ATOMIC ORTHOCOMPLEMENTED LATTICES

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Introduction. The lattice of all closed subspaces of a separable Hilbert space has the following properties. It is complete, atomic, irreducible, semi-modular, and orthocomplemented. The primary purpose of this paper is to investigate lattices with these properties.

If L is such a lattice, there is a representation theorem for L. The elements in L of finite dimension or finite deficiency form an orthocomplemented modular lattice. It follows that if the dimension of L is high enough, then there is a dual pair of vector spaces U and W such that L is isomorphic to the lattice of W closed subspaces of U. Because L is orthocomplemented the spaces U and W are isomorphic. This isomorphism establishes a "semi-inner product" on U, and L may be described as being the lattice of closed subspaces of a semi-inner product space.

The contents of the paper are as follows. Section 1 contains some definitions and establishes notation. Section 2 is concerned with the completion of an orthocomplemented lattice and § 3 with the center of such a lattice. With the exception of Theorem 3.2 the techniques used in §§ 2 and 3 are standard, and many of the results are widely known. To the best of the author's knowledge, however, the theorems have not previously appeared in print. Therefore we state and prove them in some detail. The representation theorem and other results centering about the semi-modularity condition are proved in § 4. With the other conditions holding for L, semi-modularity is equivalent to certain covering conditions. Because this is not true for arbitrary complete atomic lattices, the results seem to be of some interest. Finally, in § 5, semi-inner product spaces are discussed. A theorem is given relating the existence of a semi-inner product on U to the existence of an orthocomplemented lattice of subspaces of U. This is an easy generalization of a theorem of Birkhoff and von Neumann [4] (Appendix). In two other theorems we investigate the exact relation between the semi-inner product on U and the orthocomplemented lattice L.

1. Definitions and some elementary lemmas. Let S be a partially-ordered set. If a and b are elements of S, we denote the least upper bound or join of a and b by $a \vee b$, provided that the join exists. We denote the greatest lower bound or meet of a and b by ab, provided

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that the meet exists. If A is a subset of S which has a least upper bound, we denote the least upper bound by $\bigvee A$. If the elements of A are indexed by a set J, we may also write $A = \bigvee_j a_j$. If A has a greatest lower bound, it is denoted by $\bigwedge A$ or $\bigwedge_j a_j$.

The symbols $\bigcup$ and $\bigcap$ will be used to denote set union and set intersection respectively.

If a and b are elements of a partially ordered set S with $a \leqq b$, we will denote the set of all $x \in S$ such that $a \leqq x \leqq b$ by $[a, b]$.

A partially-ordered set S is said to be *orthocomplemented* if it contains at least element 0 and a greatest element 1, $1 \neq 0$, and if there exists a map $a \to a'$ of S onto itself which satisfies

(1.1) $a \leqq b$ implies $a' \geqq b'$,

(1.2) $a'' = a$,

(1.3) a' is a complement of a, i.e., $aa' = 0$ and $a \vee a' = 1$.

The mapping $a \to a'$ is called an orthocomplementation, and a' is called the orthocomplement of a.

Two elements a and b of S are said to be *orthogonal* if $a \leqq b'$. In this case we write $a \perp b$. The relation of being orthogonal is obviously symmetric.

We will use the following simple lemmas throughout this paper.

LEMMA 1.1. *Let S be an orthocomplemented partially-ordered set, and let $\{a_j\}$ be a subset of S such that $\bigvee_j a_j$ exists. Then $\bigwedge_j a_j'$ exists, and $(\bigvee_j a_j)' = \bigwedge_j a_j'$.*

LEMMA 1.2. *Let A be a subset of an orthocomplemented partially-ordered set S, and suppose that $\bigvee A$ exists. Then if $b \perp a$ for all $a \in A$, $b \perp \bigvee A$.*

An *isomorphism* of a partially-ordered set S onto a partially-ordered set R is a one-to-one mapping θ from S onto R, such that $\theta(x) \leqq \theta(y)$ if and only if $x \leqq y$. An isomorphism preserves any meets and joins which exist. When S and R are orthocomplemented we will say an isomorphism θ is an *ortho-isomorphism* if $\theta(x') = \theta(x)'$ for all x in S.

LEMMA 1.3. *Let S and R be orthocomplemented lattices, and let θ be a one-to-one map of S onto R. Then θ is an ortho-isomorphism if and only if* (1) *$\theta(x') = \theta(x)'$ for all x in S and* (2) *$\theta(xy) = \theta(x)\theta(y)$ for all x and y in S or $\theta(x \vee y) = \theta(x) \vee \theta(y)$ for all x and y in S.*

Let a and b be elements in a partially-ordered set S. a is said to *cover* b if $a > b$, and there does not exist c in S with $a > c > b$. If S has a least element 0, an *atom* is an element of S which covers

0. A lattice S is *atomic* if every element of S is the join of some set of atoms.

2. Completion of an orthocomplemented partially-ordered set.

A partially-ordered set S is said to be *complete* if $\bigwedge A$ and $\bigvee A$ exist for all subsets A of S. If S has both a least element and a greatest element, then $\bigvee A$ exists for all subsets A of S if and only if $\bigwedge A$ exists for all subsets A of S. The standard method for embedding a partially-ordered set S in a complete lattice is to use the completion of S by cuts.[1] If S is orthocomplemented, the completion can be constructed in another way by using the orthogonality relation. This is just the construction used in the standard proof that the completion of a Boolean algebra is a Boolean algebra.

In Theorems 2.1 and 2.2 we show that the partial ordering in an orthocomplemented partially-ordered set can be found if one knows only which elements are orthogonal. This fact suggests that we define an abstract notion of an orthogonality relation.

Let S be any set. We will say that the binary relation $\perp$ is an *orthogonality relation* if it has the following properties.

(1) $a \perp b$ implies $b \perp a$.

(2) $a \perp a$ implies $a \perp b$ for all b in S.

(3) ($c \perp a$ if and only if $c \perp b$) implies $a = b$.

THEOREM 2.1. *If S is a set with an orthogonality relation ($\perp$), then a partial ordering ($\leqq$) may be defined on S: $a \leqq b$ if and only if $d \perp b$ implies $d \perp a$.*

Proof. If $a \leqq b$ and $b \leqq a$, then $d \perp a$ if and only if $d \perp b$. Therefore $a = b$, by the definition of an orthogonality relation. If $a \leqq b$ and $b \leqq c$, $a \leqq c$ by the definition of $\leqq$ in S.

THEOREM 2.2. *If S is an orthocomplemented partially-ordered set, then the relation $\perp$, where $a \perp b$ if and only if $a \leqq b'$, is an orthogonality relation. Further the partial ordering induced by this orthogonality relation coincides with the original partial ordering.*

Proof. The relation $\perp$ is symmetric, because $a \leqq b'$ if and only if $b \leqq a'$. If $a \perp a$, $a \leqq a'$. Hence $a \leqq aa' = 0$, and $0 \perp x$ for all x in S. Finally suppose that a and b are elements of S such that $c \perp a$ implies $c \perp b$. Then $a' \perp b$, i.e., $b \leqq a$. Thus the relation $\perp$ induces the original partial ordering in S. Further if a and b are such that $d \perp a$ if and only if $d \perp b$, we have $b \leqq a$ and $a \leqq b$, i.e., $a = b$.

[1] See Birkhoff [**3**], Ch. 4, sec. 7.

We will now assume that S is a set with an orthogonality relation $(\perp)$, and that the partial ordering of Theorem 2.1 has been defined on S. S may be an orthocomplemented partially-ordered set, but it does not have to be. For any subset A of S let $A^\perp$ be the set of all x in S such that $x \perp a$ for all a in A. Let $A^- = A^{\perp\perp}$, and call a subset *closed* if $A = A^-$.

We will use the following simple lemma throughout this paper. Its proof uses only familiar arguments.

LEMMA 2.1. *Let S be a set with an orthogonality relation. Let A and B be subsets of S. Then the following relations hold.*

(1) *If* $A \subseteqq B$, $B^\perp \subseteqq A^\perp$

(2) $A^\perp = A^{\perp\perp\perp}$

(3) $A \subseteqq A^-$

(4) $A^{--} = A^-$

(5) *If* $A \subseteqq B$, $A^- \subseteqq B^-$

(6) $(A \cup B)^\perp = A^\perp \cap B^\perp$.

An orthogonality relation is just a special type of polarity as defined by Birkhoff.[2] Thus in the following theorem the assertion that the closed subsets of S form a complete orthocomplemented lattice follows from Theorem 9 and Corollary Ch. 4, of [3]. Since the rest of the proof is quite standard, we omit it.

THEOREM 2.3. *Let S be a set with an orthogonality relation and with the partial ordering induced by the orthogonality relation. Then the closed subsets of S, partially ordered by inclusion, form a complete lattice $L(S)$. If $\{A_j\}$ is a family of closed subsets, $\bigwedge_j A_j$, the meet of the A_j in $L(S)$, is just $\bigcap_j A_j$. The mapping $A \rightarrow A^\perp$ is an orthocomplementation in $L(S)$. Further there exists a one-to-one mapping of S into $L(S)$ which preserves orthogonality, order, and all existing meets in S. If S is orthocomplemented this map also preserves orthocomplements and all joins existing in S.*

From now on we will always use $L(S)$ to denote the lattice of closed subsets of S. The following theorem justifies our calling $L(S)$ the completion of S.

THEOREM 2.4. *If S is an orthocomplemented partially-ordered set, $L(S)$ is the completion of S by cuts.*

Proof. If A is a subset of S, let $A^* = \{x \in S \mid x \geqq a$ for all a in $A\}$ and let $A^\circ = \{x \in S \mid x \leqq a$ for all a in $A\}$. The completion by cuts

[2] [3], Ch. 4, sec. 5.

of S is the lattice of all subsets A of S such that $A^{*\circ} = A$[3]. We need only show that $A^{\perp\perp} = A^{*\circ}$ for all subsets A of S. Let $y \in A^*$. Then $y \geqq a$ for all a in A, and hence $y' \leqq a'$ for all a in A. This means y' is in $A^\perp$. Therefore if $x \in A^{\perp\perp}$, $x \perp y'$, i.e., $x \leqq y$. Thus if $x \in A^{\perp\perp}$, $x \leqq y$ all y in A^*, i.e., $x \in A^{*\circ}$. This proves that $A^{\perp\perp} \subseteqq A^{*\circ}$. Now if $y \in A^\perp$, $y' \geqq a$ for all a in A, i.e., $y' \in A^*$. Thus $x \in A^{*\circ}$ implies $x \leqq y'$ all $y \in A^\perp$, i.e., $x \in A^{\perp\perp}$. Therefore $A^{*\circ} \subseteqq A^{\perp\perp}$; $A^{*\circ} = A^{\perp\perp}$.

We will say that a subset I of a partially-ordered set S is *join dense* if every element of S is the join, perhaps infinite, or elements in I. The advantage of using the orthogonality relation to construct the completion of S is that only a join-dense subset of S is actually needed for the construction.

THEOREM 2.5. *Let S be an orthocomplemented partially-ordered set, and let I be a join-dense subset of S. Let $\perp$ be the orthogonality relation in S. Then restricted to I, $\perp$ is an orthogonality relation. Further the partial ordering induced in I by the orthogonality relation $\perp$ coincides with the partial ordering inherited from S. Finally $L(S)$ and $L(I)$ are ortho-isomorphic.*

Proof. (2.1) and (2.2) in the definition of an orthogonality relation are obviously satisfied, because $\perp$ is an orthogonality relation in S. Let a and b be elements of I such that for $c \in I$, $c \perp a$ if and only if $c \perp b$. Let d be an element of S such that $d \perp a$. Since I is join-dense in S, there exist c_j in I such that $d = \bigvee_j c_j$. For each c_j, $c_j \perp a$. Hence $c_j \perp b$. Therefore in S $d \perp b$. Similarly $d \perp b$ implies $d \perp a$. Therefore $a = b$. This proves that (2.3) is satisfied by the relation restricted to I and thus proves that it is an orthogonality relation on I. Now let $\leqq$ be the partial ordering in S, and let $\prec$ be the partial ordering induced in I by the orthogonality relation. If x, y, z are all in I with $x \leqq y$ and $z \perp y$, then $z \perp x$. Thus if x and y are in I with $x \leqq y$ we have $x \prec y$. Now suppose that $x \prec y$. Since I is join-dense $y' = \bigvee_j z_j$ with $z_j \in I$. Clearly $z_j \perp y$ for each j, and hence $z_j \perp x$. Therefore $x \perp \bigvee_j z_j = y'$, i.e. $x \leqq y$. This proves that the two partial orderings $\leqq$ and $\prec$ are the same. To complete the proof of the theorem we must prove that $L(S)$ and $L(I)$ are ortho-isomorphic. We first show that if A is a closed subset of S, $A \cap I$ is closed in I. Note that if B is any set closed in S, and if $x \perp y$ for all $y \in B \cap I$, then $x \perp y$ for all $y \in B$, because I is join dense. In other words, if $B \in L(S)$, $(B \cap I)^\perp = B^\perp$. Now the closure of $A \cap I$ in I is $((A \cap I)^\perp \cap I)^\perp \cap I$. If $A \in L(S)$, we have $((A \cap I)^\perp \cap I)^\perp \cap I = (A^\perp \cap I)^\perp \cap I = A^{\perp\perp} \cap I = A \cap I$, i.e. $A \cap I$ is closed in I. Now for

[3] See [3], Ch. 4., Sec. 7.

any subset B of S, let B^- denote the closure of B in S. We next show that if B is closed in I, $B^- \cap I = B$. Suppose that $x \in B^- \cap I$. Then $x \perp y$ for all $y \in B^\perp$. In particular $x \perp y$ for all $y \in B^\perp \cap I$. Therefore $x \in B$, because B is closed in I. Clearly $B \subseteqq B^- \cap I$, so we have $B = B^- \cap I$. Now define a map θ from $L(S)$ to $L(I)$ by $\theta(A) = A \cap I$. We have shown above that if $A \in L(S)$, $A \cap I$ is in $L(I)$. If $B \in L(I)$, $\theta(B^-) = B^- \cap I = B$, so θ is onto. If $A \in L(S)$, $(A \cap I)^- = A$, because I is join dense in S. Therefore θ is one-to-one. Clearly $\theta^{-1}(B) = B^-$, and therefore $\theta(A) \leqq \theta(B)$ if and only if $A \leqq B$. Thus θ is an isomorphism. Finally note that the orthocomplementation in $L(I)$ is $A \rightarrow A^\perp \cap I$. But $\theta(A^\perp) = A^\perp \cap I = (A \cap I)^\perp \cap I$. Therefore $\theta(A^\perp) = (\theta(A))^\perp \cap I$. Thus θ preserves orthocomplements.

3. Center of an orthocomplemented lattice. Let $S_j (j \in J)$ be a family of lattices. Let P be the Cartesian product of the S_j, i.e., P is the set of all functions f from J to $\bigcup_j S_j$ such that $f(j) \in S_j$ for all j in J. P has a natural partial ordering: $f \leqq g$ in P if and only if $f(j) \leqq g(j)$ for all j in J. It is easy to verify that this ordering makes P into a lattice. Meets and joins are: $(fg)(j) = f(j)g(j)$ and $(f \vee g)(j) = f(j) \vee g(j)$. P is sometimes called the cardinal product of the S_j, but we will follow von Neumann and call P the direct sum of the S_j. We will write $P = \Sigma \oplus S_j$. We will denote the direct sum of S_1 and S_2 by $S_1 \oplus S_2$. $S_1 \oplus S_2$ may be regarded as the set of all ordered pairs (x_1, x_2) with $x_1 \in S_1$ and $x_2 \in S_2$. The following Theorem is obvious.

THEOREM 3.1. *Let S_j be a family of lattices. Then $P = \Sigma \oplus S_j$ is orthocomplemented if and only if each S_j is orthocomplemented. P is complete if and only if each S_j is complete.*

Now suppose that P is ortho-isomorphic to the direct sum of two orthocomplemented lattices, $P \cong S_1 \oplus S_2$. Let a be the element of P corresponding to $(1, 0)$. Then a' corresponds to $(0, 1)$, S_1 is ortho-isomorphic to $[0, a]$, and S_2 is ortho-isomorphic to $[0, a']$. In this case it will be convenient to write $P = [0, a] \oplus [0, a']$. The *center* of P is the set of all elements a such that $P = [0, a] \oplus [0, a']$. The elements 0 and 1 are always in the center. If the center of P contains only 0 and 1, will say that P is *irreducible*. The next theorem is suggested by a similar result of von Neumann on the center of a continuous geometry.

THEOREM 3.2. *Let P be an orthocomplemented lattice. Then for an element a of P the following three conditions are equivalent.*

(1) $x = xa \vee xa'$ *for all x in P.*

(2) *$(x \vee a)y = xy \vee ay$ for all x and y in P.*
(3) *a is in the center of P.*

Proof. That (1) and (2) hold for central elements is well known. Suppose that a has property (1). Let $x \leqq a$ and $y \leqq a'$. Then $x' \geqq a'$ and $y' \geqq a$, so $x'y'a' = y'a'$ and $x'y'a = x'a$. Thus $x'y' = x'y'a \vee x'y'a' = x'a \vee y'a'$. Taking orthocomplements, we get that for $x \leqq a$ and $y \leqq a'$, $(x \vee y) = (x \vee a')(y \vee a)$. Now define a map θ from P to $[0, a] \oplus [0, a']$ by $\theta(x) = (xa, xa')$. If $\theta(x) = \theta(y)$, $xa = ya$ and $xa' = ya'$. Hence by (1) $x = xa \vee xa' = ya \vee ya' = y$. Thus θ is one-to-one. Let (x, y) be an element of $[0, a] \oplus [0, a']$. Then $x \leqq a$ and $y \leqq a'$. $x' \vee a' = x'a \vee x'a' \vee a' = x'a \vee a'$. Taking orthocomplements, we get $xa = (x \vee a')a$. As was shown above, $(x \vee y) = (x \vee a')(y \vee a)$. Thus $(x \vee y)a = (x \vee a')(y \vee a)a = (x \vee a')a = xa$. Similarly $(x \vee y)a' = y$, i.e., $\theta(x \vee y) = (x, y)$. It is now clear that $\theta^{-1}((x, y)) = x \vee y$, and that $\theta(x) \leqq \theta(y)$ if and only if $x \leqq y$. Thus θ is an isomorphism, and obviously it is an ortho-isomorphism. $P = [0, a] \oplus [0, a']$, i.e., a is in the center. Now if a has property (2), $(a \vee a')x = ax \vee a'x$ for all x in P. Thus a has property (1); a is in the center.

THEOREM 3.3. *If P is a complete, atomic orthocomplemented lattice, the center of P is a complete, atomic Boolean algebra.*

Proof. To prove that the center is a complete Boolean algebra, we need only show that for any subset A of the center $\bigvee A$ is in the center. Let $b = \bigvee A$, and let p be an atom such that $pb = 0$. Then $pa = 0$ for all a in A. Therefore $p' \geqq a$ for all a in A, because $p = pa \vee pa'$ for all a in A. Thus $p' \geqq b$, i.e., $p \leqq b'$. If $pb \neq 0, p \leqq b$, because p is an atom. Thus for every atom $p, p \leqq b$ or $p \leqq b'$. Because P is atomic this means $x = xb \vee xb'$ for all x in P, i.e., b is in the center. To show that the center is atomic, let p be any atom in P, let A be the set of all central elements a such that $p \leqq a$, and let $b = \bigwedge A$. Then b is in the center, and $b \neq 0$. Further b must be an atom of the center, for if not there exists c in the center such that $0 < c < b$. Then $p = pb = pbc \vee pbc'$, so either $p \leqq c$ or $p \leqq c'b$. Thus either $c \in A$ or $c'b \in A$, so either $b \leqq c$ or $b \leqq c'b$. This contradicts the assumption that $0 < c < b$.

LEMMA. *Let L be a complete orthocomplemented lattice, and let a be in the center of L. Then for any family $\{x_j\}$ of elements in L, $a(\bigvee_j x_j) = \bigvee_j (ax_j)$.*

Proof.
$$a(\textstyle\bigvee_j x_j) = a(\bigvee_j (x_j a \vee x_j a')) = a(\bigvee_j (x_j a) \vee \bigvee_j (x_j a'))$$
$$= a(\textstyle\bigvee_j (x_j a)) \vee a(\bigvee_j (x_j a')) = \bigvee_j (x_j a) .$$

THEOREM 3.4. *Let L be a complete, atomic, orthocomplemented lattice. Then L is ortho-isomorphic to the direct sum of irreducible, atomic, orthocomplemented lattices.*

Proof. Let $\{a_j\}$ be the set of all atoms of the center. Let $S_j = [0, a_j]$. Define a mapping θ from L to $\Sigma \oplus S_j$ by $\theta(x)(j) = xa_j$. Let y be in $\Sigma \oplus S_j$, and let $x = \bigvee_j y(j)$. Then $\theta(x)(k) = (\bigvee_j y(j))a_k = \bigvee_j (y(j)a_k) = y(k)$. Thus $\theta(x) = y$; θ is onto. Let p be an atom, and a_j be an atom of the center. Then $p = pa_j$, or $p \perp a_j$. Since $\bigvee_j a_j = 1$, $p = \bigvee_j (pa_j)$. Since L is atomic it follows that $x = \bigvee_j (xa_j) = \bigvee_j \theta(x)(j)$. Therefore θ is one-to-one. Clearly $\theta(x) \leqq \theta(y)$ if and only if $x \leqq y$, so θ is an isomorphism. Further $\theta(x)'(j) = (xa_j)'a_j = x'a_j = \theta(x')(j)$. Thus $\theta(x') = \theta(x)'$. To complete the proof we need only show that each S_j is irreducible. Suppose that $0 \leqq b \leqq a_j$, and that b is in the center of S_j. Then for $x \leqq a_j$, $x = xb \vee xb'a_j$. Hence for all x in L,

$$x = xa_jb \vee xa_jb' \vee xa_j' = xb \vee (a_jb' \vee a_j')x = xb \vee xb' .$$

Thus b is in the center of L. Since a_j is an atom of the center, this means that $b = 0$ or $b = a_j$. This proves that $[0, a_j]$ is irreducible.

4. Semi-modular, atomic, orthocomplemented lattices. Let S be an atomic lattice. Let $x_0 < x_1 < \cdots < x_n$ be a finite chain of elements in S. We will call the integer n the *length* of chain. The chain is a *covering* chain if x_{i+1} covers x_i for $i = 0, 1, \cdots$. We define a function d on the set of ordered pairs (x, y) of elements in S with $x \leqq y$ as follows. If there exists a finite covering chain connecting x and y, $d(x, y)$ is the length of the shortest such covering chain. If no such covering chain exists, $d(x, y) = \infty$.

We will call an element $x \in S$ finite if x is the join of a finite number of atoms. Clearly if $d(0, x)$ is finite, then x is finite. We will let $F(S)$ denote the set of $x \in S$ such that x is finite or x' is finite.

THEOREM 4.1. *Let S be an atomic lattice such that if a and b are finite elements of S which both cover ab, then $a \vee b$ covers a and b. Then the set of all finite elements of S is an ideal. For any finite elements $a \leqq b$, $d(a, b)$ is finite, and all covering chains connecting a and b have the same length.*

Proof. We first show that if p is an atom, $d(0, a)$ is finite, and $pa = 0$, then $p \vee a$ covers a. We will prove this by induction on $d(0, a)$. If $d(0, a) = 1$, a and p both cover $ap = 0$. Therefore $a \vee p$

covers a and p. Suppose the statement is true for $d(0, a) \leqq n$. Let $d(0, a) = n + 1$, and let p be an atom with $pa = 0$. We need only prove that $p \vee a$ covers a. Because $d(0, a) = n + 1$, there exists b such that a covers b, and $d(0, b) = n$. Now $p \vee b$ covers b, and $(p \vee b)a = b$. Therefore $(p \vee b) \vee a$ covers a, i.e., $p \vee a$ covers a. Next we show that if a is finite $d(0, a)$ is finite. For finite a let $N(a)$ be the smallest number N such that a is the join of N atoms. We will prove this lemma by induction on $N(a)$. Clearly if $N(a) = 1$, $d(0, a) = 1$. Suppose the statement is true for $N(a) \leqq n$, and let a be a finite element with $N(a) = n + 1$. There exists atoms $p_1, \cdots, p_{n+1}$ such that $a = p_1 \vee \cdots \vee p_{n+1}$. Let $b = p_1 \vee \cdots \vee p_n$. Then $b < a$, and $N(b) = n$. Therefore $d(0, b)$ is finite, and $a = p_{n+1} \vee b$ covers b. Therefore $d(0, a) \leqq d(0, b) + 1$, i.e., $d(0, a)$ is finite. Now it follows from the above that $d(a, b)$ is finite if a and b are finite with $a \leqq b$. To complete the proof of the theorem we need only prove the statement, "if $d(a, b) = n$, then all chains connecting a to b have length at most n." That the finite elements form an ideal follows immediately from this. We will prove the statement by induction on n. If $d(a, b) = 1$, b covers a and the statement is clearly true. Suppose the statement is true for $d(a, b) \leqq n$. Let $d(a, b) = n + 1$. Then there exists a covering chain $a < x_1 < \cdots < x_n < b$. Note that $d(x_1, b) = n$. Let $a < y_1 < \cdots < y_m < b$ be any chain connecting a to b. We need only prove that $m \leqq n$. If y_1 does not cover a, there exists an atom p such that $pa = 0$ and $p \leqq y_1$. Then $p \vee a$ covers a. Replacing y_1 by $p \vee a$ we get another chain of length m. Thus we may assume that y_1 covers a. If $y_1 = x_1$, we have $d(y_1, b) = d(x_1, b) = n$. Therefore by the inductive hypothesis $m \leqq n$. If $y_1 \neq x_1$, y_1 and x_1 both cover $y_1 x_1 = a$. Therefore $y_1 \vee x_1$ covers y_1 and x_1. Now let $y_1 \vee x_1 < w_1 < \cdots < w_k = b$ be any chain joining $y_1 \vee x_1$ to b. Then $x_1 < y_1 \vee x_1 < w_1 < \cdots < w_k$ joins x_1 to b. Since $d(x_1, b) = n$, $k + 1 \leqq n$, i.e., $k \leqq n - 1$. It follows that $d(y_1 \vee x_1, b) \leqq n - 1$. Since $y_1 \vee x_1$ covers y_1, this means that $d(y_1, b) \leqq n$. But $m \leqq d(y_1, b)$, so $m \leqq n$.

LEMMA 4.1. *Let S be an atomic orthocomplemented lattice. Then the following covering conditions on S are equivalent.*

(*) If a and b are in $F(S)$, and both cover ab, then $a \vee b$ covers both a and b.

(**) If a and b are in $F(S)$, and $a \vee b$ covers both a and b, then a and b both cover ab.

Proof. Suppose that (*) holds in S, that a and b are in $F(S)$, and that $a \vee b$ covers a and b. Then a' and b' are in $F(S)$, and both cover $a'b'$. Therefore by (*), $a' \vee b'$ covers a' and b'. Hence a and b cover ab. Thus (*) implies (**). A dual argument shows that (**) implies (*).

LEMMA 4.2. *Let S be an atomic orthocomplemented lattice in which the covering condition* (*) *holds. Then the finite elements of S form an atomic modular lattice.*

Proof. This follows immediately from Theorem 4.1, Lemma 4.1, and Theorem 3, Ch. 5 of Birkhoff [3].

Two elements (b, c) in a lattice are said to form a modular pair if for all $a \leqq c$, $(a \vee b)c = a \vee bc$. A lattice S is semi-modular if the relation of being a modular pair is symmetric in S. Two elements (a, b) form a d-modular pair if for all $c \geqq a$, $(a \vee b)c = a \vee bc$. S is dual semi-modular if the relation of being a d-modular pair is symmetric.

In general semi-modularity is stronger than the covering condition (*). We want to show that with one additional condition (*) implies semi-modularity. Our proof is suggested by the proofs of Theorems III–1 and III–6 of Mackey [5]. We introduce the following notation. If x is in the atomic orthocomplemented lattice S, $\mathscr{A}(x)$ is the set of all atoms p such that $p \leqq x$. $\mathscr{A}(x) + \mathscr{A}(y)$ is the set of all atoms p such that for some $q \in \mathscr{A}(x)$ and $r \in \mathscr{A}(y)$, $p \leqq q \vee r$. If X is a set of atoms, $X^{\perp}$ is the set of all atoms p such that $p \perp q$ for all q in X. It is easy to verify the rules $\mathscr{A}(x') = \mathscr{A}(x)^{\perp}$, $\mathscr{A}(xy) = \mathscr{A}(x) \cap \mathscr{A}(y)$, $(\mathscr{A}(x) + \mathscr{A}(y))^{\perp} = \mathscr{A}(x)^{\perp} \cap \mathscr{A}(y)^{\perp}$.

LEMMA 4.3. *Let S be an atomic orthocomplemented lattice in which the covering condition* (*) *holds. Assume further that if a and b are atoms in S with $a \neq b$, $a'(a \vee b) \neq 0$. Then if p is an atom in S, $\mathscr{A}(p \vee x) = \mathscr{A}(x) + \mathscr{A}(p)$ for all x in S.*

Proof. We need only show that $\mathscr{A}(x \vee p) \subseteqq \mathscr{A}(x) + \mathscr{A}(p)$. Let p be an atom with $px = 0$. First note that if q and r are atoms with $q \neq r$, then $p'(q \vee r) \neq 0$. This is immediate if $p \leqq q \vee r$. If $p \nleqq q \vee r$, let $c = p \vee q \vee r$. Let $t_1 = p'(p \vee q)$, and $t_2 = p'(p \vee r)$. Then $[0, c]$ is a modular lattice of length 3, $d(0, t_1 \vee t_2) = 2$, and $d(0, q \vee r) = 2$. Hence $(t_1 \vee t_2)(q \vee r) \neq 0$, which means $p'(q \vee r) \neq 0$. Now let s be any atom, y any element such that $y > ys'$, and r any atom in $\mathscr{A}(y)$ but not in $\mathscr{A}(ys')$. If $q \in \mathscr{A}(y)$, and $q \neq r$, then $x = s'(q \vee r)$ is in $\mathscr{A}(ys')$, and $q \leqq r \vee x$. Thus $\mathscr{A}(y) = \mathscr{A}(ys') + \mathscr{A}(r)$. Applying this to x' and p, we have $\mathscr{A}(x') = \mathscr{A}(x'p') + \mathscr{A}(r)$ for some r. Now $\mathscr{A}(x) = \mathscr{A}(x')^{\perp} = \mathscr{A}(x'p')^{\perp} \cap \mathscr{A}(r)^{\perp} = \mathscr{A}((x \vee p)) \cap \mathscr{A}(r') = \mathscr{A}((x \vee p)r')$. But $\mathscr{A}((x \vee p)) = \mathscr{A}((x \vee p)r') + \mathscr{A}(p)$, so $\mathscr{A}(x \vee p) = \mathscr{A}(x) + \mathscr{A}(p)$.

THEOREM 4.2. *Let S be an atomic orthocomplemented lattice satisfying the covering condition* (*). *Assume further that, if a and b are atoms in S, $a \neq b$, then $a'(a \vee b) \neq 0$.*[4] *Then if a is finite (a, x) is a d-modular pair for all x in S.*

Proof. We need only show that for $c \geqq a$, $(a \vee x)c \leqq a \vee xc$. It follows from Lemma 4.3 that $\mathscr{A}(x \vee a) = \mathscr{A}(x) + \mathscr{A}(a)$. Let $a \leqq c$, and let $p \in \mathscr{A}((a \vee x)c)$. Then $p \in \mathscr{A}((a \vee x))$, and hence $p \leqq q \vee r$ where $q \in \mathscr{A}(a)$ and $r \in \mathscr{A}(x)$. If $p = q$ or $p = r$, $p \in \mathscr{A}(xc) + \mathscr{A}(a)$. If p is different from q and r, then $r \leqq p \vee q \leqq c$. Thus $r \in \mathscr{A}(xc)$, which means $p \in \mathscr{A}(a) + \mathscr{A}(xc)$. This proves that $(a \vee x)c \leqq a \vee xc$.

THEOREM 4.3. *Let S satisfy the hypotheses of Theorem* 4.2. *Then $F(S)$ is an atomic, orthocomplemented, modular lattice.*

Proof. We need only show that $F(S)$ is modular. Let a, b, c be in $F(S)$ with $a \leqq c$. If a is finite, $(a \vee b)c = a \vee bc$ by the preceding theorem. Otherwise a' is finite, which means c' is finite. Then we have $[(a \vee b)c]' = (c' \vee b')a' = c' \vee b'a'$. Thus $(a \vee b)c = a \vee bc$.

Let V be a left vector space over a division ring R. Let V^* be the space of all linear functions from V to R. If W is a total subspace of V^*, i.e., $f(x) = 0$ for all $f \in W$ implies $x = 0$, then we say that V, W is a dual pair. If X is a subspace of V, let $X' = \{f \in W\colon f(x) = 0 \text{ for all } x \in X\}$. Similarly define Y' for Y a subspace of W. Then we say that a subspace X of V is W-closed if $X = X''$.

McLaughlin [6] has given a representation theorem for the completion by cuts of a complemented modular point lattice. Since a complete atomic orthocomplemented lattice S is the completion by cuts of $F(S)$ we can apply the theorem to obtain:

THEOREM 4.4. *Let S be a complete, irreducible, atomic, orthocomplemented lattice in which the covering condition* (*) *holds. Assume further that the $d(0, 1) \geqq 4$, and that if p and q are atoms with $p \neq q$, $p'(p \vee q) \neq 0$. Then there exist a pair of dual vector spaces U, W over a division ring D such that S is isomorphic to the lattice of W-closed subspaces of U.*

COROLLARY. *Let S be a complete, irreducible, atomic, orthocomplemented lattice. Then the following three statements about S are equivalent.*

(1) *S is semi-modular*

(2) *If p is an atom in S and $pa = 0$ then $p \vee a$ covers a.*

[4] Note that this condition holds if L is weakly modular, i.e., if (a, a') is a d-modular pair for all a in L.

(3) *Covering condition* (*) *holds in* S; *and if* p *and* q *are atoms with* $p \neq q$, *then* $p'(p \vee q) \neq 0$.

Proof. It is well known that (1) implies (2) and (2) implies (*). Suppose (2) holds and that p and q are atoms with $p \neq q$. Then $p \vee p'q'$ covers $p'q'$. But $q < p \vee q$, so $p'q' < q'$, which shows that 1 does not cover $p'q'$. Therefore, $p \vee p'q' \neq 1$, i.e. $p'(p \vee q) \neq 0$. This proves that (2) implies (3). Now suppose that (3) holds. If $d(0, 1)$ is finite, S is actually modular (Theorem 4.3). If $d(0, 1)$ is infinite, we can apply the theorem above. Mackey ([5], Theorem III–6) has shown that in such a lattice of closed subspaces the relation of being a d-modular pair is symmetric. Since S is orthocomplemented, this means that the relation of being a modular pair is also symmetric.

THEOREM 4.5. *The completion of a semi-modular atomic orthocomplemented lattice is semi-modular.*

Proof. If S is semi-modular the covering condition (*) holds in $F(S)$. Also if p is an atom and $px = 0$, $p \vee x$ covers x. If p and q are atoms, $p \neq q$, p' covers $p'q'$. Hence $p \vee p'q' < 1$, which gives $p'(p \vee q) \neq 0$. Let $L(S) = \Sigma \oplus R_j$ be the direct sum decomposition of the completion $L(S)$ into irreducible components. Since $F(L(S))$ and $F(S)$ are ortho-isomorphic, the covering condition (*) and the condition $p'(p \vee q) \neq 0$ hold in each R_j. If the dimension of R_j is finite, R_j is actually modular. If the dimension of R_j is infinite, R_j satisfies the hypotheses of Theorem 4.4, so R_j is semi-modular. Thus $L(S)$ is the direct sum of semi-modular lattices; $L(S)$ is semi-modular.

5. Semi-Inner Product Spaces. Let V be a left vector space over a division ring R. A *semi-bilinear functional* B on V is a map $(x, y) \rightarrow B(x, y)$ of $V \times V$ into R such that

(1) for all x_1, x_2, y_1, and y_2 in V and α in R, $B(\alpha x_1 + x_2, y_1 + y_2) = \alpha B(x_1, y_1) + \alpha B(x_1, y_2) + B(x_2, y_1) + B(x_2, y_2)$, and

(2) There exists an anti-automorphism θ of R such that for all x and y in V and α in R, $B(x, \alpha y) = B(x, y)\theta(\alpha)$. We will say that a semi-bilinear functional B is a *semi-inner product* if it satisfies the following conditions.

(1) The anti-automorphism θ associated with B is involutory.

(2) $B(x, y) = \theta(B(y, x))$ for all x and y.

(3) $B(x, x) = 0$ implies $x = 0$.

(4) For some x $B(x, x) = 1$.

A left vector space together with a semi-inner product will be called a *semi-inner product space*.

If V is a semi-inner product space, define an orthogonality relation in V by $x \perp y$ if and only if $B(x, y) = 0$. If X is a subset of V define

$X^{\perp}$ just as in §2. It is easy to verify that $X^{\perp}$ is always a subspace. The orthocomplemented lattice of all closed subspaces of V will be denoted by $L(V)$.

THEOREM 5.1. *Let V be a left vector space over a division ring R. Then V is a semi-inner product space if and only if there exists a dual space W such that V, W is a dual pair and the lattice, S, of all W-closed subspaces of V is orthocomplemented.*

Proof. Suppose that W exists and that S is orthocomplemented. Let R^* be the ring which is identical with R as an additive group and in which multiplication ($\circ$) is $\alpha \circ \beta = \beta\alpha$. Then W is a left vector space over R^*. If $\lambda \in R^*, f \in W, (\lambda f)(x) = f(x)\lambda$. For $x \in V$, let $[x]$ denote the one-dimensional subspace spanned by x. Let $[x]^*$ be the one-dimensional subspace of W spanned by those linear functionals whose nullspaces is $[x]^{\perp}$. In an obvious way one verifies that $[x] \rightarrow [x]^*$ is a one-to-one map of the one-dimensional subspaces of V onto those of W which preserve linear dependence and independence. Hence there exists a semi-linear transformation T from V onto W such that $[x]^* = [T(x)]$. Clearly $[x] \perp [y]$ if and only if $T(y)(x) = 0$. Thus if $x_0 \neq 0$, $T(x_0)(x_0) \neq 0$.

Let φ be the isomorphism from R to R^* associated with T. Then φ may also be regarded as an anti-automorphism of R. Let θ be the inner automorphism of R: $\beta \rightarrow (T(x_0)(x_0))\beta(T(x_0)(x_0))^{-1}$. Let $B(x, y) = (T(y)(x))(T(x_0)(x_0))^{-1}$. It is a matter of routine to verify that B is a semi-inner product with anti-automorphism $\sigma = \theta \circ \varphi$. If A is any finite-dimensional subspace of V containing x_0, B defines an orthocomplementation in the lattice of all subspaces of A, and $B(x_0, x_0) = 1$. Hence by proposition 1, page 110 of Baer [1], the anti-automorphism σ associated with B is involutory and $\sigma(B(x, y)) = B(y, x)$, for all x and y in A. Thus B is a semi-inner product. Now suppose that B is a semi-inner product on V. For x in V let f_x be the member of $V^*: f_x(y) = B(y, x)$. It is clear that the set W of all such f_x is a total subspace of V^* and that $L(V)$ is identical with the lattice of W-closed subspaces of V.

Suppose B and B' are two semi-inner products on V which determine the same orthogonality relation. Then there exists α in R such that $B(x, y) = B'(x, y)\alpha$ for all x and y in V[5]. It is quite possible, however, to have two semi-inner products on V, which are not equivalent in this way, but whose associated lattices are ortho-isomorphic. Our last two theorems explore this possibility.

THEOREM 5.2. *Let V_1 be a semi-inner product space over a division*

[5] Baer [**4**], page 105, Proposition 3.

ring R. Let B_1 be the semi-inner product in V_1, let θ be the anti-automorphism associated with B_1, let σ be an automorphism of R, and let τ be an inner automorphism of R. Then there exists a semi-inner product space V_2 over R whose semi-inner product B_2 has anti-automorphism $\varphi = \tau \circ \sigma \circ \theta \circ \sigma^{-1}$ such that $L(V_1)$ and $L(V_2)$ are orthoisomorphic.

Proof. Let x_j ($j \in J$, where J is some indexing set) be a maximal set of nonzero mutually orthogonal vectors in V_1. For $y \in V_1$, let $T(y)$ be the function from J to R such that $T(y)(j) = \sigma(B_1(y, x_j))$. Let V_2 be the set of all such functions $T(y)$. It is clear that V_2 is a left vector space over R, and that T is a semi-linear transformation with automorphism σ from V_1 onto V_2. Further T is one-to-one. For if $T(y) = 0$, then $y \perp x_j$ all $j \in J$; and this means $y = 0$, because $\{x_j\}$ was a maximal orthogonal set. Let the inner automorphism τ be $\tau(\beta) = \alpha^{-1}\beta\alpha$. For f and g in B_2, let $B_2(f, g) = \sigma(B_1(T^{-1}f, T^{-1}g))\alpha$. It is easy to verify that B_2 is a semi-inner product. We include only the proof that $B_2(f, \beta g) = B_2(f, g)\varphi(\beta)$. We have

$$\begin{aligned} B_2(f, \beta g) &= \sigma(B_1(T^{-1}f, T^{-1}(\beta g)))\alpha = \sigma(B_1(T^{-1}f, \sigma^{-1}(\beta)(T^{-1}g)))\alpha \\ &= \sigma(B_1(T^{-1}f, T^{-1}g)\theta(\sigma^{-1}(\beta)))\alpha \\ &= \sigma(B_1(T^{-1}f, T^{-1}g))\sigma(\theta(\sigma^{-1}(\beta)))\alpha \\ &= B_2(f, g)\alpha^{-1}\sigma(\theta(\sigma^{-1}(\beta)))\alpha = B_2(f, g)\varphi(\beta) . \end{aligned}$$

Since $B_2(f, g) = 0$ if and only if $B_1(T^{-1}f, T^{-1}g) = 0$, it is clear that T induces an ortho-isomorphism between the $L(V_1)$ and $L(V_2)$.

THEOREM 5.3. *Let V_1 and V_2 be semi-inner product spaces of dimension greater than two, over division rings R_1 and R_2 respectively, such that $L(V_1)$ and $L(V_2)$ are ortho-isomorphic. Let B_1 and B_2 be the semi-inner products in V_1 and V_2 respectively, and let θ and φ be the associated anti-automorphisms. Then there exists an isomorphism σ from R_1 onto R_2 and a semi-linear transformation T from V_1 to V_2 with isomorphism σ such that T induces the lattice isomorphism. Further there exists an inner automorphism τ of R_2 such that $\varphi = \tau \circ \sigma \circ \theta \circ \sigma^{-1}$.*

Proof. Since $L(V_1)$ and $L(V_2)$ are isomorphic, the lattice of all finite-dimensional subspaces of V_1 is isomorphic to the lattice of all finite dimensional subspaces of V_2. It follows from this that the lattice of all subspaces of V_1 is isomorphic to the lattice of all subspaces of V_2. Therefore the isomorphism σ and the semi-linear transformation T exists. To prove the final assertion, let x be a vector in V_1 such that $B_1(x, x) = 1$. Let y be a nonzero vector in V_1 with $y \perp x$.

Let $x' = T(x)$, and $y' = T(y)$. Since T induces the lattice ortho-isomorphism, $x' \perp y'$. Now for any $\lambda \in R$ with $\lambda \neq 0$,

$$x + \lambda y \perp x - \theta(\lambda^{-1})B_1(y, y)^{-1}y \ .$$

Therefore $T(x + \lambda y) \perp T(x - \theta(\lambda^{-1})B_1(y, y)^{-1}y)$, i.e.,

$$x' + \sigma(\lambda)y' \perp x' - \sigma(\theta(\lambda^{-1}))\sigma(B_1(y, y)^{-1})y' \ .$$

Therefore $B_2(x' + \sigma(\lambda)y', x' - \sigma(\theta(\lambda^{-1}))\sigma(B_1(y, y)^{-1})y') = 0$ for all $\lambda \neq 0$ in R_1. Since $x' \perp y'$, this gives

$$B_2(x', x') + \sigma(\lambda)B_2(y', y')\varphi(\sigma(B_1(y, y)^{-1}))\varphi(\sigma(\theta(\lambda^{-1}))) = 0$$

for all $\lambda \neq 0$ in R_1. Now let $\alpha = B_2(x', x')$. Taking $\lambda = 1$, we get $\alpha - B_2(y', y')\varphi(\sigma(B_1(y, y)^{-1})) = 0$. Thus $\alpha - \sigma(\lambda)\alpha\varphi(\sigma(\theta(\lambda^{-1}))) = 0$ for all $\lambda \neq 0$ in R_1. Let τ be the inner automorphism of R_2: $\tau(\beta) = \alpha^{-1}\beta\alpha$ for all β in R_2. Then taking $\lambda^{-1} = \sigma^{-1}(\beta)$, we get $\beta = (\tau^{-1}\circ\varphi\circ\theta\circ\sigma^{-1})(\beta)$ for all $\beta \neq 0$ in R_2, i.e., $\tau^{-1}\circ\varphi\circ\sigma\circ\theta\circ\sigma^{-1}$ is the identity automorphism of R_2. Since $\sigma\circ\theta\circ\sigma^{-1}$ is an involutory anti-automorphism of R_2, this gives $\varphi = \tau\circ\sigma\circ\theta\circ\sigma^{-1}$.

COROLLARY. *Let n be an integer greater than 2. Then there exist semi-inner product spaces V_1 and V_2 of dimension n over the complex numbers such that $L(V_1)$ and $L(V_2)$ are not ortho-isomorphic.*

Proof. There exists a real closed subfield K of the complex numbers C such that $K(i) = C$, and K is not isomorphic to the real numbers.[6] Let φ be the involutory automorphism of C which has K as its field of fixed points. Let θ be the usual conjugacy automorphism. Let V_1 be an n-dimensional Hilbert space over the complex numbers. Let V_2 be the set of all ordered n-tuples of complex numbers. Let $B((\alpha_1, \cdots, \alpha_n), (\beta_1, \cdots, \beta_n)) = \Sigma\alpha_i\varphi(\beta_i)$. It is easy to verify that B is a semi-inner product with anti-automorphism φ. If $L(V_1)$ and $L(V_2)$ were ortho-isomorphic, we would have $\varphi = \sigma\circ\theta\circ\sigma^{-1}$ for some automorphism σ of C. But this would mean that K was isomorphic to the field of fixed points of θ, i.e., the field of real numbers. This contradiction proves the corollary.

This corollary points up the fact that lattices $L(V_1)$ and $L(V_2)$ may be isomorphic without being ortho-isomorphic.

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