STRONGLY RECURRENT TRANSFORMATIONS

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Let $(X, \mathscr{B}, m)$ be a finite or σ-finite and non-atomic measure space. A set B is said to be measurable if it is a member of $\mathscr{B}$. Two measures on $\mathscr{B}$, finite or σ-finite (one may be finite and the other σ-finite), are said to be equivalent if they have the same null sets. In this paper we consider a one-to-one, nonsingular, measurable transformation ϕ of X onto itself. By a nonsingular transformation ϕ we mean $m(\phi B) = m(\phi^{-1}B) = 0$ for every measurable set B with $m(B) = 0$, and by a measurable transformation ϕ we mean $\phi B \in \mathscr{B}$ and $\phi^{-1}B \in \mathscr{B}$ for every $B \in \mathscr{B}$. We shall say that the transformation ϕ is measure preserving (with respect to a measure μ) or equivalently, μ is an invariant measure (with respect to the transformation ϕ) if $\mu(\phi B) = \mu(\phi^{-1}B) = \mu(B)$ for every measurable set B.

A recurrent transformation is a common notion in ergodic theory. This is a measurable transformation ϕ defined on a finite or σ-finite measure space $(X, \mathscr{B}, m)$ with the following property: if A is a measurable set of positive measure, then for almost all $x \in A$ $\phi^n x$ belongs to A for infinitely many integers n. It is not difficult to see that every measurable transformation which preserves a finite invariant measure μ equivalent to m is recurrent. The converse statement is not in general true; for example an ergodic transformation which preserves an infinite and σ-finite measure is always recurrent yet it does not preserve a finite invariant equivalent measure. In this paper we restrict the notion of a recurrent transformation. We introduce the notion of a strongly recurrent set and define a strongly recurrent transformation. We show that a transformation ϕ is strongly recurrent if and only if there exists a finite invariant measure μ equivalent to m (Theorem 2). This is accomplished by showing the connection between strongly recurrent sets and weakly wandering sets (Theorem 1). Weakly wandering sets were introduced in [1], and the condition that a transformation ϕ does not have any weakly wandering set of positive measure was further strengthened (see condition $(W)^*$ below). It was shown in [1] that this stronger condition was again a necessary and sufficient condition for the existence of a finite invariant measure μ equivalent to m. We show that a similar strengthening for a strongly recurrent transformation is false for a wide class of measure preserving transformations defined on a finite measure space (Theorem 3).

DEFINITION. A measurable set S is said to be *strongly recurrent*

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(with respect to ϕ) if the set of all integers n such that $m(\phi^n S \cap S) > 0$ is relatively dense, i.e., if there exists a positive integer k such that

$$(1) \qquad \max_{0 \leqq i \leqq k-1} m(\phi^{n-i} S \cap S) > 0$$

for $n = 0, \pm 1, \pm 2, \cdots$. This condition is obviously equivalent to the following:

$$(2) \qquad m\Big(\bigcup_{i=0}^{k-1} \phi^{n-i} S \cap S\Big) > 0$$

or

$$(3) \qquad m\Big(\phi^n S \cap \Big[\bigcup_{i=0}^{k-1} \phi^i S\Big]\Big) > 0$$

for $n = 0, \pm 1, \pm 2, \cdots$, This last condition means that there exists a finite number of images of S by the powers of ϕ such that any image of S by any power of ϕ has an intersection of positive measure with at least one of them.

The transformation ϕ is said to be strongly recurrent if every set of positive measure is strongly recurrent. We note that the property of a transformation ϕ being strongly recurrent is preserved under equivalent measures.

The following notion was introduced in [1]: A measurable set W is said to be weakly wandering (with respect to ϕ) if there exists a sequence of integers $\{n_k : k = 1, 2, \cdots\}$ such that the sets $\phi^{n_k} W$, $k = 1, 2, \cdots$ are mutually disjoint.

THEOREM 1. *Let* $(X, \mathscr{B}, m)$ be a finite or σ-finite measure space, and let ϕ be a one-to-one, nonsingular, measurable transformation of X onto itself. Then the following two conditions are equivalent:

(W) $m(A) > 0$ implies that there exists at most a finite number of mutually disjoint images of A by the powers of ϕ; in other words, A is not weakly wandering.

(S) $m(A) > 0$ implies that A is strongly recurrent.

We first prove a Lemma which is by itself of some interest.

LEMMA 1. *Let* $(X, \mathscr{B}, m)$ *and* ϕ *be as in Theorem* 1, *and let* A *be a measurable set of positive measure such that*

$$(4) \qquad \liminf_{n \to \infty} m(\phi^n A) = 0 .$$

Then given ε with $0 < \varepsilon < m(A)$, there exists a measurable subset A' of A with $m(A') < \varepsilon$ such that the set $S = A - A'$ is not strongly recurrent.

Proof. Let A be a measurable set with $m(A) = \alpha > 0$ and $\liminf_{n\to\infty} m(\phi^n A) = 0$. Let ε be a positive number with $0 < \varepsilon < \alpha$. Let

$$\varepsilon_k = \frac{\varepsilon}{k2^k}$$

for $k = 1, 2, \cdots$. Next, for each $k = 1, 2, \cdots$ we choose a positive integer n_k such that

$$m(\phi^{n_k - i}A) < \varepsilon_k$$

for $i = 0, 1, 2, \cdots, k - 1$. This is possible since ϕ is nonsingular and (4) is satisfied by A. Let us put

$$A' = \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{k-1} \phi^{n_k - i}A \cap A .$$

Then

$$m(A') \leqq \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} m(\phi^{n_k - i}A) < \sum_{k=1}^{\infty} k\varepsilon_k = \varepsilon .$$

Let $S = A - A'$, then it is easy to see that

$$\phi^{n_k - i}S \cap S \subset \phi^{n_k - i}A \cap (A - A') = \phi$$

for $i = 0, 1, 2, \cdots, k - 1$ and $k = 1, 2, \cdots$. This shows that S is not strongly recurrent.

Proof of Theorem 1. If a measurable set S of positive measure is not strongly recurrent, then it is possible to find a measurable subset N of S with $m(N) = 0$ such that $S' = S - N$ is weakly wandering. This is easy, since S not strongly recurrent means that for each positive integer n_k there exists another positive integer n_{k+1} such that

$$m\Big(\phi^{n_{k+1}}S \cap \bigcup_{i=0}^{n_k} \phi^i S\Big) = 0 .$$

In this way we may obtain a sequence of integers $\{n_k : k = 1, 2, \cdots\}$ such that

$$m(\phi^{n_k}S \cap \phi^{n_j}S) = m(S \cap \phi^{n_k - n_j}S) = 0 \text{ for } k \neq j .$$

It follows that $S' = S - N$ is weakly wandering, where

$$N = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{k-1} \phi^{n_k - n_j}S \cap S$$

and $m(N) = 0$.

Conversely, let W be a weakly wandering set of positive measure.

Since the measure space is σ-finite we can find a sequence of measurable sets $\{A_i : i = 1, 2, \cdots\}$ which are mutually disjoint, such that $0 < m(A_i) < \infty$ for $i = 1, 2, \cdots$ and $X = \bigcup_{i=1}^{\infty} A_i$. We let

$$m'(B) = \sum_{i=1}^{\infty} \frac{m(A_i \cap B)}{2^i m(A_i)} \text{ for } B \in \mathscr{B} .$$

It follows that m' and m are equivalent. Since $\phi^{n_k} W, k = 1, 2, \cdots$ are mutually disjoint and $m'(X) < \infty$ it follows that $\liminf_{n \to \infty} m'(\phi^n W) = 0$. Thus, whether m is finite or σ-finite, the set W satisfies (4) with m replaced by the equivalent and finite measure m'. By applying Lemma 1 we obtain a measurable subset S of W such that $m'(S) > 0$ and S is not strongly recurrent. Since m and m' are equivalent, this proves the theorem.

THEOREM 2. *Let* $(X, \mathscr{B}, m)$ *and* ϕ *be as in Theorem* 1. *Then condition* (*S*) *is equivalent to the existence of a finite invariant measure* μ *equivalent to* m.

Proof. Theorem 2 is an immediate consequence of Theorem 1 above and Theorem 1 of [1], where it was shown that condition (W) is equivalent to the existence of a finite invariant measure μ equivalent to m.

In [1] it was further shown that the following condition:

(W)* Given $\varepsilon > 0$, there exists a positive integer N such that $m(A) \geqq \varepsilon$ implies that there exists at most N mutually disjoint images of A by the powers of T,

is again a necessary and sufficient condition for the existence of a finite invariant measure μ equivalent to m (see condition $(V)^*$, § 3 of [1]).

Condition $(W)^*$ is in appearance a stronger condition than condition (W). We note that in condition (W)* the positive integer N depends on ε only and not on the measurable set A. However, it turns out that these two conditions are equivalent to each other and are in turn necessary and sufficient conditions for the existence of a finite invariant measure μ equivalent to m (see Theorem 1 of [1]). By analogy, we may attempt to strengthen condition (S) in the following manner:

(S)* Given $\varepsilon > 0$, there exists a positive integer N such that $m(A) \geqq \varepsilon$ implies

$$m\left(\phi^n A \cap \bigcup_{i=0}^{N-1} \phi^i A\right) > 0 \quad \text{for } n = 0, \pm 1, \pm 2, \cdots$$

We show that condition (S)* is not a necessary condition for the existence of a finite invariant measure μ equivalent to m. In fact, we shall show that for any ergodic measure preserving transformation ϕ defined on a finite measure space $(X, \mathscr{B}, \mu)$ condition (S)* is not satisfied.

We say that a transformation ϕ is ergodic if $\phi A = A$ implies $m(A) = 0$ or $m(X - A) = 0$.

LEMMA 2. *Let $(X, \mathscr{B}, \mu,)$ be a finite or σ-finite measure space, and let ϕ be an ergodic measure preserving transformation defined on it. Then given $\varepsilon > 0$ and a positive integer $N > 0$, there exists a measurable set C with $\mu(C) \leqq \varepsilon$ such that*

$$X - C = \bigcup_{i=0}^{N-1} \phi^i E$$

for some measurable set E where $E, \phi E, \cdots, \phi^{N-1}E$ are mutually disjoint.

Proof. Given $\varepsilon > 0$ and an integer $N > 0$, let F be any measurable set with $0 < \mu(F) \leqq \varepsilon/N$. Let

$$F_0 = F$$
$$F_1 = \phi^{-1}F - F_0$$
$$F_2 = \phi^{-2}F - F_0 \cup F_1$$

and in general

$$F_n = \phi^{-n}F - \bigcup_{j=0}^{n-1} F_j \quad \text{for } n = 1, 2, \cdots .$$

It follows that F_n, $n = 0, 1, 2, \cdots$ are mutually disjoint, and furthermore;

$$\phi^k F_n \subset F_{n-k} \qquad \begin{array}{l}\text{for } k = 0, \cdots, n \\ \text{and } n = 0, 1, 2, \cdots .\end{array}$$

We let

$$E_i = F_{iN} = \phi^{-iN}F - \bigcup_{j=0}^{iN-1} F_j = \phi^{-iN}F - \bigcup_{j=0}^{iN-1} \phi^{-j}F$$

then

$$\phi^k E_i \subset F_{iN-k} \quad \text{for } k = 0, 1, \cdots, iN; \text{ and } i = 1, 2, \cdots$$

which implies that the sets

$$\phi^k E_i \quad \text{for } k = 0, 1, \cdots, iN; \text{ and } j = 1, 2, \cdots \tag{5}$$

are mutually disjoint.

Next we let

$$E = \bigcup_{i=1}^{\infty} E_i$$

and

$$C = X - \bigcup_{k=0}^{N-1} \phi^k E .$$

It follows from (5) that $E, \phi E, \cdots, \phi^{N-1}E$ are mutually disjoint, and

$$\mu(C) = \mu\Big(X - \bigcup_{k=0}^{N-1} \phi^k E\Big) \leqq N\mu(F) \leqq \varepsilon .$$

THEOREM 3. *Let ϕ be on ergodic measure preserving transformation defined on a finite measure space $(X, \mathscr{B}, \mu)$ with $\mu(X) = 1$. Then condition $(S)^*$ is not satisfied.*

Proof. Let $\varepsilon = 1/(q+1)$ for some positive integer $q > 3$. Let $k > 1$ be an arbitrary positive integer. We show that there exists a measurable set A with $\mu(A) \geqq \varepsilon$ and

$$\mu\Big(\phi^{n_k} A \cap \bigcup_{i=0}^{k-1} \phi^i A\Big) = 0$$

for some integer $n_k > k$. Let us put $N = qk$. Then by Lemma 2 there exists a measurable set E with $E, \phi E, \cdots, \phi^{N-1}E$ mutually disjoint and

$$\mu\Big(X - \bigcup_{k=0}^{N-1} \phi^k E\Big) \leqq \varepsilon = \frac{1}{q+1} .$$

Since $\mu(X) = 1$, this implies $1 - N\mu(E) \leqq \varepsilon$ or $\mu(E) \geqq (1-\varepsilon)/N$. Let

$$A = \bigcup_{i=0}^{k-1} \phi^i E .$$

Since $k = N/q$ we have

$$\mu(A) = k\mu(E) \geqq \frac{N}{q}\,\frac{(1-\varepsilon)}{N} = \frac{1 - \dfrac{1}{q+1}}{q} = \frac{1}{q+1} = \varepsilon$$

and

$$\mu\Big(\phi^{n_k} A \cap \bigcup_{i=0}^{k-1} \phi^i A\Big) = \mu\Big(\bigcup_{i=n_k}^{n_k+k-1} \phi^i E \cap \bigcup_{i=0}^{2k-2} \phi^i E\Big) = 0$$

for some n_k where $2k < n_k < (q-1)k = N - k$.

This shows that condition (S)* is not satisfied since ε is fixed, k is arbitrary, and $n_k > k$.

REFERENCE

1. A. Hajian and S. Kakutani, *Weakly wandering sets and invariant measures*, Transactions A. M. S. **110** (1964), 136–151.

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