ON THE RELATIVE GROWTH OF DIFFERENCES OF PARTITION FUNCTIONS

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1. Introduction. Let A be an arbitrary set of positive integers (finite or infinite) other than the empty set or the set consisting of the single element unity. Let $p(n) = P_A(n)$ denote the number of partitions of the integer n into parts taken from the set A, repetitions being allowed. Generally, for any integer k we define $p^{(k)}(n) = p_A^{(k)}(n)$ by the formal power series relation

$$(1) \qquad f_k(x) = \sum_{n=0}^{\infty} p^{(k)}(n)X^n = (1 - X)^k \sum_{n=0}^{\infty} p(n)X^n = (1 - X)^k \prod_{a \in A} (1 - X^a)^{-1} .$$

Thus $p^{(k)}(n)$ is the kth difference of $p(n)$ if $k > 0$, $p(n)$ itself if $k = 0$, and the $(-k)$th order summatory function of $p(n)$ if $k < 0$. P. T. Bateman and P. Erdös proved (see [1]) that $p^{(k)}(n)$ is positive for all sufficiently large positive integer n if and only if A has the following property which is denoted by P_k: There are more than k elements in A, and if we remove an arbitrary subset of k elements in A, the remaining elements have greatest common divisor unity. When k is negative we agree that any set A has property P_k. Further, they conjectured that if A has property P_k then

$$(2) \qquad p^{(k+1)}(n)/p^{(k)}(n) = O(n^{-1/2})$$

for an arbitrary k.

Since for a finite set A which has property P_k we know that

$$p^{(k+1)}(n)/p^{(k)}(n) = O(1/n) \qquad \text{(see [1])}$$

i.e., this conjecture is true for such a set A we need only to consider when A is an infinite set.

The purpose of this paper is to study the asymptotic behavior of the ratio $p^{(k+1)}(n)/p^{(k)}(n)$ under rather strong restrictions on the regularity of the sequence $a_1 < a_2 < a_3 < \cdots$ formed by writing the elements of A in increasing order. Our restrictions are those used by Roth and Szekeres in [7], namely:

$$(\mathrm{I}) \qquad \lim_{u \to \infty} \frac{\log n(u)}{\log u} = \alpha$$

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where $n(u) = \sum_{\substack{a \in A \\ a \leqq u}} 1$ and $0 < \alpha \leqq 1$, and

(II) $$(\log m)^{-1} \inf \left\{ \sum_{j=1}^{m} \| \beta a_j \|^2 \right\} \to \infty \qquad \text{as } m \to \infty ,$$

where $\| x \|$ denotes the distance of x from the nearest integer and the lower bound is taken over those β satisfying $(2a_m)^{-1} < \beta \leqq 1/2$.

The assumption (I) is a smoothness assumption on the growth of the counting function of the set A, while (II) is an arithmetical condition implying P_k for every k. Roth and Szekeres showed that many frequently occurring sets have these two properties. Under these conditions we shall show that

(3) $$p^{(k+1)}(n)/p^{(k)}(n) \sim \sigma_n ,$$

where σ_n is defined as the unique solution of

$$n = \sum_{a \in A} a(e^{\sigma_n a} - 1)^{-1} .$$

Actually this result follows from the arguments of Roth and Szekeres [7], but we intend to give a direct proof using Hayman's method [4]. By a slight modification in the argument used in our proof one can obtain

(3*) $$p^{*(k+1)}(n)/p^{*(k)}(n) \sim \sigma_n^* \qquad \text{(see [7, p. 246])} ,$$

where $p^*(n) = p_A^*(n)$ denotes the number of partitions of n into distinct parts taken from the set A and $p^{*(k)}(n)$ is defined by

$$f_k^*(X) = \sum_{n=0}^{\infty} p^{*(k)}(n) X^n = (1 - X)^k \sum_{n=0}^{\infty} p^*(n) X^n = (1 - X)^k \prod_{a \in A} (1 + X^a) ,$$

and where σ_n^* is defined by $n = \sum_{a \in A} a(e^{\sigma_n a} + 1)^{-1}$. Probably (3) and (3*) hold under much weaker conditions than (I) and (II), but we have been unable to make much progress in this direction.

Furthermore, if we replace (I) by the following more stringent condition:

(I*) $$n(u) \sim u^\alpha L(u) \qquad \text{as } u \to \infty ,$$

where $0 < \alpha \leqq 1$ and L is a slowly oscillating function in the sense of Karamata [5], then we shall have

(4) $$p^{(k+1)}(n)/p^{(k)}(n) \sim n^{-1/(1+\alpha)} L_1(n) \qquad \text{as } n \to \infty ,$$

where L_1 is a slowly oscillating function related to L. This relation can be expressed in term of de Bruijn's concept of conjugate slowly oscillating function [2].

In any event we can derive under these conditions the Bateman-Erdös conjecture from (3) or (4), since $\sigma_n \leqq \pi(6n)^{-1/2}$. See the final section of the paper.

REMARK. An example of a set A of positive integers having property P_k for every k but not satisfying (II) is the following: Let A include all even numbers but very few odd numbers, say only odd numbers of the form $4^{4^n} + 1$, where n is a positive integer. Then for $x > e$ the number of odd numbers in A not exceeding x is less than $\log\log x$. Hence for $m \geqq 3$ we have

$$\sum_{j=1}^{m} \left\| \frac{1}{2} a_j \right\|^2 = \text{one-forth of the number of odd integers among } a_1, a_2, a_3, \cdots, a_m.$$
$$\leqq \frac{1}{4} \log\log a_m$$
$$\leqq \frac{1}{4} \log\log 2m ,$$

so that

$$(\log m)^{-1} \inf \left\{ \sum_{j=1}^{m} \left\| \frac{1}{2} a_j \right\|^2 \right\} \to 0 \qquad \text{as } m \to +\infty .$$

A similar example could be constructed by taking the multiples of any prime p and a very thin set of integers not divisible by p.

2. Outline of proof of (3). Let $s = \sigma + it$. Then our function

$$f_k(e^{-s}) = \sum_{n=0}^{\infty} p^{(k)}(n) e^{-sn}$$

is analytic in $\sigma > 0$. Define a function $\phi_k(s)$ as that branch of $\log f_k(e^{-s})$ given by the formula

$$\phi_k(s) = k \log(1 - e^{-s}) - \sum_{a \in A} \log(1 - e^{-as}) ,$$

where each term is defined by the principal branch of the logarithm. To this function $f_k(e^{-s})$ we shall apply the following lemma, due to Hayman [4], which is the main tool of this paper.

LEMMA 1. *Suppose that* $F(s) = \sum_{n=0}^{\infty} q_n e^{-sn}$ *converges for* $Re\, s = \sigma > 0$ *and* $F(\sigma) > 0$ *for all sufficiently small positive* σ. *Define* $q_n = 0$ *for* $n < 0$. *Suppose* $F(s)$ *satisfies the following three conditions for some* $\delta(\sigma)$, $0 < \delta(\sigma) < \pi$:

(a) $\phi''(\sigma) \to +\infty$ *as* $\sigma \to 0$,
where $\phi(\sigma) = \log F(\sigma)$,

(b) $$F(\sigma + it) \sim F(\sigma)e^{it\phi'(\sigma)-(t^2/2)\phi''(\sigma)} \qquad as\ \sigma \to 0$$
uniformly for $|t| \leqq \delta(\sigma)$, *and*

(c) $$F(\sigma + it) = o(F(\sigma))/\phi''(\sigma)^{1/2} \qquad as\ \sigma \to 0$$
uniformly for $\delta(\sigma) \leqq |t| \leqq \pi$.
Then we have uniformly for all n

$$q_n e^{-\sigma n} = \frac{F(\sigma)}{(2\pi\phi''(\sigma))^{1/2}}\left\{\exp\left[-\frac{(\phi'(\sigma)+n)^2}{2\phi''(\sigma)}\right] + o(1)\right\}$$

as $\sigma \to 0$.

We shall prove later that our function $f_k(e^{-s})$ satisfies (a), (b) and (c) for $\delta(\sigma) = \sigma^{1+(2\alpha/5)}$. Thus we will have uniformly for all n

$$(5) \qquad p^{(k)}(n)e^{-\sigma n} = \frac{f_k(e^{-\sigma})}{(2\pi\phi_k''(\sigma))^{1/2}}\left\{\exp\left[-\frac{(\phi_k'(\sigma)+n)^2}{2\phi_k''(\sigma)}\right] + o(1)\right\}$$

as $\sigma \to 0$. Denote by σ_n the unique root of

$$(6) \qquad \phi_0'(\sigma) + n = 0\,.$$

By (11) below this exists because $\phi_0'(\sigma)$ goes monotonically from 0 to $-\infty$ when σ goes from $+\infty$ to 0 through positive values. Then from (5)

$$(7) \qquad \begin{aligned} \frac{p^{(k+1)}(n)}{p^{(k)}(n)} &= (1 - e^{-\sigma_n})\left(\frac{\phi_k''(\sigma_n)}{\phi_{k+1}''(\sigma_n)}\right)^{1/2} \\ &\cdot\left\{\exp\left[\frac{(\phi_k'(\sigma_n) - \phi_0'(\sigma_n))^2}{2\phi_k''(\sigma_n)} - \frac{(\phi_{k+1}'(\sigma_n) - \phi_0'(\sigma_n))^2}{2\phi_{k+1}''(\sigma_n)}\right] + o(1)\right\}\end{aligned}$$

as $n \to \infty$.

We shall show in the next section that

$$(8) \qquad \phi_{k+1}''(\sigma_n) = \phi_k''(\sigma_n)(1 + o(1)) \qquad \text{as } n \to \infty \text{ i.e., as } \sigma_n \to 0$$

and

$$(9) \qquad (\phi_k'(\sigma_n) - \phi_0'(\sigma))^2 = o(\phi_k''(\sigma_n)) \qquad \text{as } n \to \infty\,.$$

Then from (7)

$$(10) \qquad \frac{p^{(k+1)}(n)}{p^{(k)}(n)} = (1 - e^{-\sigma_n})(1 + o(1)) = \sigma_n(1 + o(1)) \qquad \text{as } n \to \infty\,.$$

3. Proof that $\phi_k''(\sigma) \to \infty$ as $\sigma \to 0$ and proof of (8) and (9). From

$$\phi_k(\sigma) = \log f_k(e^{-\sigma}) = \log\left\{(1 - e^{-\sigma})^k \prod_{a\in A}(1 - e^{-\sigma a})^{-1}\right\}$$

we have

$$\phi_k'(\sigma) = \frac{ke^{-\sigma}}{1-e^{-\sigma}} - \sum_{a\in A} \frac{ae^{-\sigma a}}{1-e^{-\sigma a}} ,$$

(11)

$$\phi_k''(\sigma) = -\frac{ke^{-\sigma}}{(1-e^{-\sigma})^2} + \sum_{a\in A} \frac{a^2e^{-\sigma a}}{(1-e^{-\sigma a})^2} .$$

Since condition (I) implies that A is infinite, we have

(12)

$$\begin{aligned}\phi_k''(\sigma) &= -\frac{ke^{-\sigma}}{(1-e^{-\sigma})^2} + \sum_{a\in A}\left(\frac{a}{1+e^{-\sigma}+\cdots+e^{-\sigma(a-1)}}\right)^2 \frac{e^{-\sigma a}}{(1-e^{-\sigma})^2} \\ &> \frac{e^{-\sigma}}{(1-e^{-\sigma})^2}\left(-k + \sum_{a\in A} e^{-\sigma(a-1)}\right) \to +\infty \qquad \text{as } \sigma \to 0 ,\end{aligned}$$

so that (a) holds. And (8) is immediate since

$$\phi_{k+1}''(\sigma) = -\frac{e^{-\sigma}}{(1-e^{-\sigma})^2} + \phi_k''(\sigma)$$

and since

$$\frac{e^{-\sigma}}{(1-e^{-\sigma})^2} = o\left\{\frac{e^{-\sigma}}{(1-e^{-\sigma})^2}\left(-k + \sum_{a\in A} e^{-\sigma(a-1)}\right)\right\} = o(\phi_k''(\sigma)) \qquad \text{as } \sigma\to 0 .$$

Also from (11)

$$(\phi_k'(\sigma) - \phi_0'(\sigma))^2 = \frac{k^2e^{-2\sigma}}{(1-e^{-\sigma})^2}$$

and so by (12)

$$\frac{(\phi_k'(\sigma)-\phi_0'(\sigma))^2}{\phi_k''(\sigma)} < \frac{k^2e^{-\sigma}}{\left(-k+\sum_{a\in A} e^{-\sigma(a-1)}\right)} \to 0$$

as $\sigma \to 0$. Hence (9) holds.

4. Proof of (b) for $f_k(e^{-s})$.

4.1. First we obtain a result for $|t| \leqq \sigma/4$ and then specialize it to obtain (b). For $|z-\sigma| \leqq \sigma/4$ and σ sufficiently small we have for some constant B

(13)

$$|\phi_k''(z)| < B\phi_k''\left(\frac{3}{4}\sigma\right) .$$

For

$$\begin{aligned}|\phi_k''(z)| &= \left|\sum_{a\in A}\frac{a^2e^{-za}}{(1-e^{-za})^2} - \frac{ke^{-z}}{(1-e^{-z})^2}\right| \\ &= \left|\sum_{a\in A} a^2\sum_{j=1}^{\infty} je^{-zaj} - k\sum_{j=1}^{\infty} je^{-zj}\right| \\ &\leqq \sum_{a\in A} a^2\sum_{j=1}^{\infty} je^{-(Re\, z)aj} + |k|\sum_{j=1}^{\infty} je^{-(Re\, z)j} \\ &\leqq \phi_k(Re\, z) + \frac{2|k|e^{-Re\, z}}{(1-e^{-Re\, z})^2} \\ &\leqq B\phi_k''(Re\, z) \qquad \text{for a constant } B\end{aligned}$$

and

$$|\phi_k''(z)| \leqq B\phi_k''(Re\, z) \leqq B\phi_k''\left(\frac{3}{4}\sigma\right),$$

since $(3/4)\sigma \leqq Re\, z \leqq (5/4)\sigma$ and $\phi_k''(\sigma)$ is monotonically decreasing for sufficiently small σ (by an argument like those of the preceding section).

Thus we have a power series development

$$\phi_k''(z) = \sum_{n=0}^{\infty} c_n(z-\sigma)^n, \qquad |z-\sigma| \leqq \sigma/4,$$

where by Cauchy's inequality and (13)

$$|c_n| \leqq \frac{B\phi_k''\left(\frac{3}{4}\sigma\right)}{(\sigma/4)^n}.$$

Now we integrate both sides of the above power series and we have

$$\phi_k'(z) = \phi_k'(\sigma) + \sum_{n=0}^{\infty}\frac{c_n}{n+1}(z-\sigma)^{n+1}.$$

Since $c_0 = \phi_k''(\sigma)$, by integrating again we have

$$\begin{aligned}\phi_k(z) &= \phi_k(\sigma) + (z-\sigma)\phi_k'(\sigma) + \frac{1}{2}(z-\sigma)^2\phi_k''(\sigma) \\ &\quad + \sum_{n=1}^{\infty}\frac{c_n}{(n+1)(n+2)}(z-\sigma)^{n+2}.\end{aligned} \tag{14}$$

Now we have for $|z-\sigma| \leqq \sigma/4$, $n \geqq 1$,

$$|c_n(z-\sigma)|^{n+2} \leqq \frac{B\phi_k''\left(\frac{3}{4}\sigma\right)}{(\sigma/4)^n}|z-\sigma|^{n+2} \leqq \frac{B\phi_k''\left(\frac{3}{4}\sigma\right)|z-\sigma|^3}{\sigma/4}.$$

We now specialize the above by putting $z = s = \sigma + it$ with $|t| \leqq \sigma/4$. Then (14) gives

$$\log f_k(e^{-s}) = \log f_k(e^{-\sigma}) + it\phi_k'(\sigma) - \frac{1}{2}t^2\phi_k''(\sigma) + R_k(s)\ , \tag{15}$$

where

$$|R_k(s)| \leqq \frac{B\phi_k''\left(\frac{3}{4}\sigma\right)|t|^3}{\sigma/4}\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)} = \frac{2B\phi_k''\left(\frac{3}{4}\sigma\right)|t|^3}{\sigma}\ .$$

4.2. Now we use the condition (I) to estimate $\phi_k''(\sigma)$ in terms of σ. First

$$\begin{aligned}\phi_0(\sigma) &= \log f_0(e^{-\sigma}) = \log \sum_{m=1}^{\infty}(1-e^{-\sigma a_m})^{-1} \\ &= \sum_{m=1}^{\infty}\log(1-e^{-\sigma a_m})^{-1} = \sum_{m=1}^{\infty}\sigma\int_{a_m}^{\infty}\frac{e^{-\sigma u}}{1-e^{-\sigma u}}du \\ &= \sigma\int_{a_1}^{\infty}\frac{e^{-\sigma u}}{1-e^{-\sigma u}}\Big(\sum_{a_m\leqq u}1\Big)du = \sigma\int_0^{\infty}\frac{n(u)}{e^{\sigma u}-1}du\ .\end{aligned} \tag{16}$$

Let $K(v) = v/(e^v-1)$. Then we have

$$\begin{aligned}\phi_0(\sigma) &= \int_0^{\infty}\frac{K(\sigma u)}{u}n(n)du\ , \\ \phi_0'(\sigma) &= \int_0^{\infty}K'(\sigma u)n(u)du\ , \\ \phi_0''(\sigma) &= \int_0^{\infty}K''(\sigma u)un(u)du\ .\end{aligned} \tag{17}$$

Here $K(v)$, $-K'(v)$ and $K''(v)$ have positive limits as $v\to 0$ and are all $O(ve^{-v})$ as $v\to\infty$. Now for any positive ε we have from (I) that if u_0 is sufficiently large, then

$$n(u) \leqq u^{\alpha+\varepsilon} \qquad \text{for} \quad u \geqq u_0\ .$$

Here for suitable constants C and D, depending on ε, we have

$$\begin{aligned}\phi_0''(\sigma) &= \int_0^{\infty}K''(\sigma u)un(u)du \\ &\leqq \int_0^{\infty}K''(\sigma u)u^{1+\alpha+\varepsilon}du + \int_0^{u_0}K''(\sigma u)un(u)du \\ &\leqq \sigma^{-(2+\alpha+\varepsilon)}\int_0^{\infty}K''(v)v^{1+\alpha+\varepsilon}dv + C\int_0^{u_0}un(u)du \\ &= \sigma^{-(2+\alpha+\varepsilon)}\Big\{-(1+\alpha+\varepsilon)\int_0^{\infty}K'(v)v^{\alpha+\varepsilon}dv\Big\} + D \\ &= \sigma^{-(2+\alpha+\varepsilon)}(1+\alpha+\varepsilon)(\alpha+\varepsilon)\int_0^{\infty}K(v)v^{-1+\alpha+\varepsilon}dv + D \\ &= (\alpha+\varepsilon)(1+\alpha+\varepsilon)\Gamma(1+\alpha+\varepsilon)\zeta(1+\alpha+\varepsilon)\sigma^{-(2+\alpha+\varepsilon)} + D\ .\end{aligned} \tag{18}$$

Hence

$$\phi_0''(\sigma) = O(\sigma^{-(2+\alpha+\varepsilon)}) .$$

Since from (11)

$$\phi_k''(\sigma) \sim \phi_0''(\sigma) \qquad \text{as } \sigma \to 0 ,$$

$$\phi_k''(\sigma) = O(\sigma^{-(2+\alpha+\varepsilon)}) .$$

Returning to (15) we have now

$$| R_k(s) | = O(\sigma^{-(3+\alpha+\varepsilon)}) | t |^3 .$$

Finally for $| t | \leqq \delta(\sigma) = \sigma^{1+(2\alpha/5)}$ and for $\varepsilon < \alpha/5$ we have

$$| R_k(s) | = O(\sigma^{(\alpha/5)-\varepsilon}) = o(1) \qquad \text{as } \sigma \longrightarrow 0 .$$

Thus (15) yields (b) for $f_k(e^{-s})$ with this value of $\delta(\sigma)$.

5. Proof of (c) for $f_k(e^{-s})$. Let us define m to be such that

(19) $$a_m < 1/\sigma \leqq a_{m+1} .$$

In the sequel we shall express the magnitudes of $\phi_k''(\sigma)$ and $f_k(e^{-s})/f_k(e^{-\sigma})$ in terms of m and we shall compare them with the help of the condition (II).

Since from (I)

(20) $$\lim_{m\to\infty} \frac{\log a_{m+1}}{\log m} = \lim_{m\to\infty} \frac{\log a_m}{\log m} = \lim_{u\to\infty} \frac{\log u}{\log n(u)} = \frac{1}{\alpha} ,$$

we have from (19)

(21) $$\log \sigma^{-1} \sim \frac{1}{\alpha} \log m \qquad \text{as } \sigma \to 0 .$$

Thus it follows from (18) that

(22) $$\log (\phi_k''(\sigma))^{1/2} = O(\log m) .$$

Now consider

(23) $$\log \left\{ \frac{| f_k(e^{-s}) |}{f_k(e^{-\sigma})} \right\} = \log \left\{ \frac{| 1 - e^{-s} |^k}{(1 - e^{-\sigma})^k} \prod_{a \in A} \frac{1 - e^{-\sigma a}}{| 1 - e^{-sa} |} \right\} .$$

Here

$$\log \left\{ \frac{| 1 - e^{-\sigma - it} |^k}{(1 - e^{-\sigma})^k} \right\} = \frac{k}{2} \log \left\{ \frac{(1 - e^{-\sigma - it})(1 - e^{-\sigma + it})}{(1 - e^{-\sigma})^2} \right\}$$

$$= \frac{k}{2} \log \left\{ \frac{1 - 2e^{-\sigma} \cos t + e^{-2\sigma}}{(1 - e^{-\sigma})^2} \right\} .$$

Hence for small σ

$$\left|\log\left\{\frac{|1-e^{-\sigma-it}|^k}{(1-e^{-\sigma})^k}\right\}\right| \leqq \frac{|k|}{2}\log\frac{(1+e^{-\sigma})^2}{(1-e^{-\sigma})^2}$$
$$\leqq |k|\log\frac{2}{1-e^{-\sigma}} \leqq 2|k|\log\sigma^{-1} .$$

Thus by (21) we have for any fixed k

$$\log\frac{|1-e^{-\sigma-it}|^k}{(1-e^{-\sigma})^k} = O(\log m) . \tag{24}$$

Now

$$\begin{aligned}
&\log\prod_{a\in A}\frac{1-e^{-\sigma a}}{|1-e^{-\sigma a-ita}|}\\
&= \frac{1}{2}\sum_{a\in A}\log\frac{(1-e^{-\sigma a})^2}{(1-e^{-\sigma a-ita})(1-e^{-\sigma a+ita})}\\
&= \frac{1}{2}\sum_{a\in A}\log\left\{1-\frac{2e^{-\sigma a}(1-\cos ta)}{(1-e^{-\sigma a})^2+2e^{-\sigma a}(1-\cos ta)}\right\}\\
&< -\frac{1}{2}\sum_{a\in \alpha}\frac{2e^{-\sigma a}}{(1+e^{-\sigma a})^2}(1-\cos ta)\\
&< -\frac{1}{2}\sum_{a<1/\sigma}\frac{2e^{-\sigma a}}{(1+e^{-\sigma a})^2}(1-\cos ta)\\
&< -\frac{e^{-1}}{(1+e^{-1})^2}\sum_{j=1}^{m}(1-\cos ta_j) .
\end{aligned}$$

Write $E = e/(1+e)^2$ and $t = 2\pi\beta$. Then

$$\begin{aligned}
&-E\sum_{k=1}^{m}(1-\cos 2\pi\beta a_j)\\
&\quad = -E\sum_{j=1}^{m}(1-\cos 2\pi\|\beta a_j\|)\\
&\quad = -E\sum_{j=1}^{m}2\sin^2\pi\|\beta a_j\|\\
&\quad \leqq -8E\sum_{j=1}^{m}\|\beta a_j\|^2 .
\end{aligned}$$

Hence by the condition (II) for $(2a_m)^{-1} < \beta \leqq 1/2$ we have

$$\frac{1}{\log m}\log\left\{\prod_{a\in A}\frac{1-e^{-\sigma a}}{|1-e^{-\sigma a-ita}|}\right\} \to -\infty \qquad \text{as } m\to\infty . \tag{25}$$

From (19) $m\to\infty$ if and only if $\sigma\to 0$. Therefore from (22), (23), (24) and (25) we conclude now that for $\pi a_m^{-1} < |t| \leqq \pi$ we have

$$\frac{|f_k(e^{-s})|}{f_k(e^{-\sigma})}(\phi_k''(\sigma))^{1/2}$$

$$= \exp\left\{\log\frac{|f_k(e^{-s})|}{f_k(e^{-\sigma})} + \log(\phi_k''(\sigma))^{1/2}\right\} \to 0 \qquad \text{as } \sigma \to 0 .$$

On the other hand, if $\delta(\sigma) \leqq |t| \leqq \pi a_m^{-1}$, we have

$$\delta(\sigma)/(2\pi) \leqq |\beta| = |t|/(2\pi) \leqq (2a_m)^{-1} .$$

Thus if ε is a given positive number less than $\alpha/(8\alpha + 40)$, then by (20) $\sigma^{-1} \leqq a_{m+1} < m^{(1+\varepsilon)/\alpha}$ for sufficiently small σ and so

$$\sum_{j=1}^{m} \| \beta a_j \|^2 = \beta^2 \sum_{j=1}^{m} a_j^2 \geqq c_1\delta(\sigma)^2 \sum_{j=1}^{m} j^{(2/\alpha)(1-\varepsilon)}$$

$$\geqq c_1\delta(\sigma)^2 \int_0^{\sigma^{-\alpha/(1+\varepsilon)}} x^{(2/\alpha)(1-\varepsilon)}dx \geqq c_2\delta(\sigma)^2\sigma^{-2[(1-\varepsilon)/(1+\varepsilon)]-[\alpha/(1+\varepsilon)]}$$

$$\geqq c_2\sigma^{-(\alpha/5)+4\varepsilon[1+(\alpha/5)]} \geqq c_2\sigma^{-\alpha/10}$$

for sufficiently small σ, where the constants c_1 and c_2 depend on ε. Now from (21)

$$\frac{\sum_{j=1}^{m} \| \beta a_j \|^2}{\log m} > \frac{c_2\sigma^{-\alpha/10}}{2\alpha \log \sigma^{-1}} \to \infty \qquad \text{as } \sigma \to 0 .$$

As in the previous case this implies that in the case when $\delta(\sigma)/(2\pi) \leqq |\beta| \leqq (2a_m)^{-1}$ also (c) holds.

Thus we have completed the proof of (c) for $\delta(\sigma) \leqq |t| \leqq \pi$. Note that the uniformity in (b) and (c) is clear from our proofs.

6. Application to the Bateman-Erdös conjecture. In this section we shall estimate σ_n in (3) in term of n under each of the conditions (I) and (I*).

6.1. Under (I), for given any positive ε we have

$$u^{\alpha-\varepsilon} \leqq n(u) \leqq u^{\alpha+\varepsilon} \qquad \text{for large } u .$$

Recall that σ_n was defined by (6):

$$n + \phi_0'(\sigma_n) = 0 .$$

Now from (17)

$$\phi_0'(\sigma_n) = \int_0^\infty K'(\sigma_n u)n(u)du .$$

Hence

$$-\int_0^\infty K'(\sigma_n u)u^{\alpha-\varepsilon}du \leqq n \leqq -\int_0^\infty K'(\sigma_n u)u^{\alpha+\varepsilon}du\ .$$

Then a computation similar to (18) gives

$$(\alpha-\varepsilon)\Gamma(1+\alpha-\varepsilon)\zeta(1+\alpha-\varepsilon)\sigma_n^{-(1+\alpha-\varepsilon)} \\ \leqq n \leqq (\alpha+\varepsilon)\Gamma(1+\alpha+\varepsilon)\zeta(1+\alpha+\varepsilon)\sigma_n^{-(1+\alpha+\varepsilon)}\ .$$

Thus we have

$$\log n = -(1+\alpha+o(1))\log\sigma_n \qquad \text{as } n\to\infty\ .$$

Hence

$$\sigma_n = n^{-1/[1+\alpha+o(1)]}\ . \tag{26}$$

Furthermore, we have always $n(u) \leqq u$ and so

$$n \leqq -\int_0^\infty K'(\sigma_n n)udu = -\frac{1}{\sigma_n^2}\int_0^\infty K'(v)vdv \\ = \frac{1}{\sigma_n^2}\Gamma(2)\zeta(2) = \frac{1}{\sigma_n^2}\frac{\pi^2}{6} \qquad \text{(see (18))}\ .$$

Hence always $\sigma_n \leqq (\pi/\sqrt{6})(1/\sqrt{n})$. Since we obtained (3) under the conditions (I) and (II), under these conditions

$$p^{(k+1)}(n)/p^{(k)}(n) \leqq \left(\frac{\pi}{\sqrt{6}}+\varepsilon\right)\frac{1}{\sqrt{n}} \tag{27}$$

for sufficiently large n. Of course this is weaker than (26) when $\alpha < 1$. However note that (27) implies the Bateman-Erdös conjecture under these conditions. In fact it would be reasonable to conjecture that (27) holds for any set A of positive integers.

6.2. Under (I*) we have

$$n(u) \sim u^\alpha L(u) \qquad \text{as } u\to\infty\ (0<\alpha\leqq 1)\ .$$

LEMMA 2. *The condition* (I*) *implies*

$$-\phi_0'(\sigma) \sim \alpha\Gamma(1+\alpha)\zeta(1+\alpha)(1/\sigma)^{1+\alpha}L(1/\sigma)$$

as $\sigma\to 0$.

Proof. Suppose L is defined on $[a,\infty)$, $a>0$. Choose $0<\gamma<\alpha$. Then by Karamata's representation theorem for slowly oscillating function [3] there exists $b \geqq a$ such that

$$0 < n(u) < 2u^\alpha L(u)\ , \qquad u \geqq b$$

and

$$\frac{1}{2}\left(\frac{x}{u}\right)^{-\gamma} < \frac{L(x)}{L(u)} < 2\left(\frac{x}{u}\right)^{\gamma} \qquad \text{for } x \geqq u \geqq b\,.$$

Now from (17)

$$\begin{aligned}\frac{\phi_0'(\sigma)}{(1/\sigma)^{1+\alpha}L(1/\sigma)} &= \frac{\int_0^\infty K'(\sigma u)n(u)du}{(1/\sigma)^{1+\alpha}L(1/\sigma)}\\ &= \frac{\sigma\int_0^b K'(\sigma u)n(u)du + \int_{b\sigma}^\infty K'(v)n(v/\sigma)dv}{(1/\sigma)^{\alpha}L(1/\sigma)}\\ &= O\left(\frac{\int_0^b n(u)du}{(1/\sigma)^{1+\alpha}L(1/\sigma)}\right) + \int_{b\sigma}^\infty \frac{n(v/\sigma)L(v/\sigma)}{(v/\sigma)^{\alpha}L(v/\sigma)L(1/\sigma)}K'(v)v^{\alpha}du\\ &= O\left(\frac{1}{(1/\sigma)^{1+\alpha}L(1/\sigma)}\right) + \int_0^\infty g(v,\sigma)dv\end{aligned} \tag{28}$$

where

$$g(v,\sigma) = \begin{cases} 0 & \text{if } v < b\sigma\,,\\ \dfrac{n(v/\sigma)}{(v/\sigma)^2L(v/\sigma)}\dfrac{L(v/\sigma)}{L(1/\sigma)}K'(v)v^{\alpha} & \text{if } v \geqq b\sigma\,.\end{cases}$$

For fixed positive v

$$\lim_{\sigma\to 0} g(v,\sigma) = K'(v)v^{\alpha}\,.$$

Also if $\sigma \leqq 1/b$ and $v \geqq b\sigma$ we have

$$\frac{L(v/\sigma)}{L(1/\sigma)} < \begin{cases} 2v^{-\gamma} & \text{if } v \leqq 1\\ 2v^{\gamma} & \text{if } v \geqq 1\,.\end{cases}$$

And if $v \geqq b\sigma$,

$$0 < \frac{n(v/\sigma)}{(v/\sigma)^{\alpha}L(v/\sigma)} < 2\,.$$

Thus $|g(v,\sigma)| \leqq h(v)$, where

$$h(v) = \begin{cases} 4|K'(v)|v^{\alpha-\gamma} & \text{if } v \leqq 1\\ 4|K'(v)|v^{\alpha+\gamma} & \text{if } v \geqq 1\,.\end{cases}$$

Since

$$\int_0^\infty h(v)dv < \infty\,,$$

the Lebesgue dominated convergence theorem gives

$$\lim_{\sigma\to 0}\int_0^\infty g(v,\sigma)dv = \int_0^\infty K'(v)v^\alpha dv = -\alpha\Gamma(1+\alpha)\zeta(1+\alpha) \qquad \text{(see (18))} .$$

Therefore from (28) we have since

$$O\Big(\frac{1}{(1/\sigma)^\alpha L(1/\sigma)}\Big) = o(1) \qquad \text{as } \sigma\to 0 ,$$

$$\lim_{\sigma\to 0}\frac{\phi_0'(\sigma)}{(1/\sigma)^{1+\alpha}L(1/\sigma)} = -\,\alpha\Gamma(1+\alpha)\zeta(1+\alpha) .$$

Thus we have now

$$n = -\phi_0'(\sigma_n) \sim \alpha\Gamma(1+\alpha)\zeta(1+\alpha)L(1/\sigma_n)(1/\sigma_n)^{1+\alpha} \qquad \text{as } n\to\infty .$$

Hence

$$\sigma_n \sim n^{-1/(1+\alpha)}\{\alpha\Gamma(1+\alpha)\zeta(1+\alpha)L(1/\sigma_n)\}^{1/(1+\alpha)} \qquad \text{as } n\to\infty . \tag{29}$$

To obtain (4) from (29) we introduce the following result, due to de Bruijn [2],: If $M(x)$ is a slowly oscillating function, then there exists a slowly oscillating function M^*, called the conjugate of M, such that

$$M^*(xM(x))M(x)\to 1 \qquad \text{as } x\to\infty ,$$
$$M(xM^*(x))M^*(x)\to 1 \qquad \text{as } x\to\infty .$$

Moreover M^* is asymptotically uniquely determined.

Put

$$M(x) = \{\alpha\Gamma(1+\alpha)\zeta(1+\alpha)L(x)\}^{1/(1+\alpha)} .$$

Then

$$M^*(M(1/\sigma_n)1/\sigma_n)M(1/\sigma_n)\to 1 \qquad \text{as } n\to\infty .$$

But since from (29)

$$M^*(M(1/\sigma_n)1/\sigma_n) \sim M^*(n^{1/(1+\alpha)}) ,$$

we have

$$M^*(n^{1/(1+\alpha)})M(1/\sigma_n)\to 1 \qquad \text{as } n\to\infty .$$

Thus we have from (29)

$$\sigma_n \sim n^{-1/(1+\alpha)}M^*(n^{1/(1+\alpha)})^{-1} \qquad \text{as } n\to\infty .$$

Since by the property of a slowly oscillating function $M^*(n^{1/(1+\alpha)})^{-1}$ is a slowly oscillating function of n, by letting

$$L_1(n) = M^*(n^{1/(1+\alpha)})^{-1}$$

we obtain (4) from (3) and (29).

Note that

$$\limsup_{x\to+\infty} L_1(x) = \{\alpha\Gamma(1+\alpha)\zeta(1+\alpha) \limsup_{x\to+\infty} L(x)\}^{1/(1+\alpha)}$$

and similarly for lim inf. This remark gives (27) from (4), but only under the present more stringent conditions.

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