i}(x^*, x_u^*, x_v^*) \,.$$

Since φ and ψ are arbitrary, we obtain

$$(1) \qquad \iint_{\mathscr{C}} (-A\varphi_u + B\varphi_v) = 0 \quad \text{and} \quad \iint_{\mathscr{C}} (A\psi_v + B\psi_u) = 0$$

where $A = E^* - G^*$ and $B = -2F^*$. By (1) and Haar's lemma [13]

$$(2) \qquad \int_{\partial R} (Adv - Bdu) = 0$$

for almost all rectangles $R \subset \mathscr{C}$. For each $h > 0$ let

$$\mathscr{C}_h = \{(u, v) \in \mathscr{C} \,|\, [u - h, u + h] \times [v - h, v + h] \subset \mathscr{C}\}$$

and let A_h and B_h be the h-average functions of A and B defined on $\mathscr{C}_h$. These functions are continuous and satisfy (2) for every rectangle $R \subset \mathscr{C}_h$. Let $\zeta_h(u, v) = \int_{(0, 0)}^{(u, v)} (A_h dv - B_h du)$, the integral being independent of the (rectifiable) path joining $(0, 0)$ to (u, v). Now $\zeta_{hu} = -B_h$ and $\zeta_{hv} = A_h$. Using the other part of (1) in a similar fashion we obtain $\eta_{hu} = -A_h$ and $\eta_{hv} = -B_h$. Thus ζ_h and η_h are harmonic on $\mathscr{C}_h$. By [13; Th. 4.2, p. 74], ζ_h and η_h both vanish on $\partial\mathscr{C}_h$. Thus

$E^* = G^*$ and $F^* = 0$ almost everywhere in $\mathscr{C}$, and by Lemma 5.5, $A\bar{H}(x^*, x_u^* \wedge x_v^*) = \bar{H}(x^*, x_u^*, x_v^*)$ wherever these equalities hold.

If $\bar{\psi} \in \mathscr{D}$ and x is a D-map on an open set G then x is $A\bar{\psi}$-quasi-conformal on G if $\bar{\psi}(x, x_u, x_v) = A\bar{\psi}(x, x_u \wedge x_v)$ almost everywhere on G. If J is a Jordan region then we say that x is $A\bar{\psi}$-quasi-conformal on J rather than on $\text{Int } J$.

LEMMA 6.2. *Let $\mathscr{P}$ be an open non-degenerate polyhedron and suppose that $\bar{\psi} \in \mathscr{D}$. If p_i and q_i, $i = 1, 2, 3$, are distinct points of $\partial\mathscr{C}$ and $\partial\mathscr{P}$, respectively, then $\mathscr{P}$ has an $A\bar{\psi}$-quasi-conformal representation x on $\mathscr{C}$ such that $x(p_i) = q_i$.*

Proof. There exist $\bar{H}_n$ as in the last lemma with $\bar{H}_n \leq \bar{H}_{n+1}$ and $\bar{\psi} = \lim \bar{H}_n$. By Dini's Theorem, we can suppose that $\bar{\psi} < (1 + 1/n)\bar{H}_n$ on $|\mathscr{P}| \times B_1 \times B_1$, where $|\mathscr{P}|$ is the compact set covered by $\mathscr{P}$. Let $x_n \in \mathscr{P}$ be $A\bar{H}_n$-quasi-conformal with $x_n(p_i) = q_i$. Then, as before, there exists a D-map $x \in \mathscr{P}$ and we can suppose that x_n converges uniformly to x. Hence $I(\bar{\psi}, x) \leq \liminf I(\bar{\psi}, x_n) \leq \liminf (1 + 1/n)I(\bar{H}, x_n) = \liminf (1 + 1/n)I(AH_n, x_n) = \liminf (1 + 1/n)I(AH_n, x) \leq I(A\bar{\psi}, x)$.

THEOREM 6.1. *If $\mathscr{S}$ is an open non-degenerate surface of finite Lebesgue area, if $\bar{\psi} \in \mathscr{D}$, and if p_i and q_i, $i = 1, 2, 3$, are distinct points of $\partial\mathscr{C}$ and $\partial\mathscr{S}$, then $\mathscr{S}$ has an $A\bar{\psi}$-quasi-conformal representation x on $\mathscr{C}$ such that $x(p_i) = q_i$.*

Proof. There exist open non-degenerate polyhedra $\mathscr{P}_n \to \mathscr{S}$ such that $\mathscr{E}(A\bar{\psi}, \mathscr{P}_n) \to L(A\bar{\psi}, \mathscr{S})$. Let $q_{i,n}$, $i = 1, 2, 3$, be distinct points of $\partial\mathscr{P}_n$ such that $q_{i,n} \to q_i$. There exist $A\bar{\psi}$-quasi-conformal representations x_n of $\mathscr{P}_n$ such that $x_n(p_i) = q_{i,n}$. Hence, as before, there exists a D-map $x \in \mathscr{S}$ and we can suppose that x_n converges uniformly to x. Thus

$$I(\bar{\psi}, x) \leq \liminf I(\bar{\psi}, x_n) = \liminf I(A\bar{\psi}, x_n) = \liminf \mathscr{E}(A\bar{\psi}, \mathscr{P}_n)$$
$$= L(A\bar{\psi}, \mathscr{S}) = I(A\bar{\psi}, x) .$$

7. A subset $\mathscr{K}$ of m is *c-closed* if $a \in \mathscr{K}$ whenever there exists $\{a_n\}$ in $\mathscr{K}$ such that $a_n^i \to a^i$ for each i. If $\mathscr{K} \cap S$ is c-closed for each sphere S then $\mathscr{K}$ is *locally c-closed*. Evidently $\mathscr{K}$ is locally c-closed if $\mathscr{K}$ is locally compact.

If $\mathscr{S}$ is a Fréchet surface of the type of a two-cell, let $\partial\mathscr{S}$ be the Fréchet curve defined by $x | \partial\mathscr{C}$ where x is a representation of $\mathscr{S}$ on $\mathscr{C}$. Let $d(\gamma_1, \gamma_2)$ be the Fréchet distance between the Fréchet curves γ_1 and γ_2, each of the type of the circle.

If $\mathscr{K}$ is a convex locally c-closed subset of m, if γ is a simple

closed curve in $\mathscr{K}$ and if $\psi \in \mathscr{A}$ then

$$m(\psi, \mathscr{K}, \gamma) = \liminf_{d(\partial\mathscr{P}, \gamma) \to 0} \mathscr{E}(\psi, \mathscr{P})$$

and

$$a(\psi, \mathscr{K}, \gamma) = \inf_{\partial\mathscr{S} = \gamma} L(\psi, \mathscr{S})$$

where $\mathscr{P}$ is a polyhedron and $\mathscr{S}$ is a surface, each of the type of the disc, and each is contained in $\mathscr{K}$. By Lemma 3.4 we can suppose that each $\mathscr{P}$ in the definition of $m(\psi, \mathscr{K}, \gamma)$ is open non-degenerate.

Let $T(\mathscr{K})$ be the set of all D-maps on $\mathscr{C}$ with range in $\mathscr{K}$ and $T(\mathscr{K}, \gamma) = \{x \in T(\mathscr{K}) \mid x \mid \partial\mathscr{C} \in \gamma\}$.

LEMMA 7.1. *Let $\mathscr{S}$ be contained in $\mathscr{K}$. Then there exists a sequence of polyhedra $\{\mathscr{P}_n\}$ contained in $\mathscr{K}$ with $\mathscr{P}_n \to \mathscr{S}$ and $\mathscr{E}(\psi, \mathscr{P}_n) \to L(\psi, \mathscr{S})$.*

Proof. Let us suppose that $\{a \in m \mid \|a\| < \delta\} \subset \mathscr{K}$ for some $\delta > 0$. If $\mathscr{T}$ is a surface with representation x and if ρ is a positive number, let $\rho\mathscr{T}$ be the surface determined by ρx. Now suppose that $\{\mathscr{Q}_n\}$ is a sequence of polyhedra with $\mathscr{Q}_n \to \mathscr{S}$ and $\mathscr{E}(\psi, \mathscr{Q}_n) \to L(\psi, \mathscr{S})$. Then $\{\mathscr{P}_n\}$ can be chosen by $\mathscr{P}_n = (\delta\mathscr{Q}_n)/(\delta + 2d(\mathscr{Q}_n, \mathscr{S}))$. If $\mathscr{K}$ has an interior point other than 0, a translation reduces the problem to the preceding. If $\mathscr{K}$ has no interior point, let B be the space spanned by $\mathscr{K}$ and let $L_B(\psi, \mathscr{S})$ be the area defined by restricting sequences of polyhedra approximating $\mathscr{S}$ to be in B. Since $L_B(\psi, \mathscr{S}) = L(\psi, \mathscr{S})$ as in [17] and the argument applies to L_B, the lemma is proved.

Now suppose $\mathscr{S}$ is contained in $\mathscr{K}$ and $\partial\mathscr{S} = \gamma$. There exists a sequence $\mathscr{P}_n$ of polyhedra contained in $\mathscr{K}$ such that $\mathscr{E}(\psi, \mathscr{P}_n) \to L(\psi, \mathscr{S})$. Hence, [12], $m(\psi, \mathscr{K}, \gamma) \leqq \liminf \mathscr{E}(\psi, \mathscr{P}_n) = L(\psi, \mathscr{S})$ and, consequently, $m(\psi, \mathscr{K}, \gamma) \leqq a(\psi, \mathscr{K}, \gamma)$.

If G is an open connected set in the plane, if $x = \{x^i\}$ where each x^i is of class $\mathscr{L}_2$ on G and if $D(x)$ is finite, then x is of class $\mathscr{P}_2$ on G [13]. If G is of class K, in particular, if G is the interior of a circle, or the intersection of the interiors of two circles, then there exists a function φ on ∂G which plays the role of the boundary value function for x. We shall write $x \mid \partial G$ for φ and $x(p)$ for $\varphi(p)$ for $p \in \partial G$.

Let x be of class $\mathscr{P}_2$ on Int $\mathscr{C}$. Then z is a *simple cone-function* for x on Int $\mathscr{C}$ if

$$z(p) = \begin{cases} \|p\| x(p/\|p\|) + (1 - \|p\|)q\,, & p \neq (0, 0) \\ q\,, & p = (0, 0) \end{cases}$$

whenever q is contained in the convex hull of $x(\partial\mathscr{C})$. Now let x be

of class $\mathscr{P}_2$ on a region J of class K. Then z is a *cone-function* for x on J if there exists a conformal transformation T from J onto $\mathscr{C}$ such that $z = wT$ and w is a simple cone-function for $xT^\vee$. By some remarks in § 5,

$$D(w)/4 \leqq D(z) \leqq 4D(w) .$$

We require some slight modification of one of Morrey's results. Let $C(P, r)$ be the open circle with center P and radius r.

LEMMA 7.2. *Let x be class $\mathscr{P}_2$ on $C(0, R)$ with $D(x) = M < \infty$. Suppose there is a number $k > 0$ such that*

$$D(x \mid C(0, r)) \leqq kD(H(x, r)) , \qquad 0 < r \leqq R ,$$

whenever $H(x, r)$ is a simple cone-function for x over $C(0, r)$. Then

$$D(x \mid C(0, r)) \leqq M(r/R)^{1/(6k)} , \qquad 0 \leqq r \leqq R .$$

We use polar coordinates and let

$$\psi(r) = D(x \mid C(0, r)) = \int_0^r \rho \left\{ \int_0^{2\pi} \| x_\rho(\rho, \theta) \|^2 + \rho^{-2} \| x_\theta(\rho, \theta) \|^2 \, d\theta \right\} d\rho .$$

Since ψ is absolutely continuous on $[0, R]$,

$$r\psi'(r) = \int_0^{2\pi} [r^2 \| x_\rho(r, \theta) \|^2 + \| x_\theta(r, \theta) \|^2] d\theta$$

for almost all r in $[0, R]$.

Now we compute

$$D(H(x, r)) = r^{-2} \int_0^r \int_0^{2\pi} \rho [\| x(r, \theta) \|^2 + \| x_\theta(r, \theta) \|^2] d\rho d\theta$$

$$\leqq r^{-2} \int_0^r \int_0^{2\pi} \rho \left[2^{-1} \int_0^{2\pi} \| x_\theta(r, \varphi) \| d\varphi \right]^2 d\rho d\theta + r^{-2} \int_0^r \int_0^{2\pi} \rho \| x_\theta(r, \theta) \|^2 \, d\rho d\theta$$

$$\leqq \frac{\pi r^{-2}}{2} \int_0^r \int_0^{2\pi} \rho \int_0^{2\pi} \| x_\theta(r, \varphi) \|^2 \, d\varphi d\rho d\theta + 2^{-1} \int_0^{2\pi} \| x_\theta(r, \theta) \|^2 \, d\theta$$

$$\leqq \frac{\pi^2 + 1}{2} \int_0^{2\pi} \| x_\theta(r, \varphi) \|^2 \, d\varphi \leqq 6r\psi'(r)$$

for almost all r, where we have made use of the fact that $\operatorname{diam} x(\partial \mathscr{C}) \leqq$ (length $x \mid \partial \mathscr{C}$)/2.

Thus $\psi(r) \leqq 6kr\psi'(r)$ and $(r^{-1/(6k)} \psi)' \geqq 0$.

The following lemma is usually stated for surfaces in Euclidean space, but the proof, with trivial modifications, shows that it is true for surfaces in m.

LEMMA 7.3. *Let $\{\mathscr{S}_n\}$ be a sequence of Fréchet surfaces such that $d(\partial \mathscr{S}_n, \gamma) \to 0$ where γ is a simple closed curve. Suppose that x_n is*

a *D-representation* of $\mathscr{S}_n$ on $\mathscr{C}$ such that $\{x_n\}$ satisfies the three point condition and such that $\{D(x_n)\}$ is uniformly bounded. Then $\{x_n \mid \partial \mathscr{C}\}$ is equicontinuous.

THEOREM 7.1. *If $T(\mathscr{K}, \gamma)$ is not empty and if $\bar{\psi} \in \mathscr{D}$ then there exists $x \in T(\mathscr{K}, \gamma)$ such that $I(\bar{\psi}, x) = \inf \{I(\bar{\psi}, y) \mid y \in T(\mathscr{K}, \gamma)\}$. In addition, x is open non-degenerate, $A\bar{\psi}$-quasi-conformal and can be chosen to take three distinct points of $\partial \mathscr{C}$ into three distinct points of γ.*

Proof. If $\mathscr{K}$ were a finite dimensional subspace of m then the first statement would follow from the remarks [13; p. 45]. The proof there is sufficient to permit $\mathscr{K}$ to be a convex subset of a finite dimensional space. This last condition may be deleted by replacing [13; Th. 6.1] with Lemma 7.2. This part of the proof is essentially the proof required for Theorem 7.2, and it is outlined after the statement of that theorem.

If x were not open non-degenerate, there would exist a retraction y of x, thus $y \in T(\mathscr{K}, \gamma)$, which would be open non-degenerate. By the representation theorem there would exist an $A\bar{\psi}$-quasi-conformal conformal map z, Fréchet equivalent to y and taking three distinct points of $\partial \mathscr{C}$ into three distinct points of γ, and we would have $I(\bar{\psi}, z) = I(A\bar{\psi}, z) = I(A\bar{\psi}, y) < L(A\bar{\psi}, x) = I(A\bar{\psi}, x) \leqq I(\bar{\psi}, x)$.

Similarly, the assumption that x is not $A\bar{\psi}$-quasi-conformal leads to a contradiction.

The theorem shows that there exists an open non-degenerate surface $\mathscr{S}$ and a representation x of $\mathscr{S}$ which is $A\bar{\psi}$-quasi-conformal such that $L(A\bar{\psi}, \mathscr{S}) = a(A\bar{\psi}, \mathscr{K}, \gamma)$. Furthermore, under suitable conditons on $\bar{\psi}$, x has certain differentiability properties [13; Ch. VII].

THEOREM 7.2. *Let $\bar{\psi} \in \mathscr{D}$. If $m(A\bar{\psi}, \mathscr{K}, \gamma) < \infty$ then there exists an open non-degenerate surface $\mathscr{S}$ with an $A\bar{\psi}$-quasi-conformal representation x on $\mathscr{C}$ such that $x \in T(\mathscr{K}, \gamma)$ and $a(A\bar{\psi}, \mathscr{K}, \gamma) = m(A\bar{\psi}, \mathscr{K}, \gamma) = L(A\bar{\psi}, \mathscr{S})$.*

Proof. Let $\{\mathscr{P}_n\}$ be a sequence of open non-degenerate polyhedra in $\mathscr{K}$ with $\partial \mathscr{P}_n \to \gamma$ and $\mathscr{E}(A\bar{\psi}, \mathscr{P}_n) \to m(A\bar{\psi}, \mathscr{K}, \gamma) = I$. Then there exist $A\bar{\psi}$-quasi-conformal representations y_n of $\mathscr{P}_n$ such that $\{y_n\}$ satisfies the three point condition and $I(\bar{\psi}, y_n) < I + 1$, for sufficiently large n. Thus $\{D(y_n)\}$ is uniformly bounded and, by deleting some terms if necessary, there is a function y such that y_n^i converges weakly in $\mathfrak{L}_2$ to y^i. Since $I(\bar{\psi}, \cdot)$ is lower semicontinuous with respect to this convergence, $I(\bar{\psi}, y) = I$ and thus $D(y)$ is finite and y is in $\mathscr{P}_2$. That there exists $k > 0$ for which $D(x \mid J) \leqq kD(H(x, J))$, and $H(x, J)$ is

a cone-function for x over J, follows as in [13; p. 45]. With only trivial changes, [13; p. 13, Th. 2.1] holds in our situation and it follows that y is continuous on each closed subset of Int $\mathscr{C}$. That y is continuous on a neighborhood of $\partial\mathscr{C}$ is proved as in [13; pp. 43–44] except that we must replace the harmonic functions used there by cone-functions. Thus y is continuous on $\mathscr{C}$ and is in $\mathscr{P}_2$, that is, y is a D-map. That range $y \subset \mathscr{K}$ and $y \mid \partial\mathscr{C} \in \gamma$ follows from the fact that range y is contained in a suitably large sphere and $\mathscr{K}$ is locally c-closed. Hence $y \in T(\mathscr{K}, \gamma)$.

It is obvious that y is open non-degenerate. Hence there exists an $A\bar{\psi}$-quasi-conformal function z which is Fréchet equivalent to y and takes three distinct points of $\partial\mathscr{C}$ into three distinct points of γ. By Theorem 7.1 there exists $x \in T(\mathscr{K}, \gamma)$ taking the same three points into the proper image points such that

$$I(\bar{\psi}, x) = \inf\{I(\bar{\psi}, w) \mid w \in T(\mathscr{K}, \gamma) \leqq I(\bar{\psi}, z) = I(A\bar{\psi}, z) = I(A\bar{\psi}, y)$$
$$= m(A\bar{\psi}, \mathscr{K}, \gamma) \leqq a(A\bar{\psi}, \mathscr{K}, \gamma) \leqq I(A\bar{\psi}, x) \leqq I(\bar{\psi}, x) .$$

Let $\mathscr{S}$ be the surface determined by x.

Since each surface can be approximated arbitrarily closely in both the Frechet metric and in $A\bar{\psi}$-Lebesgue area, by polyhedra, it follows that $m(A\bar{\psi}, \mathscr{K}, \gamma) \leqq \liminf L(A\bar{\psi}, \mathscr{S}_n)$ whenever $\{\mathscr{S}_n\}$ is a sequence of surfaces in $\mathscr{K}$ such that $d(\partial\mathscr{S}_n, \gamma) \to 0$.

COROLLARY. *Let $\{\gamma_n\}$ be a sequence of Jordan curves in $\mathscr{K}$ with $d(\gamma_n, \gamma) \to 0$. Then $m(A\bar{\psi}, \gamma) \leqq \liminf m(A\bar{\psi}, \mathscr{K}, \gamma_n)$.*

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