DOUBLY INVARIANT SUBSPACES

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1. Our theme is a theorem on doubly invariant subspaces attributed to Wiener in the folk lore; our discussion was inspired by that of Helson-Lowdenslager [2] on simply invariant subspaces and a course of lectures by Professor Helson on this subject. Let $\mathscr{M}$ denote a closed subspace of L^2 of the circle $|z| = 1$, which we shall denote as $L^2(e^{ix})$. Let λ denote the function on $|z| = 1$ defined by $\lambda(e^{ix}) = e^{ix}$. Say that $\mathscr{M}$ is *doubly invariant* if $f \in \mathscr{M} \Rightarrow \lambda f, \lambda^{-1} f \in \mathscr{M}$. An example of such a subspace is the set of all $f \in L^2(e^{ix})$ which vanish on a fixed measurable subset E. Wiener's theorem asserts that every doubly invariant $\mathscr{M}$ is of this form. A similar result holds for L^2 of the real line too (which we shall denote as $L^2(dt)$). In this case a doubly invariant subspace is any closed subspace $\mathscr{M}$ of $L^2(dt)$ such that $f \in \mathscr{M} \Rightarrow e^{iut} f \in \mathscr{M}$ for all real u, and every such subspace consists precisely of all functions in $L^2(dt)$ which vanish on a fixed measurable subset E of the line. In either case—the circle or the line—$\mathscr{M}$ determines E uniquely. We shall refer to either of these cases as the scalar case.

Wiener's theorem extends to L^2 spaces of vector valued functions on the circle or the line. Let $\mathscr{H}$ be any separable Hilbert space and $L^2_{\mathscr{H}}$ denote the set of all functions on $|z| = 1$ with values in $\mathscr{H}$ which are weakly measurable and whose norms are square integrable. $L^2_{\mathscr{H}}$ is a Hilbert space for the inner product

$$(f, g) = \int_{-\pi}^{\pi} (f(e^{ix}), g(e^{ix}))d\sigma$$

where the inner product on the right is the one in $\mathscr{H}$ and $d\sigma = (1/2\pi)dx$. The doubly invariant subspaces of $L^2_{\mathscr{H}}$ are defined exactly as before. An example of such a subspace in this case can be given as follows:

Let $\mathscr{J}$ be a *range function* meaning a function on $|z| = 1$ to the family of closed subspaces of $\mathscr{H}$, defined a.e. Two range functions which agree a.e. are regarded as the same function. Let $P(e^{ix})$ be the self adjoint projection on $\mathscr{J}(e^{ix})$. Say that that $\mathscr{J}$ is "measurable" if P is weakly measurable. Given $\mathscr{J}$ measurable, let $\mathscr{M}_{\mathscr{J}}$ be the set of all functions $f \in L^2_{\mathscr{H}}$ for which $f(e^{ix}) \in \mathscr{J}(e^{ix})$ a.e. Then $\mathscr{M}_{\mathscr{J}}$ is a doubly invariant subspace of $L^2_{\mathscr{H}}$. The version of Wiener's theorem in this case will be that every doubly invariant subspace of $L^2_{\mathscr{H}}$ is obtained as above from a measurable range function $\mathscr{J}$ and

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$\mathscr{M}_{\mathscr{J}}$ determines $\mathscr{J}$ uniquely. The scalar case corresponds to one dimensional $\mathscr{H}$ in which case $\mathscr{J}(e^{ix})$ can have only one of two values, either $\{0\}$ or the whole space, so that specifying $\mathscr{J}(e^{ix})$ is merely prescribing the set on which all functions in $\mathscr{M}_{\mathscr{J}}$ vanish. Thus the above indeed generalizes the scalar case for the circle. The generalization of the line case to the vector context is now obvious.

In both the scalar and vector cases, the circle or the line and the associated Lebesgue measure are inessential. Let X be any locally compact space and m a regular Borel measure on X and let P be any subspace of $L_\infty(dm)$ which is weak* dense. Say that a closed subspace $\mathscr{M}$ of $L^2(dm)$ or $L^2_{\mathscr{H}}(dm)$ is *doubly invariant* if it is invariant for multiplication by functions in P. Then the doubly invariant subspaces of $L^2(dm)$ or $L^2_{\mathscr{H}}(dm)$ have precisely the same structure as in the circle or the line case. The circle corresponds to the situation '$m(X) < \infty$' and the line to '$m(X) = \infty$'; the subspace P corresponds in either case to the set of all trignometric polynomials.

In this paper we first give a proof of Wiener's theorem for the scalar circle case and show that essentially the same proof applies to the line case too. We then generalize our proof to yield the vector case. Our proof for the (scalar and vector) circle case applies word for word (with obvious changes) to the context of finite regular measure spaces mentioned above; our proof of the line case could be adapted to the context of infinite measure spaces. By modifying our proof for the vector case we obtain a theorem (Theorem 5) on range functions of constant dimension which incidentally gives a characterization of range functions associated with *simply invariant* subspaces with no *remote past* (Theorem 6). Finally we show that in the scalar case the $L^2(dm)$ theorem implies a corresponding $L^p(dm)$ theorem (Theorem 7), $1 \leqq p \leqq \infty$[1].

The Wiener L^2 theorem is known. In the scalar case, direct proofs are also known; our proof seems to be simpler. In the vector case our version of the theorem was suggested by Professors Helson and Lowdenslager; we have not seen in the literature a direct proof of the theorem in this case. It could be derived as a corollary from the following general theorem in the theory of 'rings of operators':

Any bounded operator $T: L^2_{\mathscr{H}} \to L^2_{\mathscr{H}}$ which commutes with multiplication by bounded scalar functions is multiplication by a bounded operator valued function. [cf: 1, p. 167, Theorem 1; 3, p. 301, Lemma 1] The proof this way would be more involved. Our L^p theorem and Theorem 6, we believe, are new.

We may point out in passing that the general theorem on multiplication operators quoted above can itself derived from Wiener's

[1] The $L^p(dm)$ theorem for $p \neq 2$ is of interest as it exhibits a class of subspaces of $L^p(dm)$ which admit bounded projections.

theorem by an application of the spectral theorem for self adjoint operators. We omit the proof of this.

We have benefitted considerably by our discussion with Professor Helson in the course of preparation of this paper and our thanks are due to him.

2. THEOREM 1. *Let $\mathscr{M}$ be a doubly invariant subspace of $L^2(e^{ix})$. Then $\mathscr{M} = C_E L^2(e^{ix})$ for some measurable sebset E (where C_E denotes the characteristic function of E).*

Proof. Let $\mathscr{M}^{\perp}$ be the orthogonal complement of $\mathscr{M}$ in $L^2(e^{ix})$ and let q be the orthogonal projection on $\mathscr{M}$ of the constant function 1. Then $1 - q \in \mathscr{M}^{\perp}$, and because of double invariance of $\mathscr{M}$ and hence of $\mathscr{M}^{\perp}$, $\lambda^n(1-q) \in \mathscr{M}^{\perp}$ for all n. So $\int (q - |q|^2)e^{-nix}d\sigma = 0$ for all n so that $|q|^2 = q$ a.e. Hence $q = C_E$ for some measurable subset E.

Trivially $qL^2(e^{ix}) \subset \mathscr{M}$. This inclusion is in fact an equality. For if $g \in \mathscr{M} \ominus qL^2(e^{ix})$ then $g \perp \lambda^n q$ for all n, also $g \perp \lambda^n(1-q)$ (which lies in $\mathscr{M}^{\perp}$), so $g \perp \lambda^n$ for all n and hence $g = 0$ a.e. Thus $\mathscr{M} = qL^2(e^{ix}) = C_E L^2(e^{ix})$. We pass now to the line case:

THEOREM 2. *Let $\mathscr{M}$ be a doubly invariant subspace of $L^2(dt)$, $-\infty < t < \infty$. Then $\mathscr{M} = C_E L^2(dt)$ for some measurable subset E of the line.*

Proof. Let $\tilde{L}^2 = (1 - it)L^2(dt)$ and $\tilde{\mathscr{M}} = (1 - it)\mathscr{M}$. $\tilde{L}^2$ is a Hilbert space for the inner product

$$(f, g) = \int_{-\infty}^{\infty} f\bar{g}\frac{1}{1+t^2}dt$$

and $\tilde{\mathscr{M}}$ is a closed subspace of $\tilde{L}^2$ invariant under multiplication by all e^{iut}. Let $\tilde{\mathscr{M}}^{\perp}$ be the orthogonal complement of $\tilde{\mathscr{M}}$ in $\tilde{L}^2$ and let q be the projection of the constant function 1 (which belongs to $\tilde{L}^2$) on $\tilde{\mathscr{M}}$. Now the arguments are the same as in the circle case:

$(1 - q)e^{iut} \in \tilde{\mathscr{M}}^{\perp}$ for all u and hence $q \perp (1-q)e^{iut}$ for all u. That is

$$\int_{-\infty}^{\infty} (q - |q|^2)\frac{1}{1+t^2}e^{-iut}dt = 0 \qquad \text{for all } u.$$

Hence $(q - |q|^2)(1/(1+t^2)) = 0$ a.e. Thus $|q|^2 = q$ a.e. and $q = C_E$ for some E. Then as in the circle case, $\tilde{\mathscr{M}} = q\tilde{L}^2 = C_E\tilde{L}^2$, i.e. $(1 - it)\mathscr{M} = (1 - it)C_E L^2$. Hence $\mathscr{M} = C_E L^2$. The uniqueness of E is trivial in both the cases.

3.1. We deal with the vector case for the circle. Let $\mathscr{H}$ be a separable Hilbert-space and $L^2_{\mathscr{H}}$ be defined as in § 1. Then we have

THEOREM 3. *For every doubly invariant subspace $\mathscr{M}$ of $L^2_{\mathscr{H}}$ there exists a unique measurable range function $\mathscr{J}$ such that $\mathscr{M} = \mathscr{M}_{\mathscr{J}}$.*

Proof. Let $\{e_k\}$ $k = 1, 2, \cdots$ be an orthonormal basis for $\mathscr{H}$ and let q_k be the projection of the constant function e_k on $\mathscr{M}$. Then $q_k \in L^2_{\mathscr{H}}$ and of course is measurable. Each q_k is defined a.e. on the circle and hence also all q_k's together. Let $\mathscr{J}(e^{ix})$ be the closed subspace of $\mathscr{H}$ spanned by $\{q_k(e^{ix})\}_k$. Then $\mathscr{J}(e^{ix})$ is defined a.e. We shall show that

(a) $\mathscr{J}$ is measurable

(b) $\mathscr{M} = \mathscr{M}_{\mathscr{J}}$

Proof of (a). Let $P(e^{ix})$ be the orthogonal projection on $\mathscr{J}(e^{ix})$. We have only to show that $P(e^{ix})e_k$ is measurable for all k. We shall actually show that $P(e^{ix})e_k = q_k(e^{ix})$ a.e. Let $\mathscr{M}^\perp = L^2_{\mathscr{H}} \ominus \mathscr{M}$. Now $q_k \in \mathscr{M}$ and $e_k - q_k \in \mathscr{M}^\perp$. Because of double invariance then, $\lambda^n q_r \in \mathscr{M}$ for all n, and is $\perp e_k - q_k$ for all k. Thus $\int (e_k - q_k(e^{ix}), q_r(e^{ix}))e^{-nix}d\sigma = 0$ for all n and hence $e_k - q_k(e^{ix}) \perp q_r(e^{ix})$ a.e. for every r so that $e_k - q_k(e^{ix}) \perp q_r(e^{ix})$ for all r, a.e. This means $e_k - q_k(e^{ix}) \perp \mathscr{J}(e^{ix})$ a.e. Since $q_k(e^{ix}) \in \mathscr{J}(e^{ix})$ it follows that $P(e^{ix})e_k = q_k(e^{ix})$ a.e.

Proof of (b). Let $\mathscr{N}$ be the closed span of $\{\lambda^n q_k\}$ in $L^2_{\mathscr{H}}$, $k \geqq 1$, $n = 0, \pm 1, \pm 2, \cdots$. Then $\mathscr{N}$ is doubly invariant and $\mathscr{N} \subset \mathscr{M}$. If $\mathscr{N} \neq \mathscr{M}$ let $g \in \mathscr{M} \ominus \mathscr{N}$. Then, using the invariance, we have

(i) $g \perp \lambda^n q_k$ for all k, n

(ii) $\lambda^n g \perp e_k - q_k$ for all k, n.

It follows as in the proof of (a) that

(i) $g(e^{ix}) \perp q_k(e^{ix})$ a.e.

(ii) $g(e^{ix}) \perp e_k - q_k(e^{ix})$ a.e.

Hence $g(e^{ix}) \perp e_k$ a.e. for every k so that $g(e^{ik}) \perp e_k$ for all k, a.e. Hence $g(e^{ix}) = 0$ a.e. This shows that $\mathscr{M} = \mathscr{N}$.

If $f \in \mathscr{N}$ then $f(e^{ix}) \in \mathscr{J}(e^{ix})$ a.e. Hence $\mathscr{M} \subset \mathscr{M}_j$. Let now $g \in \mathscr{M}_{\mathscr{J}} \ominus \mathscr{M}$. Then $g \perp \lambda^n q_k$ for all k, n, so $g(e^{ix}) \perp q_k(e^{ix})$ a.e. for every k and hence $g(e^{ix}) \perp \mathscr{J}(e^{ix})$ a.e. But $g(e^{ix}) \in \mathscr{J}(e^{ix})$ a.e. as $g \in \mathscr{M}_{\mathscr{J}}$. Hence $g = 0$. Thus $\mathscr{M} = \mathscr{M}_{\mathscr{J}}$.

Only the uniqueness part of the theorem remains to be proved. This we prove independently as a lemma.

LEMMA. *If $\mathscr{J}$ and $\mathscr{K}$ are measurable range functions and $\mathscr{M}_{\mathscr{J}} = \mathscr{M}_{\mathscr{K}}$ then $\mathscr{J} = \mathscr{K}$ a.e.*

Proof. Let as before $P(e^{ix})$ be the orthogonal projection on $\mathscr{J}(e^{ix})$ and let $q_k(e^{ix}) = P(e^{ix})e_k$, $k = 1, 2 \cdots$ where $\{e_k\}$ is an o.n. basis for $\mathscr{J}$. q_k is measurable as $\mathscr{J}$ is and $\| q_k(e^{ix}) \|^2 \leqq \| e_k \|^2 = 1$ so that $q_k \in L^2_{\mathscr{H}}$. Also $\{q_k(e^{ix})\}_k$ generate $\mathscr{J}(e^{ix})$ as $\{e_k\}$ generate $\mathscr{H}$. Now $q_k \in \mathscr{M}_{\mathscr{J}} = \mathscr{M}_{\mathscr{K}}$ so that $q_k(e^{ix}) \in \mathscr{K}(e^{ix})$ a.e. for all k. It follows that $\mathscr{J}(e^{ix}) \subset \mathscr{K}(e^{ix})$ a.e. Interchanging $\mathscr{J}$ and $\mathscr{K}$ we conclude that $\mathscr{J} = \mathscr{K}$ a.e.

3.2. The functions $\{q_k\}$ defined in § 3.1 provide a measurable basis pointwise a.e. for $\mathscr{J}$. We shall show that we can secure the $\{q_k\}$ to be orthogonal a.e. The usual orthogonalization process can be applied at every point but the measurability of the resulting functions needs to be proved. This can be avoided by a slight modification of our construction of the q_k's which while preserving their other properties also ensures their pointwise orthogonality. The modification is the following:

Let q_1 be the orthogonal projection of e_1 on $\mathscr{M}$ and let $\mathscr{N}_1$ be the closed span of $\{\lambda^n q_1\}_n$. Then $\mathscr{N}_1$ is doubly invariant and so is $\mathscr{M}_1 = \mathscr{M} \ominus \mathscr{N}_1$. Let now q_2 be the projection of e_2 on $\mathscr{M}_1$ and let $\mathscr{N}_2 \subset \mathscr{M}_1$ be the closed span of $\{\lambda^n q_2\}$. Having obtained $q_1, q_2, \cdots, q_{k-1}$ and $\mathscr{N}_1, \mathscr{N}_2, \cdots \mathscr{N}_{k-1}$ as above, define q_k as the projection of e_k on $\mathscr{M} \ominus \sum_{i=1}^{k-1} \mathscr{N}_i$. The q_k's are easily seen to be mutually orthogonal a.e. If $\mathscr{J}(e^{ix})$ is defined to be the closed span of $\{q_k(e^{ix})\}_k$, the arguments in §3.1 which trivial modifications will show that $\mathscr{M} = \mathscr{M}_{\mathscr{J}}$. We have thus proved

THEOREM 4. *Corresponding to every measurable range function there exist functions $q_k \in L^2_{\mathscr{H}}$, $k = 1, 2, \cdots$ such that the $q_k(e^{ix})$'s are mutually orthogonal and span $\mathscr{J}(e^{ix})$ a.e.*

The question that arises next is: when does $\mathscr{J}(e^{ix})$ have a measurable o.n. basis a.e.? If $\{q_k(e^{ix})\}_k$ is an o.n. basis a.e. for $\mathscr{J}(e^{ix})$ then the dimension of $\mathscr{J}(e^{ix})$ is a constant a.e., being equal to the cardinality of the indexing k's (finite or not). Conversely also we have

THEOREM 5. *If $\mathscr{J}$ is a measurable range function of constant dimension a.e., there exist functions q'_k, $k = 1, 2, \cdots$ in $L^2_{\mathscr{H}}$ such that $\{q'_k(e^{ix})\}$ is an o.n. basis for $\mathscr{J}(e^{ix})$* a.e.

Proof. By our construction in the proof of Theorem 4 we can assume that there exist $q_k \in L_{\mathscr{H}}$, $k = 1, 2, \cdots$ such that $\| q_k(e^{ix}) \| = 1$ or 0 a.e. and $\{q_k(e^{ix})\}_k$ is orthogonal and generates $\mathscr{J}(e^{ix})$. For a given x let $q'_1(e^{ix}) = q_{i_1}(e^{ix})$ where i_1 is the smallest index such that $q_{i_1}(e^{ix}) \neq 0$; having obtained $q'_1(e^{ix}), \cdots, q'_{n-1}(e^{ix})$, let $q'_n(e^{ix}) = q_{i_n}(e^{ix})$ where i_n is the smallest index $\geqq i_{n-1}+1$ such that $q_{i_n}(e^{ix}) \neq 0$.[1] If dimension $\mathscr{J}(e^{ix}) = \infty$ a.e., this construction defines q'_n for every n; if dimension $\mathscr{J}(e^{ix}) = N < \infty$ a.e., the construction proceeds exactly N steps and defines $q'_1, q'_2, \cdots, q'_N$ a.e. The verification that the q'_k's satisfy the requirements of the theorem is not hard.

The above theorem has an interesting corollary. Say that a closed subspace $\mathscr{M} \subset L^2_{\mathscr{H}}$ is "simply invariant" if $\lambda^n \mathscr{M} \subset \mathscr{M}$ for all $n \geqq 0$ but not for all $n < 0$. The range function $\mathscr{J}$ associated with the smallest doubly invariant subspace containing $\mathscr{M}$, we shall call the "range function of $\mathscr{M}$". The subspace $\mathscr{M}_\infty = \bigcap_{n \geqq 0} \lambda^n \mathscr{M}$, we shall call the "remote past" of $\mathscr{M}$. If $\mathscr{M}_\infty = \{0\}$ (when $\mathscr{M}$ is said to be without remote past) it can be shown from the $L^2_{\mathscr{H}}$ version of a theorem of Lax [3, p. 300] that the associated range function is of constant dimension a.e. (meaning finite and equal or infinite a.e.). Conversely, if $\mathscr{J}$ is any measurable range function of constant dimension, by Theorem 5 it has a pointwise o.n. basis $\{q'_k(e^{ix})\}_k$, $q'_k \in L^2_{\mathscr{H}}$. Then $\{\lambda^n q'_k\}_{k,n}$ is an o.n. set in $L^2_{\mathscr{H}}$. If $\mathscr{N}_m$ is the closed span of $\{\lambda^m q'_k\}_k$, the $\mathscr{N}_m$'s are mutually orthogonal in $L^2_{\mathscr{H}}$ for $m = 0, \pm 1, \pm 2, \cdots$ and the orthogonal sum $\mathscr{M} = \sum_{m \geqq 0} \mathscr{N}_m$ is a simply invariant subspace of $L^2_{\mathscr{H}}$ without remote past whose range function is the given $\mathscr{J}$. Thus we have

THEOREM 6. *A measurable range function is of constant dimension a.e. if and only if it is the range function of a simply invariant subspace without remote past.*

4. The modification employed in § 2 for discussing the line case in the scalar context carries over without change to the vector situation and extends Theorems 3–5 to $L^2_{\mathscr{H}}$ over the line. Theorem 6 remains true but needs to be discussed anew; we omit the details.

5. Let m be a regular Baire measure on a locally compact space X and P a subspace of $L^\infty(dm)$ which is weak* dense. The reasoning given in § 2–3 shows that the doubly invariant subspaces $\mathscr{M}$ of $L^2(dm)$ are the subspaces of the form $C_E L^2(dm)$, $E \subset X$ measurable. Using this we wish to prove the following

THEOREM 7. *Let $\mathscr{N}$ be a subspace of $L^p(dm)$ which is invariant under multiplication by functions in P and which is closed if $1 \leqq p < \infty$ and weak* closed if $p = \infty$. Then $\mathscr{N} = C_E L^p(dm)$ for some measurable subset E of X.*

[1] This construction resulted from a discussion with Professor Ju-kwei Wang.

Proof.

Case (i) $1 \leqq p < 2$:

Let $\mathscr{M} = \mathscr{N} \cap L^2(dm)$. Then $\mathscr{M}$ is a doubly invariant subspace of $L^2(dm)$ and so $\mathscr{M} = C_E L^2(dm)$ for some measurable subset E. We shall show that $\mathscr{N} = C_E L^p(dm)$.

Let $f \in \mathscr{N}$ and $f = f_1 f_2$ be any factorization for f as a product of an L^μ function and L^2 function where $(1/\mu)+(1/2)=(1/p)$, for instance $f_2 = |f|^{p/2}$ and $f_1 = (\text{sgn.} f)\,|f|^{1-(p/2)}$. Let P_a be the subalgebra generated by P and constants in $L^\infty(dm)$. The closed subspace $[f_2 P_a]_2$ generated by $f_2 P_a$ in $L^2(dm)$ is doubly invariant and hence $[f_2 P_a]_2 = C_{E_2} L^2(dm)$ for some $E_2 \subset X$. Now

$$f_1 C_{E_2} \in f_1 C_{E_2} L^2(dm) = f_1 [f_2 P_a]_2 \subset [f_1 f_2 P_a]_p \subset \mathscr{N}$$

Trivially $f_1 C_{E_2} \in L^\mu(dm) \subset L^2(dm)$. Hence

$$f_1 C_{E_2} \in \mathscr{N} \cap L^2(dm) = \mathscr{M} = C_E L^2(dm) .$$

Let $f_1 C_{E_2} = C_E g,\ g \in L^2(dm)$. Then $g \in L^\mu(dm)$. So

$$f = f_1 f_2 = f_1 C_{E_2} g',\ g' \in L^2(dm) = C_E g \cdot g' \in C_E L^p(dm) .$$

This shows $\mathscr{N} \subset C_E L^p(dm)$. The reverse inclusion is immediate from the invariance of $\mathscr{N}$. Hence $\mathscr{N} = C_E L^p(dm)$ in this case.

Case (ii). $2 < p \leqq \infty$:

Let $\mathscr{N}' = \{f \mid f \in L^{p'}, f \perp \mathscr{N}\}$ where $(1/p') + (1/p) = 1$. Then $\mathscr{N}'$ is a doubly invariant subspace of $L^{p'}$ and $1 \leqq p' < 2$. Hence $\mathscr{N}' = C_{E'} L^{p'}$ for some $E' \subset X$. Then $\mathscr{N} = C_E L^p$ where $E = X - E'$.

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