THE RELATIONSHIP BETWEEN THE RADICAL OF A LATTICE-ORDERED GROUP AND COMPLETE DISTRIBUTIVITY

PAUL CONRAD

1. Introduction. Throughout this note let G be a lattice-ordered group (notation 1-group). G is said to be *representable* if there exists an 1-isomorphism of G onto a subdirect sum of a cardinal sum of totally ordered groups (notation 0-groups). In particular, every abelian 1-group is representable. G is said to be *completely distributive* if for $g_{ij} \in G$

$$\bigwedge_{i \in I} \bigvee_{j \in J} g_{ij} = \bigvee_{f \in J^I} \bigwedge_{i \in I} g_{if(i)}$$

provided the indicated joins and intersections exist.

For each $0 \neq g$ in G let R_g be the subgroup of G that is generated by the set of all 1-ideals of G not containing g. Then R_g is an 1-ideal of G and the radical of G is defined to be

$$R(G) = \cap R_g \qquad (0 \neq g \in G) .$$

In [2] it is shown that if G is a divisible abelian 1-group, then there exists a minimal Hahn-type embedding of G into an 1-group of real valued functions if and only if $R(G) = 0$. Thus it would be useful to identify the class of abelian 1-groups with zero radicals, and to examine the properties of non-abelian 1-groups with zero radicals. In our main theorem we show that a representable 1-group G is completely distributive if and only if $R(G) = 0$. We also show $R(G) = 0$ if and only if G has a regular representation. This settles a question raised by Weinberg [6].

With no restrictions on G we show that $R(G)$ is completely determined by the lattice $\mathscr{L}$ of all 1-ideals of G. In particular, if G is a representable 1-group, then whether or not G is completely distributive depends only on $\mathscr{L}$.

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2. Regular and essential L-ideals. If $g \in G$ and M is an 1-ideal of G that is maximal with respect to $g \notin M$, then M is called a *regular* 1-*ideal* of G. Let M^* be the intersection of all 1-ideals of G that

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properly contain M. Then since $g \in M^*$, it follows that M^* is the unique 1-ideal of G that covers M. Let Γ be an index set for the set of all pairs (G^γ, G_γ) of 1-ideals of G such that G_γ is regular and G^γ covers G_γ. Define $\alpha < \beta$ if $G^\alpha \subseteqq G_\beta$. Then Γ is a *po*-set, and we say that $\gamma \in \Gamma$ is a *value* of g if $g \in G^\gamma \backslash G_\gamma$. In particular, the set of all values of g is a trivially ordered subset of Γ. An element $\gamma \in \Gamma$ is called *essential* if there exists an $0 \neq h$ in G such that all the values of h are $\leqq \gamma$. In this case G_γ is called an *essential* 1-*ideal* of G, and if $g \in G^\gamma \backslash G_\gamma$, then we say that γ is an *essential value* of g.

Clearly the set E of all essential elements in Γ is a dual ideal of Γ ($\alpha < \beta \in \Gamma, \alpha \in E \rightarrow \beta \in E$). The following lemma shows that the radical $R(G)$ of G is completely determined by the essential ideals of G.

LEMMA 1. *The radical of G is the intersection of essential* 1-*ideals of G*: $R(G) = \cap G_\gamma (\gamma \in E)$.

Proof. If $g \notin R(G)$, then $g \notin R_h$ for some h in G and by Zorn's lemma there exists an 1-ideal M of G that is maximal with respect to $g \notin M \supseteqq R_h$. Thus $M = G_\gamma$ for some $\gamma \in E$, $g \in G^\gamma \backslash G_\gamma$ and hence g has an essential value. If $x \in \cap G_\gamma$, then x has no essential value and hence $x \in R(G)$. Therefore $\cap G_\gamma \subseteqq R(G)$. If E is the null set, then $G = \cap G_\gamma \supseteqq R(G)$ and if $\gamma \in E$, then there exists $0 \neq h_\gamma \in G$ such that if δ is a value of h_γ, then $\delta \leqq \gamma$ and hence $G_\delta \leqq G_\gamma$. Thus $R_{h_\gamma} \subseteqq G_\gamma$ and so

$$\bigcap_{\gamma \in E} G_\gamma \supseteqq \bigcap_{\gamma \in E} R_{h_\gamma} \supseteqq \bigcap_{0 \neq g \in G} R_g = R(G) .$$

COROLLARY. $R(G) = 0$ *if and only if each nonzero element in G has at least one essential value.*

We next show that $R(G)$ depends only on the lattice $\mathscr{L}$ of all 1-ideals of G. Note that a regular 1-ideal M of G is characterized by the fact that it is meet irreducible in $\mathscr{L}$. That is, if M^* is the intersection of all 1-ideals of G that properly contain M, then M is properly contained in M^*.

LEMMA 2. $\beta \in \Gamma$ *is essential if and only if* $\cap \{G_\gamma : \gamma \in \Gamma$ *and* $\gamma \not\leqq \beta\} \neq 0$.

Proof. Suppose that $0 < h \in \cap \{G_\gamma : \gamma \in \Gamma$ and $\gamma \not\leqq \beta\}$ and let α be a value of h. Then $h \notin G_\alpha$ and so $\alpha \leqq \beta$. Thus all the values of h are $\leqq \beta$, and hence β is essential. Conversely assume that G_β is essential and pick $0 < h \in G$ such that all the values of h are $\leqq \beta$. Then

$h \in \cap\{G_\gamma : \gamma \in \Gamma$ and $\gamma \not\leqq \beta\}$. For if $h \notin G_\gamma$, where $\gamma \not\leqq \beta$, then h must have a value $\alpha \geqq \gamma$ which is impossible.

COROLLARY. *$R(G)$ is an invariant of the lattice $\mathscr{L}$ of all* 1*-ideals of G.*

LEMMA 3. *For an* 1*-group G the following are equivalent.*

(1) *G/M is an* 0*-group for each regular* 1*-ideal M of G.*

(2) *G is representable.*

Proof. For each $0 \neq g$ in G pick an l-ideal M_g of G that is maximal with respect to not containing g. Then $\cap M_g = 0$, and if (1) is satisfied, then each G/M_g is an 0-group and the mapping of $x \in G$ upon $(\cdots, M_g + x, \cdots)$ is a representation of G. Conversely suppose that G has a representation, then clearly

(3) if a, $b \in G^+$ and $a \wedge b = 0$, then $a \wedge (-x + b + x) = 0$ for all $x \in G$. In fact, Sik [5] established that (2) and (3) are equivalent, but we only need that (2) implies (3). Let M be an 1-ideal of G that is maximal with respect to not containing $0 < a \in G$, and let $A = M + a$. Suppose (by way of contradiction) that G/M is not an 0-group. Then there exist strictly positive elements X and Z in G/M such that $X \wedge Z = M$.

Case I. $X \wedge A = M$. Then $P(A) = \{Y \in G/M : |Y| \wedge A = M\}$ is a convex 1-subgroup of G/M that contains X but not A. If $M < Y \in P(A)$, then $Y = M + y$, where $0 < y \in G$, and $a = a \wedge y + a'$, $y = a \wedge y + y'$, $a' \wedge y' = 0$. Moreover

$$M = A \wedge Y = M + a \wedge M + y = M + a \wedge y .$$

Thus $a \wedge y \in M$ and so $Y = M + y'$ and $A = M + a'$. But by (3), $a' \wedge (-g + y' + g) = 0$ for all g in G and hence $A \wedge -(M + g) + Y + (M + g) = M$. Thus $P(A)$ is a nonzero 1-ideal of G/M that does not contain A, and hence there exists an 1-ideal of G that properly contains M but not a, but this contradicts the maximality of M.

Case II. $X \wedge A \neq M$. Then $P(X)$ is an 1-ideal of G/M that contains Z but not A, and once again we contradict the maximality of M. Therefore G/M is an 0-group, and hence (2) implies (1).

COROLLARY. *If G is representable and $R(G) = 0$, then an element g is positive in G if and only $G_\gamma + g$ is positive for all essential values γ of g.*

Proof. If g is positive in G, then $G_\gamma + g$ is positive for all values γ of g, essential or otherwise. If g is not positive, then $g = g \vee 0 +$

$g \wedge 0 = g^+ + g^-$, where $g^- \neq 0$ and $g^+ \wedge -g^- = 0$. By the Corollary to Lemma 1 there exists an essential value γ of g^- and by Lemma 3, G/G_γ is an 0-group, and so $g^+ \in G_\gamma$. Thus γ is also an essential value of g and $G_\gamma + g = G_\gamma + g^-$ is negative.

LEMMA 4. *If* $0 < g \in \vee A_\lambda$, *where the* A_λ *are* 1*-ideals of* G, *then* $g = g_1 \vee \cdots \vee g_n$, *where* $0 \leqq g_i \in \cup A_\lambda$ *for* $i = 1, \cdots, n$.

Proof. This proof is due to T. Lloyd. Clearly $g = a_1 + \cdots + a_n$, where the $a_i \in A_{\lambda i}$ for $i = 1, \cdots, n$. Thus it suffices to show that $g \leqq a_1' \vee \cdots \vee a_n'$, where $a_1' \in A_{\lambda i}$ for $i = 1, \cdots, n$. For then

$$\begin{aligned} g &= ((a_1' \vee 0) \wedge g) \vee \cdots \vee ((a_n' \vee 0) \wedge g) \\ &= g_1 \vee \cdots \vee g_n \end{aligned}$$

where $0 \leqq g_i \in A_{\lambda i}$ for $i = 1, \cdots, n$. If $n = 2$, then

$$a_1 + a_2 \leqq 2a_1 \vee (a_1 + a_2 - a_1 + a_2) = a_1' \vee a_2'$$

because

$$\begin{aligned} 0 \leqq |a_1 - a_2| &= (a_1 - a_2) \vee (a_2 - a_1) \\ &= -a_1 + (2a_1 \vee (a_1 + a_2 - a_1 + a_2)) - a_2 \,. \end{aligned}$$

Thus $a_1 + \cdots + a_n \leqq (a_1 + \cdots + a_{n-1})' \vee a_n'$, and since $(a_1 + \cdots + a_{n-1})' \in \vee A_{\lambda i}$ $(i = 1, \cdots, n-1)$, $(a_1 + \cdots + a_{n-1})' = b_1 + \cdots + b_{n-1}$, where $b_i \in A_{\lambda i}$ for $i = 1, \cdots, n-1$. Thus by induction $b_1 + \cdots + b_{n-1} \leqq a_1' \vee \cdots \vee a_{n-1}'$ and hence $g \leqq a_1' \vee \cdots \vee a_n'$.

3. **Completely distributive** L**-groups.** Let A be a sublattice and and subdirect sum of a cardinal sum B of 0-groups $B_\lambda (\lambda \in \Lambda)$. If for each λ in Λ, the projection ρ_λ of A onto B_λ preserves infinite joins, then A is called a *regular* subgroup of B. An 1-group G is said to have a *regular representation* if it is 1-isomorphic to a regular subgroup of a cardinal sum of 0-groups. It is easy to prove that an 1-group G with a regular representation is completely distributive [6]. Weinberg has also shown ([6] Proposition 1.3) that the natural homomorphism of an 1-group G onto G/J, where J is an 1-ideal of G, preserves infinite joins if and only if J is closed $(\vee j_\lambda \in G, \{j_\lambda : \lambda \in \Lambda\} \subseteqq J \rightarrow \vee j_\lambda \in J)$. Thus it follows that G has a regular representation if and only if there exists a family of closed 1-ideals J_λ of G such that $\cap J_\lambda = 0$ and each G/J_λ is an 0-group.

LEMMA 5. (Weinberg) *An* 1*-group* G *is completely distributive if and only if for each* $0 < g$ *in* G *there exists* $0 < g^*$ *in* G *such that*

$$g = \vee g_\lambda,\ g_\lambda \in G^+ \rightarrow g^* \leqq g_\lambda \textit{ for some } \lambda.$$

Theorem. *For a representable* 1*-group* G *the following are equivalent.*

(1) $R(G) = 0$.

(2) *Each essential* 1*-ideal of* G *is closed and* $\cap G_\gamma = 0$ $(\gamma \in E)$.

(3) G *has a regular representation.*

(4) G *is completely distributive.*

Proof. By Lemma 3, for each γ in E, G/G_γ is an 0-group, and hence by the preceding discussion (2) implies (3) and (3) implies (4). Suppose that G is completely distributive, and assume (by way of contradiction) that $0 < g \in R(G)$. Then by Lemma 5 there exists $0 < g^* \in G$ such that if $g = \vee g_\alpha$ $(g_\alpha \in G^+)$, then $g^* \leqq g_\alpha$ for some α. Since $g \in R(G)$ it follows that $g \in R_{g^*} = \vee A_\lambda$. where the A_λ are the 1-ideals of G not containing g^*. Thus by Lemma 4, $g = g_1 \vee \cdots \vee g_n$, where $0 \leqq g_i \in \cup A_\lambda$. But then $g^* \leqq g_i$ for some i, and hence $g^* \in \cup A_\lambda$ *a* contradiction. Therefore (4) implies (1).

To complete the proof we must show that (1) implies (2). If (1) is satisfied, then by Lemma 1, $\cap G_\gamma = 0$ $(\gamma \in E)$. Let G_δ be an essential 1-ideal of G and assume (by way of contradiction) that G_δ is not closed. Then there exists $g \in G^+ \backslash G_\delta$ such that $g = \vee g_j (g_j \in G_\delta^+)$. Since G_δ is essential there exists $0 < h \in G$ such that all the values of h are $\leqq \delta$. We shall show that for some such h, $g - h \geqq g_j$ for all j, and hence $\vee g_j > \vee g_j - h = g - h \geqq \vee g_j$.

Case I. There exists $0 < h \in G$ such that all the values of h are $\leqq \delta$ and $G_\delta + h < G_\delta + g$. Since $g - h \notin G_\delta$ and $g_j \in G_\delta$, $g - h - g_j \neq 0$. By the Corollary to Lemma 3 it suffices to show that $G_\beta + g - h - g_j$ is positive for all values β of $g - h - g_j$ in E. If $h \in G_\beta$, then $G_\beta + g - h - g_j = G_\beta + g - g_j$ is positive. If $h \notin G_\beta$, then there exists a value γ of h such that $\gamma \geqq \beta$. But then $\beta \leqq \gamma \leqq \delta$, and since $g - h - g_j \in G^\beta \backslash G_\delta$, $\beta = \delta$. Therefore $G_\beta + g - h - g_j = G_\delta + g - h$ is positive.

Case II. For each $0 < h \in G$ such that all of the values of h are $\leqq \delta$, $G_\delta + h \geqq G_\delta + g$. If $\delta > \gamma \in E$, then we may choose $0 < k \in G$ such that all of the values of k are $\leqq \gamma < \delta$. But then $G_\delta + g > G_\delta = G_\delta + k$. Therefore δ is minimal in E. If all values of $0 < h$ are $\leqq \delta$, then $G_\delta + h \geqq G_\delta + g$ and so $G_\delta + g \wedge h = G_\delta + g$. If β is a value of $g \wedge h$ in E, then $g \wedge h \in G^\beta \backslash G_\beta$ and hence $h \notin G_\beta$. Thus there exists a value γ of h such that $\beta \leqq \gamma \leqq \delta$ and since δ is minimal in E, $\beta = \delta$. Thus without loss of generality, $0 < h \in G$, δ is the only value of h in E and $G_\delta + h = G_\delta + g$. If $g - h - g_j \neq 0$ and β is a value of $g - h - g_j$ in E then $h \in G_\beta$. Otherwise $\beta = \delta$, but $g - h - g_j \in G_\delta$. Therefore $G_\beta + g - h - g_j = G_\beta + g - g_j$ is positive for all values β

of $g - h - g_j$ in E. This completes the proof of our theorem. In proving that (4) implies (1) we did not use the hypothesis that G is representable. Thus we have

COROLLARY I. *If G is a completely distributive* 1-*group, then* $R(G) = 0$.

From the Corollary to Lemma 2 we have

COROLLARY II. *If G is a representable* 1-*group, then whether or not G is completely distributive depends only on the lattice $\mathscr{L}$ of all* 1-*ideals of G.*

4. Remarks and examples. Let P be the 1-group of all order preserving permutations of the real line (with $fg(x) = f(g(x))$ and f positive if $f(x) \geqq x$ for all x). Let

$A = \{f \in P : f$ induces the identity on $(-\infty, a]$ for some $a\}$, and

$B = \{f \in P : f$ induces the identity on $[a, \infty)$ for some $a\}$.

Let $C = A \cap B$. Then Holland [4] has shown that A, B and C are the only proper 1-ideals of G, and Higman [3] has shown that C is algebraically simple. Therefore 0 is the only essential 1-ideal of C and since $C/0$ is not an 0-group it follows from Lemma 3 that C is not representable. Therefore C satisfies property (2) of the theorem, but not property (3).

(G, B) is the only value of each element in $A \backslash B$ and $(C, 0)$ is the only value of each nonzero element in C. Thus B and 0 are essential 1-ideals of P, and in particular, P satisfies (1). For each $n = 1, 2, \cdots$ let

$$f_n(x) = \begin{cases} 2x & \text{if } x \leqq n \\ \dfrac{x + 3n}{2} & \text{if } n \leqq x \leqq 3n \\ x & \text{if } 3n \leqq x\,. \end{cases}$$

Then $(\vee f_n)(x) = 2x$, and hence the f_n belong to B, but $\vee f_n \notin B$. Therefore P satisfies (1) but not (2).

A simple application of Lemma 5 shows that P is completely distributive (or see [6] Example 3.3). Therefore (4) does not imply (2) or (3). On the other hand for arbitrary 1-groups, $(3) \rightarrow (2) \rightarrow (1)$. The remaining question is whether or not (1) or (2) implies (4) for non-representable 1-groups? Note that if $R(G) = 0$ implies complete distributivity, then every 1-group with no proper 1-ideals is completely distributive, and in particular, every 1-group that is algebraically simple is completely distributive.

If the radical used in this note is replaced by one constructed in exactly the same way, but with 1-ideals replaced by convex 1-subgroups, then if this new radical is zero, the group is completely distributive. Also the new radical is an invariant of the lattice of all convex 1-subgroups of G. The proofs of these statements are analogous to those in this paper using the fact that if C is a regular convex 1-subgroup, then the set of right cosets of C in G is totally ordered by

$$C + x \leqq C + y \text{ if } x \leqq y + c \text{ for some } c \in C.$$

Unfortunately the converse to the above is false. For example, the new radical for P is P itself and yet P is completely distributive.

Let G be an Archimedean 1-group. By Theorem 5.7 in [2], $R(G) = 0$ if and only if G has a basis, and by Theorem 7.3 in [1], G has a basis if and only if G is (isomorphic to) a subdirect sum of a cardinal sum of subgroups R_γ of the reals which contains the finite cardinal sum of the R_γ. Thus we have a new proof for one of the main results in [6].

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