ON THE REFLECTION OF HARMONIC FUNCTIONS AND OF SOLUTIONS OF THE WAVE EQUATION

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Introduction. While the analytic extension of a harmonic function across analytic differential boundary conditions is always possible for the case of two independent variables [3], no comparable global theorem exists for harmonic functions in $N > 2$ variables.

This work is concerned with the problem of global extension of a harmonic function $U(x, y, z)$ across a plane on which U satisfies a linear differential boundary condition of the form

$$B(U) \equiv \frac{\partial U}{\delta z} + P_n(x, y)U = 0 \qquad \text{on } \sigma(z = 0) ,$$

where $P_n(x, y)$ is a polynomial of degree n. It is assumed here that the given function U is C^1 in the closure of a cylindrical domain $R: \{x^2 + y^2 < \rho^2, \ -l < z < 0\}$.

The possibility of harmonic reflection is obvious for $n = 0$, $P_n =$ const. as $B(U)$ itself is harmonic. Since it vanishes on $z = 0$, it can be extended harmonically, and the harmonic extension of U can then be found by integrating with respect to z. But such procedure is no longer available in our case. We shall show, how our problem can be reduced to that of solving an initial value problem of a certain hyperbolic differential equation (1.22) of order $2n$ with distinct characteristic surfaces (of normal type).

Classical considerations yield the analyticity of U on σ and, therefore, its harmonic extensibility across σ into a neighborhood of σ. Our result asserts that this neighborhood is the whole of the mirror image of R, denoted by $\bar{R}$.

Our method consists of constructing a new function $V(x, y, z)$ from U and a differential expression in V (see (1.6) and (1.18)), which is harmonic in R and vanishes on $z = 0$. Thus, this expression in V can be first extended into $R \cup \sigma \cup \bar{R}$ as a harmonic function $\varphi(x, y, z)$. The solution of the differential equation thus obtained for V in $\bar{R}$ is impeded by its degeneracy. To remove this degeneracy we add to the differential equation the Laplacian of V and its higher derivatives in such a way as to obtain a normal hyperbolic problem (1.22), whose solution is guaranteed by a result of I. G. Petrovsky. This modification of the differential equation can be done in infinitely many ways, in particular, so as to make the characteristic surfaces close down on

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parallels to the z-axis. Local extensibility of U, together with the solution of the modified equation, then yields the global extension of U. We note, that this method works equally well for $N > 3$ independent variables.

The above described method, however, seems to fail in the case of the wave equation when σ is part of the tilmelike plane $z = 0$, and the boundary condition on it is as simple as $U_z + xU = 0$.

On the other hand, the oblique derivative problem for the wave equation $U_{xx} + U_{yy} - U_{tt} = 0$, whose solution satisfies the boundary condition

$$B'(U) \equiv U_x + \alpha U_y + (Ay + B)U = 0 \qquad \text{on } x = 0 ,$$

yields to a similarly motivated, yet formally different attack. The domain of extension in this case depends on $\alpha \neq 0$.

I would like to take this opportunity to express my gratitude to professor H. Lewy who suggested this problem and offered advice during its investigation.

1. Analytic extension of harmonic functions. We consider an open cylindrical domain $R : \{x^2 + y^2 < \rho^2, -l < z < 0\}$ and the plane region $\sigma : \{x^2 + y^2 < \rho^2, z = 0\}$. Denote by $\bar{R}$ the mirror image of R with respect to the $z = 0$ plane.

Let there be given a real function $U(x, y, z)$, $U \in C^1$ in the closure of R, such that:

$$U_{xx} + U_{yy} + U_{zz} \equiv \Delta U = 0 \qquad \text{in } R \tag{1.1}$$

$$\frac{\partial U}{\partial z} + P_n(x, y) U = 0 \qquad \text{on } \sigma \tag{1.2}$$

where $P_n(x, y)$ is a polynomial in x, y of degree n, given in the form

$$P_n(x, y) = \sum_{k=0}^{n} \sum_{m=0}^{k} A_{km}\, x^{k-m} y^m , \tag{1.3}$$

the coefficients A_{km} being real.

LEMMA 1. *If $U(x, y, z)$ is harmonic in R, $U \in C^1$ in $R \cup \partial R$, and satisfies condition* (1.2) *on σ, then U can be harmonically extended into $R \cup \sigma \cup G$, where G is the portion $z > 0$ of some neighborhood of σ.*

Proof. Since U is C^1 in $R \cup \partial R$, we have by Green's formula

$$4\pi U(X) = \iint_{\partial R} \left\{ \frac{1}{|X - \tau|} \frac{\partial U(\tau)}{\partial n} - U(\tau) \frac{\partial}{\partial n} \frac{1}{|X - \tau|} \right\} d\tau \tag{1.4}$$

where $X = (x, y, z)$, $\tau = (\xi, \eta, \zeta)$, n is the outer normal, and integration is over the surface of the cylinder $\xi^2 + \eta^2 = \rho^2$, $\zeta = -l$, $\zeta = 0$. By (1.2) this becomes

$$4\pi U(X) = A(X) - \iint_\sigma \left\{ \frac{P_n(\tau)U(\tau)}{|X-\tau|} + U(\tau)\frac{\partial}{\partial\zeta}\frac{1}{|X-\tau|} \right\} d\tau$$

where $A(X)$ stands for the integral in (1.4) taken over the lateral surface and the lower base of the cylinder. By passage to the limit as X tends to $X' \in \sigma$, one obtains in a manner familiar in potential theory,

$$2\pi U(X') = A(X') - \iint_\sigma \frac{P_n(\tau')U(\tau')}{|X' - \tau'|} d\tau'$$

where $A(X')$ is an analytic function on σ. This integral equation is an especially simple case of E. Hopf's equation (6.1) ([2], page 220), and his method yields immediately the result, that $U(x, y, 0)$ is analytic on the open disc σ.

Since, due to condition (1.2), $U_z(x, y, 0)$ is also analytic, we obtain from the Cauchy-Kowalewski theorem, that there exists an analytic solution $\tilde{U}$ of Cauchy's problem with $\tilde{U} = U$, $\tilde{U}_z = U_z$ on σ for $\Delta\tilde{U} = 0$ in some neighborhood G of σ.

If we continue U, given in $R \cup \sigma$, as $\tilde{U}$ in $G - R - \sigma$, this new function is, according to well known arguments, harmonic in $R \cup \sigma \cup G$.

We now introduce the symbolic notation

$$D_z^{-1} f(x, y, z) = \int_0^z f(x, y, \zeta) d\zeta \,, \tag{1.5}$$

and define an analytic function $V(x, y, z)$ for $(x, y, z) \in R \cup \sigma$:

$$V(x, y, z) \equiv D_z^{-(2n-1)} U(x, y, z) + \sum_{k=0}^{2n-2} \frac{z^k}{k!} F_k(x, y) \,, \tag{1.6}$$

where the functions $F_k(x, y)$ $(0 \leqq k \leqq 2n - 2)$ are solutions of the following equations on σ:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) F_{2n-2} + U_z(x, y, 0) = 0 \tag{1.7}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) F_{2n-3} + U(x, y, 0) = 0 \tag{1.8}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) F_r + F_{r+2} = 0 \qquad (0 \leqq r \leqq 2n - 4) \tag{1.9}$$

with, say, boundary values zero on $x^2 + y^2 = \rho^2$.

The choice of these functions is motivated by the requirements

$$(1.10)\qquad \Delta V = 0 \quad \text{in } R$$

$$(1.11)\qquad V_{z^{2n}} + P_n(x, y) V_{z^{2n-1}} = 0 \quad \text{on } \sigma ,$$

which are easily verified.

Let s stand for either x or for y, and denote

$$H_{s,z} = s\frac{\partial}{\partial z} - z\frac{\partial}{\partial s}, \quad H^m_{s,0} = (H_{s,z})^m|_{z=0} .$$

We then have the identities:

$$(1.12)\qquad H^{2m+1}_{s,0} = \sum_{k=0}^{m}\sum_{j=0}^{k} a^m_{jk} s^{2k-j+1}\frac{\partial^{2k-j+1}}{\partial s^j \partial z^{2k-2j+1}} \qquad (m = 0, 1, 2, \cdots)$$

$$(1.13)\qquad H^{2m}_{s,0} = \sum_{k=0}^{m}\sum_{j=0}^{k} b^m_{jk} s^{2k-j}\frac{\partial^{2k-j}}{\partial s^j \partial z^{2k-2j}} \qquad (m = 1, 2, \cdots)$$

where the coefficients a^m_{jk} and b^m_{jk} are real numbers, and $a^m_{0m} = b^m_{0m} = 1.$

Proof. Introducing new variables $t = s + iz$, $\tau = s - iz$, we may write, with $\partial/\partial t = 1/2[(\partial/\partial s) - i(\partial/\partial z)]$ and $\partial/\partial\tau = 1/2[(\partial/\partial s) + i(\partial/\partial z)]$

$$H_{s,z} = i\Big(t\frac{\partial}{\partial t} - \tau\frac{\partial}{\partial \tau}\Big) .$$

Hence,

$$(1.14)\qquad H^n_{s,0} = i^n \sum_{p=0}^{n} (-1)^p \binom{n}{p}\Big(t\frac{\partial}{\partial t}\Big)^{n-p}\Big(\tau\frac{\partial}{\partial\tau}\Big)^p\Big|_{z=0} .$$

Now, for any variable ξ (real or complex)

$$\Big(\xi\frac{\partial}{\partial\xi}\Big)^r = \sum_{h=0}^{r} B^r_h \xi^h \frac{\partial^h}{\partial\xi^h}$$

where the coefficients B^r_h are nonnegative integers. Since $\partial\tau/\partial t = \partial t/\partial\tau = 0$, and for $z = 0$ we have $t = \tau = s$, each term in (1.14) is, but for a constant coefficient, of the form

$$t^\alpha \tau^\beta \frac{\partial^{\alpha+\beta}}{\partial t^\alpha \partial\tau^\beta}\Big|_{z=0} = s^{\alpha+\beta}\frac{\partial^{\alpha+\beta}}{\partial t^\alpha \partial\tau^\beta} \qquad (1 \leqq \alpha + \beta \leqq n) .$$

Since $\partial^2/\partial t\partial\tau = 1/4[(\partial^2/\partial s^2) + (\partial^2/\partial z^2)]$, each term in (1.14) is, but for a constant coefficient, either of the form

$$s^{\alpha+\beta}\Big[\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial z^2}\Big]^{\min\alpha,\beta}\Big(\frac{\partial}{\partial t}\Big)^{|\alpha-\beta|},$$

or of the form

$$s^{\alpha+\beta}\Big[\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial z^2}\Big]^{\min\alpha,\beta}\Big(\frac{\partial}{\partial \tau}\Big)^{|\alpha-\beta|} .$$

Now, for any positive integer q, $[(\partial/\partial s) \pm i(\partial/\partial z)]^q$ has terms with imaginary coefficients only of the form $\partial^{\lambda+\mu}/\partial s^\lambda \partial z^\mu$, where μ is odd, and terms with real coefficients only of the form $\partial^{\lambda+\mu}/\partial s^\lambda \partial z^\mu$, where μ is even ($\lambda + \mu = q$). Consequently, as $H^n_{s\,0}$ must have real coefficients, it will consist of terms $s^{\lambda+\mu}(\partial^{\lambda+\mu}/\partial s^\lambda \partial z^\mu)$, where μ is odd when n is odd, and μ is even when n is even, which implies identities (1.12) and (1.13).

LEMMA 2. *There exist differential operators*

$$D_i^r \equiv \sum_{h=0}^{r} C_{irh} \frac{\partial^r}{\partial s^h \partial z^{r-h}} ,$$

where C_{irh} are real constants, such that

$$s^p \frac{\partial^{2p-1}}{\partial z^{2p-1}} = \sum_{i=1}^{P} H^i_{s,0} D_i^{p-1} \qquad \textit{for } z = 0 \tag{1.15}$$

Proof. Starting from the definition of $H^n_{s\,0}$ we see, that the above statement holds for $p = 1$ and $p = 2$, with $D_1^0 = 1$, $D_1^1 = \partial/\partial s$ and $D_2^1 = \partial/\partial z$. Assuming, that the statement holds for $p \leqq 2n$, we prove by induction, that it also holds for $p = 2n + 1$ and $p = 2n + 2$.

Since, by assumption, the lemma holds for $p \leqq 2n$, we have for any nonnegative integers α and β, and any positive integer $q \leqq 2n$

$$s^q \frac{\partial^{2q-1+\alpha+\beta}}{\partial s^\alpha \partial z^{2q-1+\beta}} = \sum_{i=1}^{q} H^i_{s,0} D_i^{q-1+\alpha+\beta} . \tag{1.16}$$

But identity (1.12) yields

$$s^{2n+1} \frac{\partial^{4n+1}}{\partial z^{4n+1}} = H^{2n+1}_{s,0} \frac{\partial^{2n}}{\partial z^{2n}} - \sum_{j=1}^{n} a^n_{jn} s^{2n-j+1} \frac{\partial^{4n-j+1}}{\partial s^j \partial z^{4n-2j+1}}$$
$$- \sum_{k=0}^{n-1} \sum_{j=0}^{k} a^n_{jk} s^{2k-j+1} \frac{\partial^{2n+2k-j+1}}{\partial s^j \partial z^{2n+2k-2m+1}} .$$

We now observe, that all terms on the right hand side of the above expression are of the form (1.16), where $q = 2n - j + 1$ $(1 \leqq j \leqq n$, i.e. $q \leqq 2n)$, $\alpha = j$, $\beta = 0$, for terms contained in the simple sum, and $q = 2k - j + 1$ $(0 \leqq j \leqq k,\ 0 \leqq k \leqq n - 1$, i.e. $q \leqq 2n - 1)$, $\alpha = j$, $\beta = 2n - 2k$, for terms contained in the double sum. Hence, the above lemma holds for $p = 2n + 1$.

A similar argument, which utilizes identity (1.13) instead of (1.12), shows that this lemma holds also for $p = 2n + 2$, and thus completes the proof.

We now introduce the differential operator of order $2p - 1$

$$Q^p_{s,z} \equiv \sum_{i=1}^{p} H^i_{s,z} D_i^{p-1} \qquad (p \geqq 1) \tag{1.17}$$

where the D_i^{p-1} are those of (1.15). Note that, for $z=0$, $Q^p_{s,0} = s^p(\partial^{2p-1}/\partial z^{2p-1})$.

Define an analytic function $\varphi(x, y, z)$ for $(x, y, z) \in R \cup \sigma$:

$$\varphi(x, y, z) \equiv V_{z^{2n}}(x, y, z) + NV(x, y, z)\ . \tag{1.18}$$

Here $V(x, y, z)$ is the function defined in (1.6), and $N = N(x, y, z)$ is a differential operator of order $2n-1$ defined by:

$$\begin{aligned} N(x, y, z) = A_{00}\frac{\partial^{2n-1}}{\partial z^{2n-1}} &+ \sum_{k=1}^{n}(A_{k0}Q^k_{x,z} + A_{kk}Q^k_{y,z})\frac{\partial^{2n-2k}}{\partial z^{2n-2k}} \\ &+ \sum_{k=2}^{n}\sum_{m=1}^{k-1} A_{km}Q^{k-m}_{x,z}Q^m_{y,z}\frac{\partial^{2n-2k+1}}{\partial z^{2n-2k+1}} \end{aligned} \tag{1.19}$$

where the coefficients A_{km} are the coefficients of the polynomial $P_n(x, y)$ defined in (1.3).

LEMMA 3. *$\Delta\varphi = 0$ in R, and $\varphi(x, y, 0) = 0$.*

Proof. Note, that $\Delta H_{x,z} = H_{x,z}\Delta$ and $\Delta H_{y,z} = H_{y,z}\Delta$. Thus, by (1.17) and (1.19), the operators Δ and N commute. Therefore, operating on both sides of (1.18) by Δ, and making use of (1.10), we obtain

$$\Delta\varphi = \left(\frac{\partial^{2n}}{\partial z^{2n}} + N\right)\Delta V = 0 \qquad \text{in } R\ .$$

Making use of (1.17) and (1.15) we may write, for $z = 0$,

$$\begin{aligned} &N(x, y, z)V(x, y, z)|_{z=0} \\ &= \Bigg\{A_{00}\frac{\partial^{2n-1}}{\partial z^{2n-1}} + \sum_{k=1}^{n}\left(A_{k0}x^k\frac{\partial^{2k-1}}{\partial z^{2k-1}} + A_{kk}y^k\frac{\partial^{2k-1}}{\partial z^{2k-1}}\right)\frac{\partial^{2n-2k}}{\partial z^{2n-2k}} \\ &\quad + \sum_{k=2}^{n}\sum_{m=1}^{k-1}A_{km}x^{k-m}\frac{\partial^{2k-2m-1}}{\partial z^{2k-2m-1}}y^m\frac{\partial^{2m-1}}{\partial z^{2m-1}}\frac{\partial^{2n-2k+1}}{\partial z^{2n-2k+1}}\Bigg\}V(x, y, z)\Bigg|_{z=0}\ , \end{aligned}$$

which becomes

$$N(x, y, z)V(x, y, z)|_{z=0} = \sum_{k=0}^{n}\sum_{m=0}^{k}A_{km}x^{k-m}y^m V_{z^{2n-1}}(x, y, 0)\ . \tag{1.20}$$

Thus, setting $z = 0$ in (1.18) and making use of (1.20) and (1.11) we obtain $\varphi\ (x, y, 0) = 0$.

Hence, if we set for $(x, y, z) \in \bar{R} \cup \sigma$

$$\varphi(x, y, z) = -\varphi(x, y, -z) \equiv -\left[\frac{\partial^{2n}}{\partial\zeta^{2n}} + N(x, y, \zeta)\right]V(x, y, \zeta)\bigg|_{\zeta=-z}\ , \tag{1.21}$$

then φ is harmonic in $R \cup \sigma \cup \bar{R}$.

Since $\varphi(x, y, -z)$ is known for $(x, y, z) \in \bar{R} \cup \sigma$, we shall seek a function $\bar{V}(x, y, z)$ for $(x, y, z) \in \bar{R} \cup \sigma$, which satisfies the following overdetermined system (S) for $\bar{V}$ on $z > 0$:

$$\left.\begin{array}{l}\bar{V}_{z^{2n}}(x, y, z) + N(x, y, z)\bar{V}(x, y, z) = -\varphi(x, y, -z) \\ \Delta\bar{V}(x, y, z) = 0 \\ \left.\dfrac{\partial^r \bar{V}}{\partial z^r}\right|_{z=0} = F_r(x, y) \quad 0 \leqq r \leqq 2n-2 \quad \bar{V}_{z^{2n-1}}(x, y, 0) = U(x, y, 0)\end{array}\right\} \text{(S)}$$

where the functions $F_r(x, y)$ are defined by the equations (1.7), (1.8) and (1.9).

Since, by Lemma 1, U can be continued into $R \cup \sigma \cup G$ as an analytic function, the formula (1.6) can be used to define a function $V^*(x, y, z)$ as an analytic function in $R \cup \sigma \cup G'$, where G' consists of all those points of G, which can be joined in G to points of σ by parallels to the z-axis. This, so defined function V^* is harmonic in $R \cup \sigma \cup G'$, satisfies the initial conditions of (S), and

$$V^*_{z^{2n}}(x, y, z) + NV^* = -\left[\frac{\partial^{2n}}{\partial\zeta^{2n}} + N(x, y, \zeta)\right]V^*(x, y, \zeta)\,|_{\zeta=-z}$$
$$= -\varphi(x, y, -z) \qquad \text{in } G' .$$

Thus, a solution $V^*(x, y, z)$ of system (S) *exists for $(x, y, z) \in G' \cup \sigma$.*

To investigate the size of the domain into which $V(x, y, z)$ can be continued, consider the solution of the following Cauchy problem:

$$\text{(1.22)} \qquad M\bar{V}(x, y, z) \equiv \prod_{i=1}^{n}\left[\frac{\partial^2}{\partial z^2} - \alpha_i\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\bar{V} + \beta N(x, y, z)\bar{V}$$
$$= -\beta\varphi(x, y, -z)$$

$$\text{(1.23)} \qquad \left.\frac{\partial^r \bar{V}}{\partial z^r}\right|_{z=0} = F_r(x, y) \ (0 \leqq r \leqq 2n-2)\ , \ \bar{V}_{z^{2n-1}}(x, y, 0) = U(x, y, 0)$$

where $\alpha_i (i = 1, 2, \cdots, n)$ are distinct positive real numbers, and $\beta = \prod_{i=1}^{n}(1 + \alpha_i)$.

Now, for distinct positive α_i, M is a normal hyperbolic operator with the distinct characteristic sheets through a point (x^0, y^0, z^0) of the form $(x - x^0)^2 + (y - y^0)^2 = \alpha_i(z - z^0)^2$. It is a result of I. G. Petrovsky (see [1]), that the Cauchy problem (1.22), (1.23) has the unique C^∞ *solution $\bar{V}(x, y, z)$ in that part R^*_α $(\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n))$ of the domain* of influence of the initial surface σ for the equation $M\bar{V}(x, y, z) = -\beta\varphi(x, y, -z)$, which lies in $\bar{R}$, so that $\varphi(x, y, -z)$ is defined.

In view of the identity

$$\prod_{i=1}^{n}(1 + \alpha_i)\frac{\partial^{2n}}{\partial z^{2n}} - \prod_{i=1}^{n}\left[\frac{\partial^2}{\partial z^2} - \alpha_i\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right] = P(\Delta)$$

where P is a polynomial in $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ the function $V^*(x, y, z)$, which solves system (S) in G' satisfies the above Cauchy problem (1.22), (1.23) in the neighborhood of the initial surface σ, and by uniqueness, the solution $\bar{V}(x, y, z) \in R_\alpha^*$ must coincide with $V^*(x, y, z)$ in that neighborhood. Consequently, $\Delta \bar{V}$ and all its derivatives vanish on σ.

Since the operators M and Δ commute, operating on equation (1.22) by Δ we obtain $M(\Delta \bar{V}) = 0$. Therefore, by uniqueness of the solution of Cauchy's problem for $M(\Delta \bar{V}) = 0$ with homogeneous initial conditions, we conclude that $\bar{V}(x, y, z)$, which solves (1.22), (1.23), is harmonic in R_α^* and solves system (S) in this domain.

Putting $U(x, y, z) = (\partial^{2n-1}/\partial z^{2n-1}) \bar{V}(x, y, z)$ for $(x, y, z) \in R_\alpha^*$ we have constructed the harmonic extension of U into $R \cup \sigma \cup R_\alpha^*$. We now observe, that as $\alpha_i \to 0$ $(i = 1, 2, \cdots, n)$ the characteristic surfaces of M close down on parallels to the z-axis. It follows, that every point of $\bar{R}$ is in some R_α^* for α_i sufficiently small. In view of the simple connectedness of $R \cup \sigma \cup \bar{R}$, the harmonic extension of U at any point of $\bar{R}$ cannot depend on α, and it follows that U can be harmonically extended into all of $R \cup \sigma \cup \bar{R}$. Thus,

THEOREM 1. *If $U(x, y, z)$ is harmonic in R, $U \in C^1$ in $R \cup \partial R$, and satisfies condition* (1.2) *on σ, then U can be harmonically extended into $R \cup \sigma \cup \bar{R}$.*

REMARK The construction of the extension of U depended on the solution of a hyperbolic problem whose order is twice the degree of the polynomial $P_n(x, y)$, the coefficient in the first order boundary condition. This illustrates the difficulty of extending our result to the case of, say, a coefficient $f(x, y)$, which is an entire function.

2. Extension of solutions of the wave equation. We consider an open domain $D: \{-m < x < 0, \ -l < y < l, \ -l < t < l\}$ and the plane region $\sigma: \{x = 0, \ -l < y < l, \ -l < t < l\}$. Denote, for any domain $\mathscr{D}$, the mirror image of $\mathscr{D}$ with respect to the $x = 0$ plane by $\bar{\mathscr{D}}$.

Let there be given a real function $U(x, y, t)$, $U \in C^4$ in the closure of D, such that:

(2.1) $$LU \equiv U_{xx} + U_{yy} - U_{tt} = 0 \quad \text{in } D$$

(2.2) $$U_x + \alpha U_y + (Ay + B)U = 0 \quad \text{on } \sigma$$

where α, A, B are real constants; $\alpha \neq 0$.

Define a function $V(x, y, t)$ for $(x, y, t) \in D \cup \sigma$:

(2.3) $$V(x, y, t) \equiv \int_0^x U(\xi, y, t) d\xi + G(y, t)$$

where $G(y, t)$ is the C^4 solution of the Cauchy problem:

(2.4) $$\left.\begin{array}{l} G_{yy} - G_{tt} + U_x(0, y, t) = 0 \\ G(y, 0) = G_t(y, 0) = 0 \end{array}\right\}$$

Let P be the parallelepiped bounded by the planes $t \pm y = \pm l$, $x = 0$, $x = -m$. Then, $V(x, y, t) \in C^4(V_x \in C^4)$ is defined in $P \cap D \cup \sigma$, and we have the relations:

$$LV = 0 \quad \text{in} \quad P \cap D \cup \sigma\,, \tag{2.5}$$

$$V_{xx} + \alpha V_{xy} + (Ay + B)V_x = 0 \quad \text{on} \quad P \cap \sigma\,, \tag{2.6}$$

which are easily verified.

We now define for $(x, y, t) \in P \cap D \cup \sigma$ the function:

$$\varphi(x, y, t) \equiv V_{xx} + \alpha V_{xy} + A\Big(y\frac{\partial}{\partial x} - x\frac{\partial}{0y}\Big)V + BV_x\,. \tag{2.7}$$

Since the operators L and $\{y(\partial/\partial x) - x(\partial/\partial y)\}$ commute, operating on both sides of (2.7) by L, and making use of (2.5), we obtain:

$$L\varphi = \Big\{\frac{\partial^2}{\partial x^2} + \alpha\frac{\partial^2}{\partial x\partial y} + (Ay + B)\frac{\partial}{\partial x} - Ax\frac{\partial}{\partial y}\Big\}(LV) = 0\,.$$

Setting $x = 0$ in (2.7), and making use of (2.6) we have $\varphi(0, y, t) = 0$.

If we now set for $(x, y, t) \in \bar{P} \cap \bar{D} \cup \sigma$

$$\varphi(x, y, t) = -\varphi(-x, y, t)$$

it follows, that $L\varphi = 0$ in $P \cap D \cup \sigma \cup \bar{P} \cap \bar{D}$, and $\varphi \in C^3$.

Since $\varphi(-x, y, t)$ is known for $(x, y, t) \in \bar{P} \cap \bar{D} \cup \sigma$, we now seek a function $\bar{V}(x, y, t)$ for $(x, y, t) \in \bar{P} \cap \bar{D} \cup \sigma$, which solves the following Cauchy problem:

$$M\bar{V}(x, y, t) \equiv \bar{V}_{xx} + \alpha\bar{V}_{xy} + (Ay + B)\bar{V}_x - Ax\bar{V}_y = -\varphi(-x, y, t) \tag{2.8}$$

$$\bar{V}(0, y, t) = G(y, t)\,, \quad \bar{V}_x(0, y, t) = U(0, y, t) \quad \text{on } \bar{P} \cap \sigma\,. \tag{2.9}$$

It is well known, that the function $\bar{V}(x, y, t) \in C^4$, which satisfies (2.8), (2.9), exists in a domain Q. Here Q is that domain, each of whose sections by a plane $t = K(-l < K < l)$ is a right triangle bounded by $x = 0$, $y = l - |K|$ and $y - \alpha x = |K| - l$ if $\alpha > 0$, or by $x = 0$, $y = |K| - l$ and $y - \alpha x = l - |K|$ if $\alpha < 0$. Note that Q does not depend on U, and is a subdomain of $\bar{P} \cap \bar{D} \cup \sigma$.

LEMMA 4. *If $\bar{V}(x, y, t) \in C^4$ in Q is the solution of the Cauchy problem* (2.8), (2.9), *then $L\bar{V} = 0$ in Q.*

Proof. We operate on both sides of (2.8) by L. Since the operators L and $\{y(\partial/\partial x) - x(\partial/\partial y)\}$ commute, and $L\varphi(-x, y, t) = 0$, we obtain:

$$M(L\bar{V}) = 0\,.$$

setting $x = 0$ in (2.8) we have,

$$\bar{V}_{xx}(0, y, t) = -\alpha \bar{V}_{xy}(0, y, t) - (Ay + B)\bar{V}_x(0, y, t)$$

and hence, making use of (2.9) and (2.2), we obtain:

$$\bar{V}_{xx}(0, y, t) = U_x(0, y, t) \,. \tag{2.10}$$

Thus, due to equations (2.9) and (2.4)

$$L\bar{V}|_{x=0} = 0 \,.$$

From (2.3) and (2.7) we have:

$$\begin{aligned}\varphi(-x, y, t) \equiv \varphi(\xi, y, t)|_{\xi=-x} &= U_\xi(\xi, y, t)|_{\xi=-x} + \alpha U_y(-x, y, t) \\ &+ (Ay + B)U(-x, y, t) + AxG_y(y, t) + Ax\int_0^{-x} U_y(\xi, y, t)d\xi\end{aligned}$$

and therefore,

$$\begin{aligned}\frac{\partial}{\partial x}\varphi(-x, y, t)|_{x=0} &= -U_{xx}(0, y, t) - \alpha U_{xy}(0, y, t) \\ &\quad -(Ay + B)U_x(0, y, t) + AG_y(y, t) \,.\end{aligned} \tag{2.11}$$

Differentiating (2.8) with respect to x, and setting $x = 0$ we obtain

$$\bar{V}_{xxx} + \alpha \bar{V}_{xxy} + (Ay + B)\bar{V}_{xx} - A\bar{V}_y = -\frac{\partial}{\partial x}\varphi(-x, y, t)|_{x=0} \quad \text{on } x = 0 \,,$$

which after substituting (2.9), (2.10) and (2.11) becomes:

$$\bar{V}_{xxx}(0, y, t) = U_{xx}(0, y, t) \,.$$

Hence, by (2.9) and (2.1),

$$\frac{\partial}{\partial x}L\bar{V}|_{x=0} = 0 \,.$$

Consequently, by uniqueness of the solution of Cauchy's problem for $M(L\bar{V}) = 0$ with homogeneous initial conditions, we have that $L\bar{V} = 0$ in Q.

We thus have:

THEOREM 2. *If $U(x, y, t) \in C^4$ in the closure of D solves the wave equation* (2.1) *and satisfies the boundary condition* (2.2) *on σ, then there exists a function $U = \bar{V}_x \in C^3$ in the subdomain Q of $\bar{D}$, which extends U across σ as C^3 solution of the wave equation.*

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