TOEPLITZ MATRICES AND INVERTIBILITY OF HANKEL MATRICES

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1. Introduction. Let $\{c_n\}$, for $n = 0, \pm 1, \pm 2, \cdots$, be a sequence of real numbers satisfying $c_0 = 0, c_{-n} = c_n$ and $0 < \sum_{n=1}^{\infty} c_n^2 < \infty$, and let $f(\theta)$ ($\not\equiv 0$) be the even function of class $L^2(-\pi, \pi)$ defined by

$$(1) \qquad f(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta} = 2 \sum_{n=1}^{\infty} c_n \cos n\theta \,.$$

Define the Toeplitz matrix T and the Hankel matrices H and K by

$$(2) \quad T = (c_{i-j}),\ H = (c_{i+j-1}) \text{ and } K = (c_{i+j}), \quad \text{where } i, j = 1, 2, \cdots .$$

Then

$$(3) \qquad T = F + K\,, \quad \text{where } F = \int_0^{\pi} f(\theta) dE_0(\theta)\,,$$

and $\{E_0(\theta)\}$ is the resolution of the identity of the matrix belonging to the quadratic form $2 \sum_{n=1}^{\infty} x_n x_{n+1}$. (See [12], p. 837.)

A self-adjoint operator A on a Hilbert space, with the spectral resolution $A = \int \lambda dE(\lambda)$, will be called absolutely continuous if $\| E(\lambda)x \|^2$ is an absolutely continuous function of λ for every element x of the Hilbert space. If the function $f(\theta)$ of (1) is (essentially) bounded then T must be bounded (Toeplitz). Since F must also be bounded, so also are H and K. It was shown in [12], p. 840, using methods involving commutators of operators, that if the function $g(\theta)$ defined by

$$(4) \qquad g(\theta) \sim \sum_{n=1}^{\infty} c_n e^{in\theta}$$

is bounded (hence $f(\theta)$ is also bounded) then T must be absolutely continuous if either

(5) 0 is not in the point spectrum of H (that is, H^{-1} exists),

or

(6) F is absolutely continuous.

Rosenblum [17] has shown, using results of Aronszajn and Donoghue [1], that in fact T is *always* (with no restrictions) absolutely continuous.

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In addition, it was shown in Putnam [12], using a theorem of Rosenblum [16], and generalized by Rosenblum in [17] using results of Kato [7], that if $\sum_{n=1}^{\infty} n\,|\,c_{n+1}| < \infty$ or, equivalently, if

$$\sum_{n=1}^{\infty} n\,|\,c_n\,| < \infty\ , \tag{7}$$

and if (6) holds, then T and F are unitarily equivalent, so that

$$T = UFU^*\ , \quad U \text{ unitary}\ . \tag{8}$$

The absolute continuity of F is equivalent to the requirement that

$$\text{meas}\,\{\,\theta : f(\theta)\,\varepsilon Z\} = 0 \quad \text{whenever meas } Z = 0\ . \tag{9}$$

In the present paper a sufficient condition, involving the *negation* of (5), for (6), that is, for the validity of (9), will be obtained.

Before stating the theorem it will be convenient to define the operators $F_k (k = 0, 1, 2, \cdots)$ by

$$F_k = \int_0^{\pi} f_k(\theta) dE_0(\theta)\ , \quad \text{where } f_k(\theta) \sim \sum_{n=1}^{\infty} c_n n^{-k} \cos n\theta\ . \tag{10}$$

(In particular, $F_0 = F$.)

There will be proved the following

THEOREM 1, *Suppose that*

$$0 \text{ is in the point spectrum of } H\ . \tag{11}$$

Then,

(a) *the point spectrum of* F *is empty, and*

(b) *each of the operators* $F_2, F_3, \cdots$ *is absolutely continuous.*

(c) *If, in addition to* (11), *it is assumed that* $\sum_{n=1}^{\infty} |\,c_n\,| < \infty$, *then* F_1 *is absolutely continuous.*

(d) *If, in addition to* (11), *relation* (7) *is assumed, then* (6) *holds.*

From part (d) of the theorem and the results mentioned earlier there follows the

COROLLARY. *Relations* (7) *and* (11) *imply* (8).

It will remain undecided whether (11) alone, without the additional assumption (7), is sufficient to imply not only the assertion of (a) but also (6). It is interesting to observe though that, if the implication (11) $\rightarrow$ (6) is valid, then either (5) or (6) must hold, and, at least if $g(\theta)$ is bounded, the absolute continuity of T (cf. [17]) can be deduced from the commutator methods of [12] (cf. also [11]) as noted above.

It is to be noted that the function $f(\theta)$ determines explicitly the operator F and its spectrum. On the other hand, the structure of T as determined by $f(\theta)$ is not so clear. It is known however that the spectrum of T, in case T is self-adjoint, is the interval $[m, M]$, where m and M denote the essential lower and upper bounds of $f(\theta)$ (Hartman and Wintner [6], pp. 868, 878). Although necessary and sufficient conditions involving $f(\theta)$, or rather $g(\theta)$, for the boundedness of H (Nehari [10]) and the complete continuity of H (Hartman [4]) are known, apparently no similar results are known relating the spectrum of H to the function $f(\theta)$. Concerning the spectrum of H in certain specific cases, see, e.g., Hartman and Wintner [6], p. 366, Magnus [8].

2. Proof of (a) of Theorem 1. Let $\{x_n\}$ and $\{d_n\}$, for $n = 1, 2, \cdots$, be two sequences of complex numbers satisfying $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ and $\sum_{n=1}^{\infty} |d_n|^2 < \infty$, let $x(\theta) \sim \sum_{n=1}^{\infty} x_n e^{in\theta}$ and $h(\theta) \sim \sum_{n=1}^{\infty} d_n e^{in\theta}$. Then it is easily verified that

$$(2\pi)^{-1}\int_{-\pi}^{\pi} x(\theta)(g^*(\theta) + h(\theta))e^{ij\theta}d\theta = \sum_{n=1}^{\infty} c_{n+j}x_n \tag{12}$$

holds for $j = 0, 1, 2, \cdots$, where the asterisk denotes complex conjugation. If $d_n = c_n$ then $g^*(\theta) + h(\theta) = f(\theta)$ and so 0 is in the point spectrum of H if and only if

$$\int_{-\pi}^{\pi} x(\theta)f(\theta)e^{ij\theta}d\theta = 0\ , \quad \text{where } j = 0, 1, 2, \cdots\ , \tag{13}$$

holds for some $x(\theta) \not\equiv 0$ as defined above. Relation (13) implies that the function $x(\theta)f(\theta)$, of class $L(-\pi, \pi)$, has a Fourier series of the form

$$x(\theta)f(\theta) \sim \sum_{n=0}^{\infty} a_n e^{in\theta}\ . \tag{14}$$

For a fixed constant p, $0 < p < \infty$, consider the class H_p (after Hardy; see, e.g., Zygmund [19], p. 158) of functions $A(z)$ analytic in the disk $|z| < 1$ and for which $\int_{-\pi}^{\pi} |A(re^{i\theta})|^p d\theta$ remains bounded for $0 \leqq r < 1$. If $p \geqq 1$, the class L^{p+} of functions $B(\theta) \varepsilon L^p(-\pi, \pi)$ with Fourier series of the form

$$B(\theta) \sim \sum_{n=0}^{\infty} b_n e^{in\theta} \quad \left(b_n = (2\pi)^{-1}\int_{-\pi}^{\pi} B(\theta)e^{-in\theta}d\theta\right)\ , \tag{15}$$

coincides with the class of boundary functions $B(\theta) = A(e^{i\theta})$; see Rogosinski and Shapiro [15], p. 293. Furthermore, it is known that if $p > 0$ and if $A(z)$ is of class H_p and if $A(z) \not\equiv$ const., then $A(e^{i\theta}) = a$, for an arbitrary constant a, can hold at most on a set of measure zero. For $p = 2$, this result is due to F. and M. Riesz ([14]); for $p \neq 2$, see F. Riesz [13].

Returning to (14), since $x(\theta)f(\theta)\varepsilon L^{1+}$, it follows that $f(\theta) \neq 0$ almost everywhere. A similar argument with $x(\theta)f(\theta)$ replaced by $x(\theta)(f(\theta) - a)$, for any constant a, shows that $f(\theta) \neq a$ almost everywhere, that is,

$$\text{meas } \{\theta : f(\theta) = a\} = 0 . \tag{16}$$

But (16) holds if and only if the operator F has no point spectrum and the proof (a) is complete.

3. Proof of (b) of Theorem 1. In order to show that F_2 is absolutely continuous, it must be shown that the set $S_2 = \{\theta : f_2(\theta) \in Z\}$ is a zero set whenever Z is a zero set. Since $\sum_{n=1}^{\infty} | c_n n^{-1} | < \infty$, $f_2'(\theta)$ is continuous and the set $\{\theta : f_2'(\theta) \neq 0\}$ is open. If its canonical decomposition is the finite or infinite union of open intervals I_n ($n = 1, 2, \cdots$), then $f_2(\theta)$ is strictly monotone on each I_n. Also, on I_n, both f_2 and its inverse g_n are absolutely continuous. Since $I_n \cap S_2$ is the image under g_n of a subset of Z, it follows (cf., e.g., Natanson [9], p. 249) that

$$I_n \cap S_2 \text{ has measure } 0 . \tag{17}$$

If it is shown that $f_2'(\theta) \neq 0$ almost everywhere, it will follow from (17) that meas $S_2 = 0$, as was to be proved.

In order to prove that $f_2'(\theta) \neq 0$ almost everywhere, note that $f_2'(\theta)$ is absolutely continuous and that $f_2''(\theta) = (-1/2)f(\theta)$ almost everywhere. Hence, if $f_2'(\theta) = 0$ on a set of positive measure, then also $f(\theta) = 0$ on a set of positive measure, a contradiction. Hence F_2 is absolutely continuous.

Next, it will be shown that F_3 is absolutely continuous. In the definition of $h(\theta)$, choose $d_n = -c_n$, so that in (12), $k(\theta) = g^*(\theta) + h(\theta) = 2i \sum_{n=1}^{\infty} c_n \sin n\theta$. The argument of § 2 shows that $x(\theta)k(\theta)$ is of class L^{1+} and hence $k(\theta) \neq 0$ almost everywhere. Since $f_3'(\theta)$ is continuous, and since $f_3'''(\theta) = (1/2i)k(\theta)$, an argument similar to that used above shows that F_3 is absolutely continuous.

In like manner, it follows that $F_4, F_5, \cdots$ are absolutely continuous and the proof of (b) is complete.

4. Proof of (c) of Theorem 1. In order to prove the absolute continuity of F_1, it must be shown that the set $S_1 = \{\theta : f_1(\theta) \in Z\}$ is a zero set whenever Z is a zero set. The hypothesis of (c) implies that $f_1'(\theta) = (-1/2i)k(\theta)$ is continuous. Since $k(\theta) \neq 0$ almost everywhere, a relation similar to (17) implies that meas $S_1 = 0$, and the proof of (c) is complete.

5. Proof of (d) of Theorem 1. Since (7) implies that $f'(\theta)$ is continuous, then $x^2(\theta)f'(\theta)$ is of class $L(-\pi, \pi)$. It will be shown that $x^2(\theta)f'(\theta)$ is also of class L^{1+}, so that

$$x^2(\theta)f'(\theta) \sim \sum_{n=0}^{\infty} b_n e^{in\theta} , \tag{18}$$

and hence (cf. the above reference to [15]) the F. and M. Riesz theorem can be applied to yield $f'(\theta) \neq 0$ almost everywhere. Once this has been shown, the absolute continuity of F follows by an argument similar to that used above.

There remains then to prove (18). Since $f(\theta)$ is now bounded, it follows from the definition of $x(\theta)$ and (14) that both $x(\theta)$ and $x(\theta)f(\theta)$ belong to L^{2+}. Let $u(z)$ and $v(z)$ denote the functions analytic in $|z| < 1$ and possessing the respective boundary functions $x(\theta)$ and $x(\theta)f(\theta)$. Let $U(\theta) = u(e^{i\theta})$ and $V(\theta) = v(e^{i\theta})$, so that $x^2(\theta)f'(\theta) = U^2(\theta)(V(\theta)/U(\theta))'$.

A heuristic argument leading to (18) is the following. Let U' and V' be defined by the formal trigonometrical series obtained by term by term differentiation of the corresponding series for U and V, and suppose that $U^2(V/U)' = UV' - U'V$ is meaningful. Since the trigonometrical series for U, V, U' and V' are of the type $\sum_{n=0}^{\infty} f_n e^{in\theta}$ then so also are the products UV' and $U'V$ as well as their difference.

A rigorous proof of (18) can be given as follows. Let the Fourier series of $U(\theta)$ and $V(\theta)$ be given by

$$U(\theta) \sim \sum_{n=0}^{\infty} a_n e^{in\theta} , \quad V(\theta) \sim \sum_{n=0}^{\infty} b_n e^{in\theta} . \tag{19}$$

Since $V(\theta) = U(\theta)f(\theta)$, where $U(\theta)$ and $f(\theta)$ each belongs to class $L^2(-\pi, \pi)$, then $\sum_{n=0}^{\infty} a_k c_{n-k} = b_n$ for $n = 0, 1, 2, \cdots$, and

$$\sum_{k=0}^{\infty} a_k c_{n-k} = 0 \text{ for } n = -1, -2, \cdots ; \tag{20}$$

cf. Zygmund [19], p. 90. Note that the convergence of the series defining the b_n is assured by the Schwarz inequality. Similarly, the Fourier series of $U^2(\theta)$ is given by

$$U^2(\theta) \sim \sum_{n=0}^{\infty} A_n e^{in\theta} , \quad A_n = \sum_{k=0}^{n} a_{n-k} a_k . \tag{21}$$

Since, by (7),

(22) $$f'(\theta) \sim \sum_{n=-\infty}^{\infty} inc_n e^{in\theta} ,$$

and, since $x^2(\theta) = U^2(\theta)$ is of class $L(-\pi, \pi)$ and $f'(\theta)$ is bounded, the Fourier series of $x^2(\theta)f'(\theta)$ is given by

(23) $$x^2(\theta)f'(\theta) \sim \sum_{n=-\infty}^{\infty} B_n e^{in\theta} , \quad B_n = i \sum_{k=-\infty}^{\infty} A_{n-k} k c_k ;$$

cf. Zygmund [19], p. 90.

Since $U^2(\theta)\varepsilon L(-\pi, \pi)$ then, by the Riemann-Lebesgue lemma, $A_n \to 0$ as $n \to \infty$, and the absolute convergence of each of the series defining the B_n is assured by (7). Also the same assertion holds for the series corresponding to the above B_n but where $U(\theta)$ is replaced by the function with the Fourier series $\sum_{n=0}^{\infty} | a_n | e^{in\theta}$. Since $B_n = i\sum_{m=0}^{\infty} A_m(n - m)c_{n-m}$, this implies that each of the iterated series

(24) $$B_n = i \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} a_{m-k} a_k (n - m) c_{n-m}$$

is absolutely convergent. Consequently, an interchange of the order of summation leads to

(25) $$B_n = i \sum_{k=0}^{\infty} a_k \Big[(n - k) \sum_{p=0}^{\infty} a_p c_{n-k-p} - \sum_{p=0}^{\infty} p a_p c_{n-k-p} \Big] .$$

On reversing the order of summation in the second iterated sum, it follows from (20) that $B_n = 0$ for $n = 0, -1, -2, \cdots$, so that (18) follows from (23). This completes the proof of Theorem 1.

6. Some dual results. A theorem similar to Theorem 1 but with the cosines replaced by sines is valid. In particular, whereas (a) of Theorem 1 states that (11) implies (16) while (d) states that (11) and (7) imply (9), the duals of these assertions become the following

THEOREM 2. *Let* $j(\theta)$ *be defined by*

(26) $$j(\theta) \sim 2 \sum_{n=1}^{\infty} c_n \sin n\theta ,$$

and suppose that (11) *holds. Then, for every constant* α,

(27) $$\text{meas } \{\theta : j(\theta) = \alpha\} = 0 .$$

If, in addition to (11), *relation* (7) *is assumed, then*

(28) $$\text{meas } \{\theta : i(\theta)\varepsilon Z\} = 0 \text{ whenever meas } Z = 0 .$$

The proof follows from the observation that the function $k(\theta) = ij(\theta)$ considered in the beginning of § 3 plays a role similar to that of $f(\theta)$.

7. Remarks. If $A(z) \varepsilon H_p$, then $B(\theta) = A(e^{i\theta})$ satisfies, for every constant α, not only

(29) $$\text{meas } \{\theta : B(\theta) = \alpha\} = 0, \text{ unless } B(\theta) \equiv \alpha \,,$$

but even

(30) $$\int_{-\pi}^{\pi} |\log |B(\theta) - \alpha|| \, d\theta < \infty \,.$$

This result was proved by Szego [18] for $p = 2$. Its validity for arbitrary $p > 0$ was pointed out by F. Riesz ([13], pp. 91–92) to be a consequence of his factorization theorem for functions of class H_p. Thus, for every constant α, relations (16) and (27), and even

(31) $$\int_{-\pi}^{\pi} |\log |f(\theta) - \alpha|| \, d\theta < \infty \text{ and } \int_{-\pi}^{\pi} |\log |j(\theta) - \alpha|| \, d\theta < \infty \,,$$

are seen to be necessary conditions in order that 0 be in the point spectrum of H, or, what is the same thing, in order that the translated sequences $(c_1, c_2, \cdots)$, $(c_2, c_3, \cdots)$, $\cdots$ fail to form a fundamental set for the Hilbert space l^2 of vectors $x = (x_1, x_2, \cdots)$ with $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. (In connection with this latter form of (11), it is interesting to compare the present situation relating to the completeness of shifted sequences, with a similar, but different one considered in the papers of Beurling [2] and Halmos [3].) That the condition (31) is not sufficient for 0 to be in the point spectrum of H can be seen for the case $c_n = 1/n$ $(n = 1, 2, \cdots)$. Then $f(\theta)$ of (1) becomes $-2 \log (2 |\sin (\theta/2)|$ and $j(\theta)$ of (26) becomes the odd function on $(-\pi, \pi)$ defined on $(0, \pi)$ by $j(\theta) = \pi - \theta$, and so (31) holds for every constant α. However, 0 is not in the point spectrum of $H = ((i + j - 1)^{-1})$; in fact, the spectrum of H is known to be purely continuous (Magnus [8]).

Since (7) holds if, say, $f''(\theta)$ is continuous, it follows from the Theorems 1 and 2 that for such functions f, in order that (11) hold, not only (16) and (27), but even the more restrictive conditions (9) and (28) must be satisfied. It is to be noted that even if, say, $f''(\theta)$ is continuous, (16) does not imply (9). In order to see this, let C denote a closed, nowhere dense (Cantor) set of *positive* measure on $[0, \pi]$, and define a function $q(\theta)$ so as to have a continuous derivative on $[0, \pi]$ and satisfy $q(\theta) = 0$ on C and $q(\theta) > 0$ on $[0, \pi] - C$.

Then $q(0) = q'(0) = 0$ and $f(\theta) = \int_0^\theta q(u)du$ is a strictly increasing function on $[0, \pi]$; hence, if $f(-\theta) = f(\theta)$ for $0 \leqq \theta \leqq \pi, f(\theta)$ is of the form (1), has a continuous second derivative, and satisfies (16). If T denotes the image under f of the set C, then T is measurable and meas $T = \int_c |df| = \int_c q(\theta)d\theta = 0$, so that (9) fails to hold with $T = Z$.

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