LINEAR TRANSFORMATIONS ON GRASSMANN SPACES

R. WESTWICK

1. Let U denote an n-dimensional vector space over an algebraically closed field F, and let G_{nr} denote the set of nonzero pure r-vectors of the Grassmann product space $\bigwedge^r U$. Let T be a linear transformation of $\bigwedge^r U$ which sends G_{nr} into G_{nr}. In this note we prove that T is nonsingular, and then, by using the results of Wei-Liang Chow in [1], we determine the structure of T.

For each $z = x_1 \wedge \cdots \wedge x_r \in G_{nr}$, we let $[z]$ denote the r-dimensional subspace of U spanned by the vectors $x_1, \cdots, x_r$. By Lemma 5 of [1], two independent elements z_1 and z_2 of G_{nr} span a subspace all of whose nonzero elements are in G_{nr} if and only if $\dim([z_1] \cap [z_2]) = r - 1$; that is, if and only if $[z_1]$ and $[z_2]$ are adjacent. If $V \subseteq \bigwedge^r U$ is a subspace such that each nonzero vector in V is in G_{nr} and if V is maximal (that is, not contained in a larger such subspace) then $\{[z] \mid z \in V, z \neq 0\}$ is a maximal set of pairwise adjacent r-dimensional subspaces of U. These sets of subspaces are of two types; namely, the set of all r-dimensional subspaces of U containing a common $(r-1)$-dimensional subspace, and the set of all r-dimensional subspaces of an $(r+1)$-dimensional subspace of U. We adopt the usual convention of calling these sets of subspaces maximal sets of the first and second kind respectively. We will let A_r denote the set of those maximal V which determine a set of pairwise adjacint subspaces of the first kind, and we will let B_r denote the set of those maximal V which determine a set of pairwise adjacent subspaces of the second kind.

2. In this section we prove that if T sends each member of B_r into a member of B_r then T is nonsingular.

Let $U_1, \cdots, U_t$ be k-dimensional pairwise adjacent subspaces of U and let $z_i \in G_{nk}$ be such that $[z_i] = U_i$ for $i = 1, \cdots, t$. Then $\{U_1, \cdots, U_t\}$ is said to be independent if and only if $\{z_1, \cdots, z_t\}$ is an independent subset of $\bigwedge^k U$. We note the following facts concerning an independent set $\{U_1, \cdots, U_t\}$. If it is of the first kind (in the sense of the previous section) then there is an independent set of vectors $\{x_1, \cdots, x_{k-1}, y_1, \cdots, y_t\}$ of U such that for $i = 1, \cdots, t$, $U_i = \langle x_1, \cdots, x_{k-1}, y_i\rangle \cdot \langle \cdots \rangle$ denotes the linear subspace spanned by the vectors enclosed. If it is of the second kind, then there is an independent set of vectors $\{x_1, \cdots, x_{k+1}\}$ such that $U_i = \langle x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k+1}\rangle$, for $i = 1, \cdots, t$. It is easily

Received July 2, 1963. The author is indebted to M. Marcus for his encouragement and help.

deduced from this that $\dim(\bigwedge^r U_1 + \cdots + \bigwedge^r U_t)$ is equal to $t\binom{k-1}{r-1} + \binom{k-1}{r}$ or $\sum_{i=0}^{t-1}\binom{k-i}{r-1}$ according as the set of subspaces $\{U_i\}$ is of the first or second kind. We adopt the usual convention that $\binom{m}{n} = 0$ if $m < n$. Finally, if the set $\{U_1, \cdots, U_t\}$ is not independent, then for some i, $\bigwedge^r U_i \subseteqq \bigwedge^r U_1 + \cdots + \bigwedge^r U_{i-1}$. In fact, the choice of i such that $\{z_1, \cdots, z_{i-1}\}$ is independent and $z_i \in \langle z_1, \cdots, z_{i-1}\rangle$ will do.

We require the

LEMMA 1. *Let $\{U_1, \cdots, U_{s+1}\}$ be a set of pairwise adjacent k-dimensional subspaces of U. Suppose further that the set is independent and is of the second kind. Let $V \subseteqq \bigwedge^r U_1 \cdots + \bigwedge^r U_{s+1}$ be a subspace with dimension $\binom{k-s}{r-s}$, where $s \leqq r \leqq k$. Then there is a set $\{V_1, \cdots, V_s\}$ of pairwise adjacent k-dimensional subspaces of U such that $V \cap (\bigwedge^r V_1 + \cdots + \bigwedge^r V_s) \neq \{0\}$.*

Proof. Let $m = \binom{k-s}{r-s}$ and let $\{z_1, \cdots, z_m\}$ be a basis of V. Choose an independent set of vectors $\{x_1, \cdots, x_{k+1}\}$ of U such that for $i = 1, \cdots, s+1$, $U_i = \langle x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k+1}\rangle$. We can write

$$z_i = z_1^i + x_1 \wedge \cdots \wedge x_{s-1} \wedge x_s \wedge z_2^i + x_1 \wedge \cdots \wedge x_{s-1} \wedge x_{s+1} \wedge z_3^i$$

where

$$z_1^i \in \bigwedge^r U_1 + \cdots + \bigwedge^r U_{s-1} \quad\text{and}\quad z_2^i, z_3^i \in \bigwedge^{r-s} \langle x_{s+2}, \cdots, x_{k+1}\rangle$$

for $i = 1, \cdots, m$. In the case that $s = 1$, we take $z_1^i \in \bigwedge^r \langle x_3, \cdots, x_{k+1}\rangle$. In the case that $s = r$, we take $z_2^i, z_3^i \in F$. If $\{z_2^1, \cdots, z_2^m\}$ or $\{z_3^1, \cdots, z_3^m\}$ is dependent, then we can form a linear combination of $z_1, \cdots, z_m$ which will be in $\bigwedge^r U_1 + \cdots \bigvee^r{}_{s-1} + \bigwedge^r U_{s+1}$ or $\bigwedge^r U_1 + \cdots + \bigwedge^r U_{s-1} + \bigwedge^r U_s$ respectively. If, on the other hand, both sets are independent then each is a basis of $\bigwedge^{r-s}\langle x_{s+2}, \cdots, x_{k+1}\rangle$ since $\dim(\bigwedge^{r-s}\langle x_{s+2}, \cdots, x_{k+1}\rangle) = \binom{k-s}{r-s} = m$. Let $z_2^i = \sum_{j=1}^m a_{ij} z_3^j$, $i = 1, \cdots, m$. Choose $\lambda \neq 0$ and $b_i \in F$, not all equal to zero, such that

$$\lambda b_j = \sum_{i=1}^m b_i a_{ij}\ , \qquad j = 1, \cdots, m\ .$$

Then

$$0 \neq \sum_{j=1}^m b_j z_j = \sum_{j=1}^m z_1^j + \sum_{j=1}^m x_1 \wedge \cdots \wedge x_{s-1} \wedge (x_s + \lambda^{-1}x_{s+1}) \wedge b_j z_2^j$$
$$\in \bigwedge^r U_1 + \cdots + \bigwedge^r U_{s-1} + \bigwedge^r V_1$$

where $V_1 = \langle x_1 \cdots, x_{s-1}, x_s + \lambda^{-1}x_{s+1}, x_{s+2}, \cdots, x_{k+1}\rangle$. The subspaces

$U_1, \cdots, U_{s-1}, V_1$ are pairwise adjacent and so the Lemma is proved.

The nonsingularity of T is now proved as follows. Let W be a subspace of U. We prove, by induction on the dimension of W, that T is one-to-one on $\bigwedge^r W$ and that the image of $\bigwedge^r W$ under T is $\bigwedge^r W'$ for some subspace W' of U with $\dim(W) = \dim(W')$. When $\dim(W) = r + 1$ this is clear since we are assuming that B_r is sent into B_r by T. Suppose that the statement has been proved for k-dimensional subspaces, and consider a $(k+1)$-dimensional subspace W of U. Let s be the largest integer such that for any set $\{W_1, \cdots, W_s\}$ of pairwise adjacent k-dimensional subspaces of W, T is one-to-one on $\bigwedge^r W_1 + \cdots + \bigwedge^r W_s$. If $s \geqq r + 1$ then T is one-to-one on $\bigwedge^r W$, since in this case, for an independent set $\{W_1, \cdots, W_s\}$ we must have $\bigwedge^r W = \bigwedge^r W_1 + \cdots + \bigwedge^r W_s$. Suppose then that $1 \leqq s \leqq r$ and let $\{U_1, \cdots, U_{s+1}\}$ be any set of $s+1$ pairwise adjacent k-dimensional subspaces of W. If the set is dependent then T is one-to-one $\bigwedge^r U_1 + \cdots + \bigwedge^r U_{s+1}$ since we may drop one of the terms. Therefore we assume that the set is independent. Choose k-dimensional subspaecs $U_1', \cdots, U_{s+1}'$ such that $T(\bigwedge^r U_i) = \bigwedge^r U_i'$ for $i = 1, \cdots, s+1$. For each $j \leqq s$, T maps $\bigwedge^r U_1 + \cdots + \bigwedge^r U_j$ onto $\bigwedge^r U_1' + \cdots + \bigwedge^r U_j'$. Therefore, since T is one-to-one on $\bigwedge^r U_1 + \cdots + \bigwedge^r U_s$, the set $\{U_1', \cdots, U_s'\}$ is independent. Furthermore, the set $\{U_1', \cdots, U_{s+1}'\}$ is also independent. If not, then the image under T of both $\bigwedge^r U_1 + \cdots + \bigwedge^r U_s$ and $\bigwedge^r U_1 + \cdots \bigwedge^r U_{s+1}$ is $\bigwedge^r U_1' + \cdots + \bigwedge^r U_s'$. But then the dimension of the null space of T in $\bigwedge^r U_1 + \cdots + \bigwedge^r U_{s+1}$ is at least as large as the difference in the dimensions of $\bigwedge^r U_1 + \cdots + \bigwedge^r U_{s+1}$ and $\bigwedge^r U_1 + \cdots + \bigwedge^r U_s$, that is, $\binom{k-s}{r-s}$. We apply Lemma 1 to contradict the choice of s. It follows that T is one-to-one on all of $\bigwedge^r W$. Finally, let $\{W_1, \cdots, W_{k+1}\}$ be an independent set of k-dimensional pairwise adjacent subspaces of W (necessarily of the second kind). Let W_i' be chosen so that $T(\bigwedge^r W_i) = \bigwedge^r W_i'$. It follows easily that $\{W_1', \cdots, W_{k+1}'\}$ is of the second kind also, so that the image of $\bigwedge^r W$ is $\bigwedge^r W'$ where W' is the $(k+1)$-dimensional subspace of U containing $W_1', \cdots, W_{k+1}'$. By taking $W = U$ we see that T is one-to-one on $\bigwedge^r U$.

3. It is necessary to investigate whether a general T does necessarily send each element of B_r into B_r. For the cases $n > 2r$, $n < 2r$, this is proved directly, using Lemma 2. The case $n = 2r$ requires a more delicate argument, given at the end of this section; there it is shown that if some element of B_r is sent into B_r by T, then T sends B_r into B_r.

LEMMA 2. *Let* $r < n$ *and let* V_1 *and* V_2 *be in* A_r *such that* $V_1 \cap V_2 \neq \{0\}$. *Then, if* $V \subseteqq V_1 + V_2$ *and* $\dim(V) = n - r$, *we have* $V \cap G_{nr} \neq \phi$.

Proof. Let U_i be the $(r-1)$-dimensional subspace of U determined by V_i for $i = 1, 2$. Since $V_1 \cap V_2 \neq \{0\}$, either $U_1 = U_2$ or $\dim(U_1 \cap U_2) = r - 2$.

If $U_1 = U_2$ then $V_1 = V_2$, so that in this case it is clear that $V \cap G_{nr} \neq \phi$.

Suppose that $\dim(U_1 \cap U_2) = r - 2$ and let $\{x_1, \cdots, x_{r-2}\}$ be a basis of this intersection. Choose y_i such that $U_i = \langle x_1, \cdots, x_{r-2}, y_i \rangle$ for $i = 1, 2$. Choose u_i and v_i in U, $i = 1, \cdots, n - r$, such that

$$\{z_i = x_1 \wedge \cdots \wedge x_{r-2} \wedge (y_1 \wedge u_i + y_2 \wedge v_i) \mid i = 1, \cdots, n - r\}$$

forms a basis of V. If

$$\{x_1, \cdots, x_{r-2}, y_1, y_2, v_1, \cdots, v_{n-r}\} \quad \text{or} \quad \{x_1, \cdots, x_{r-2}, y_1, y_2, u_1, \cdots, u_{n-r}\}$$

is dependent, then there is a linear combination of the z_i which is in V_1 or V_2 respectively. If, on the other hand, both sets are independent, then they are both bases for U and we may write

$$u_i = w_i + c_i y_2 + \sum_{j=1}^{n-r} a_{ij} v_j \,, \qquad\qquad i = 1, \cdots, n - r \,,$$

where $w_i \in \langle x_1, \cdots, x_{r-2}, y_1 \rangle$ and $c_i, a_{ij} \in F$. We note that $\det(a_{ij}) \neq 0$ so we can choose $\lambda \neq 0$ and b_j for $j = 1, \cdots, n - r$, not all zero, such that $\lambda b_j = \sum_{i=1}^{n-r} b_i a_{ij}$. Then

$$0 \neq \sum_{j=1}^{n-r} b_j z_j = x_1 \wedge \cdots \wedge x_{r-2} \wedge (y_1 + \lambda^{-1} y_2) \wedge \left[\left(\sum_{j=1}^{n-r} b_j c_j\right) y_2 + \lambda \sum_{j=1}^{n-r} b_j v_j\right]$$

is an element of $V \cap G_{nr}$. This proves the Lemma.

For $n \neq 2r$ the image under T of an element of B_r is an element of B_r. For $n < 2r$ this is clearly so since the subspaces of $\bigwedge^r U$ in B_r have dimension $r + 1$, which is greater than the dimension $(n - r + 1)$ of the subspaces in A_r.

For $n > 2r$ we proceed as follows. The image of an A_r is an A_r. Suppose that the image of a $W \in B_r$ is a subspace of a $V \in A_r$. Choose two elements V_1 and V_2 of A_r such that $V_1 \cap V_2 \neq \{0\}$ and $\dim(V_1 \cap W) = \dim(V_2 \cap W) = 2$. One does this by choosing V_1 and V_2 so that the $(r-1)$-dimensional subspaces of U determined by them are adjacent subspaces of the $(r+1)$-dimensional subspace determined by W. Now, $T(V_1) = T(V_2) = V$ since each is in A_r and each intersects V in at least two dimensions. Therefore $T(V_1 + V_2) = V$ and so the null space of T in $V_1 + V_2$ has dimension equal to $(2n - 2r + 1) - (n - r + 1) = n - r$. By Lemma 2, it follows that the null space of T intersects G_{nr} which contradicts the hypothesis that T sends G_{nr} into G_{nr}.

In the case that $n = 2r$ the image of a B_r may be an A_r since the dimensions are equal. However, we prove that if some B_r is sent into a B_r by T, then the image of each B_r is a B_r. Suppose not. Then we can choose $(r+1)$-dimensional subspaces W_1 and W_2 of U such that $T(\bigwedge^r W_1) \in A_r$ and $T(\bigwedge^r W_2) \in B_r$. Furthermore, we can choose W_1 and W_2 adjacent, so that $\dim(W_1 \cap W_2) = r$. Choose three distinct elements V_1, V_2, and V_3 of A_r such that the $(r-1)$-dimensional subspaces of U determined by these elements are contained in $W_1 \cap W_2$. Then $\dim(V_i \cap \bigwedge^r W_j) = 2$ for $i = 1, 2, 3$ and $j = 1, 2$, so that $T(V_i)$ intersects $T(\bigwedge^r W_j)$ in at least two dimensions for each i, j. This implies that each $T(V_i)$ is equal to one of $T(\bigwedge^r W_j)$ and so two of them are equal. The argument of the previous paragraph now leads to a contradiction.

4. By essentially the same argument as used by Chow in [1] to prove his Theorem 1, we can prove that; if S is a nonsingular linear transformation of $\bigwedge^r U$ sending G_{nr} into G_{nr}, and if the image of each B_r is a B_r, then S is a compound. (By a compound we mean a linear transformation of $\bigwedge^r U$ which is induced by a linear transformation of U.)

In the case that $n \neq 2r$ it follows that T is necessarily a compound. For $n = 2r$, T is a compound if some B_r is sent into a B_r. If we let T_0 denote a linear transformation of $\bigwedge^r U$ induced by a correlation of the r-dimensional subspaces of U, then T_0 is nonsingular and sends G_{nr} onto G_{nr}. The image of each A_r under T_0 is a B_r. Therefore, if a B_r is sent by T into an A_r, the $T_0 T$ is a compound. We have proved the

THEOREM. *Let U be an n-dimensional vector space over an algebraically closed field and let T be a linear transformation of $\bigwedge^r U$ which sends G_{nr} into G_{nr}. Then T is a compound except, possibly, when $n = 2r$, in which case T may be the composite of a compound and a linear transformation induced by a correlation of the r-dimensional subspaces of U.*

REFERENCE

1. Wei-Liang Chow, *On the Geometry of Algebraic Homogeneous Spaces*, Annals of Math., **50** (1949), 32-67.