AN APPROXIMATE GAUSS MEAN VALUE THEOREM

W. FULKS

1. Introduction. The mean value theorem of Gauss, and its converse, due to Koebe, have long been known to characterize harmonic functions. Since any second order homogeneous elliptic operator L can, by an appropriate linear change of variables, be reduced (at a given point) to the Laplacian, it seems reasonable to expect that solutions of $Lu = 0$ should, when averaged over appropriate small ellipsoids, satisfy an approximate Gauss-type theorem, and one could hope that such a mean value property would characterize the solutions of the equation.

It turns out that this is the case. In fact the operator need not be elliptic, but may be parabolic, or of mixed elliptic and parabolic type. While the methods used here do not permit the weak smoothness conditions on the solutions admitted by Koebe's theorem, the result is stronger than might be expected in that no smoothness, not even measurability, is required of the coefficients of L: they need only be defined.

Since the result applies to parabolic equations, it seems of interest to examine the heat equation, for it can be cast in the required form. This leads to a characterization of its solutions in terms of averages over parabolic arcs.

2. The basic theorem. In the following $D_i = \partial/\partial y_i$, $D_{ij} = \partial^2/\partial y_i \partial y_j$, $u_{,ij} = D_{ij}u$, and ∇_y is the gradient operator with respect to the components of y.

It is convenient to consider equations of the form $Lu = f$, where f need only be defined, and may depend on u and any of its derivatives.

LEMMA. *Let $A = [a_{ij}]$ be an $n \times n$ constant nonnegative definite symmetric matrix, and denote by $B = [b_{ij}]$ the unique nonnegative definite symmetric square root of A. Let u be defined in a neighborhood of a point y in E_n, and be twice differentiable at y. For this y define the quadratic function q of x by*

$$q(x) \equiv (Bx \cdot \nabla_y)^2 u(y) .$$

Then the sum of the coefficients of the squared terms of $q(x)$ is $\sum_{i,j} a_{ij} u_{ij}(y)$.

Received July 24, 1963. This paper was prepared at Oregon State University under an NSF grant.

Proof. We have

$$q(x) = (Bx\cdot \nabla)^2 u = \Big(\sum_{i,m} b_{im}x_m D_i\Big)\Big(\sum_{j,k} b_{jk}x_k D_j\Big)u = \sum_{k,m}\Big(\sum_{ij} b_{im}b_{jk}u_{,ij}\Big)x_k x_m \,.$$

The sum of the coefficients of the squared terms is then

$$\sum_k\Big(\sum_{i\,j} b_{ik}b_{jk}\Big)u_{,ij} = \sum_{i,j}\Big(\sum_k b_{ik}b_{kj}\Big)u_{,ij} = \sum_{i,j} a_{ij}u_{,ij} \,.$$

THEOREM. *Let $L = \sum_{i,j} a_{ij}(y)D_{ij}$ be a well defined symmetric differential operator with a nonnegative definite matrix $A(y) = [a_{ij}(y)]$ in an open region R in E_n. Let $B(y) = [b_{ij}(y)]$ be the unique nonnegative definite square root of A, and for $y \in R$ and r sufficiently small, define*

$$u_r(y) \equiv \frac{1}{\Omega_r}\int_{|x|=r} u(y + B(y)x)d\Omega_r \tag{1}$$

where Ω_r is the area of the sphere $\{|x| = r\}$. Let u be a function defined in a neighborhood of a point $y_0 \in R$, which is twice differentiable at y_0. Then for u to be a solution of $Lu = f$ at y_0 it is necessary and sufficient that

$$u_r(y_0) = u(y_0) + C_n r^2 f(y_0) + o(r^2) \quad as\ r \to 0\,, \tag{2}$$

where C_n is a certain constant depending only on n, in fact it is easily verified that

$$C_n = \frac{n-1}{2n}\,\frac{\Gamma(n/2)}{\Gamma((n+1)/2)}\,.$$

Proof. Denote the *constant* matrices $A(y_0)$, $B(y_0)$ by A and B respectively. Since u is twice differentiable at y_0 we have

$$\begin{aligned} u(y_0 + B(y_0)x) = u(y_0 + Bx) &= u(y_0) + (Bx\cdot\nabla_y)u(y)\,|_{y_0} \\ &+ \frac{1}{2}(Bx\cdot\nabla_y)^2 u(y)\,|_{y_0} + o(|Bx|^2)\,. \end{aligned} \tag{3}$$

But $|Bx| \leqq \|B\|\ |x|$. Thus on $\{|x| = r\}$, (3) becomes

$$\begin{aligned} u(y_0 + B(y_0)x) &= u(y_0) + (Bx\cdot\nabla_y)u(y)\,|_{y_0} \\ &+ \frac{1}{2}(Bx\cdot\nabla)^2 u(y)\,|_{y_0} + o(r^2)\,. \end{aligned} \tag{4}$$

Dividing (4) by Ω_r and integrating over $\{|x| = r\}$ we get

$$u_r(y_0) = u(y_0) + \frac{1}{2\Omega_r}\int_{|x|=r}(Bx\cdot\nabla_y)^2 u(y)\,|_{y_0}\,d\Omega_r + o(r^2)\,. \tag{5}$$

We next observe

$$\frac{1}{2\Omega_r}\int_{|x|=r} x_i x_j d\Omega_r = C_n r^2 \delta_{ij}$$

where C_n is a constant depending only on n. Thus (5) becomes, by the lemma,

$$u_r(y_0) = u(y_0) + C_n r^2 \sum_{i\,j} a_{ij}(y_0) u_{,ij}(y_0) + o(r^2)\ . \tag{6}$$

But (6) is compatible with (2) if and only if $Lu = f$ at y_0.

3. **The heat equation.** As an application of the main result let us consider the heat operator $Hu = u_{xx} - u_t$. If we make the change of variables given by $x = \xi$, $t = \tau - (1/2)\xi^2$ and set $u(x, t) = v(\xi, \tau)$ then we see that our operator takes the form $v_{\xi\xi} + 2\xi v_{\xi\tau} + \xi^2 v_{\tau\tau}$. In this case the matrix A is given by

$$A = \begin{pmatrix} 1 & \xi \\ \xi & \xi^2 \end{pmatrix}.$$

To compute B we observe that $A^2 = (1 + \xi^2)A$, so that $B = A/\sqrt{1+\xi^2}$. Then

$$B\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \frac{1}{\sqrt{1+\xi^2}}\begin{pmatrix} r\cos\theta + \xi r\sin\theta \\ \xi r\cos\theta + \xi^2 r\sin\theta \end{pmatrix}. \tag{7}$$

For each ξ, there is an α satisfying $-(\pi/2) \leqq \alpha \leqq (\pi/2)$ for which

$$\frac{\cos\theta + \xi\sin\theta}{\sqrt{1+\xi^2}} = \cos(\theta - \alpha)\ ,$$

so that (7) takes the form

$$B\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} r\cos(\theta-\alpha) \\ r\xi\cos(\theta-\alpha) \end{pmatrix}. \tag{8}$$

Then $v_r(\xi_0, \tau_0)$ becomes

$$v_r(\xi_0, \tau_0) = \frac{1}{2\pi}\int_0^{2\pi} v(\xi_0 + r\cos(\theta-\alpha), \tau_0 + r\xi_0\cos(\theta-\alpha))d\theta\ .$$

Replacing $\theta - \alpha$ by θ and using the symmetry of the cosine function this reduces to

$$v_r(\xi_0, \tau_0) = \frac{1}{\pi}\int_0^{\pi} v(\xi_0 + r\cos\theta, \tau_0 + r\xi_0\cos\theta)d\theta\ .$$

By changing back to (x, t) coordinates and defining $x_0 = \xi_0$, $t_0 =$

$\tau_0 - (1/2)\xi_0^2$ and $u_r(x_0, t_0) \equiv v_r(\xi_0, \tau_0)$ we get

$$\begin{aligned} u_r(x_0, t_0) &= \frac{1}{\pi}\int_0^\pi u\Big(x_0 + r\cos\theta, \tau_0 + rx_0\cos\theta - \frac{1}{2}(x_0 + r\cos\theta)^2\Big)d\theta\ , \\ &= \frac{1}{\pi}\int_0^\pi u\Big(x_0 + r\cos\theta, t_0 - \frac{1}{2}r^2\cos^2\theta\Big)d\theta\ , \\ &= \frac{1}{\pi}\int_{-r}^r u\Big(x_0 + z, t_0 - \frac{1}{2}z^2\Big)\frac{dz}{\sqrt{r^2 - z^2}} \end{aligned}$$

or finally

$$u_r(x_0, t_0) = \frac{1}{\pi}\int_{-1}^1 u\Big(x_0 + rz, t_0 - \frac{1}{2}r^2z^2\Big)\frac{dz}{\sqrt{1 - z^2}} \tag{9}$$

which is easily seen to be a weighted average of u over the tip of a parabola with vertex at (x_0, t_0), having the line $t = t_0$ as its axis and opening down.

This gives us the following theorem.

THEOREM. *If u is twice differentiable at a point (x_0, t_0), then a necessary and sufficient condition that $Hu = f$ at (x_0, t_0) is that*

$$u_r(x_0, t_0) = u(x_0, t_0) + C_2r^2f + o(r^2) \quad as\ r \to 0\ ,$$

where $u_r(x_0, t_0)$ is given by (9).

To study the heat equation in higher dimensions one can make similar transformations. But it is easier to guess the form the previous theorem would take and verify it directly by the methods which established our basic theorem. The result is given below where Δu is the n-dimensional Laplacian, and Ω is the area of the unit sphere in $n + 1$ dimensions.

THEOREM. *If u is twice differentiable at a point (x_0, t_0) in $n + 1$ dimensions, then a necessary and sufficient condition that $\Delta u - u_t = f$ at (x_0, t_0) is that*

$$u_r(x_0, t_0) = u(x_0, t_0) + C_{n+1}r^2f + o(r^2) \quad as\ r \to 0$$

where

$$u_r(x_0, t_0) = \frac{2}{\Omega}\int_{|z|<1} u\Big(x_0 + zr, t_0 - \frac{1}{2n}z^2r^2\Big)\frac{dz}{\sqrt{1 - |z|^2}}$$

with $dz = dz_1dz_2 \cdots dz_n$.

UNIVERSITY OF COLORADO