ON CONTINUOUS MATRIX-VALUED FUNCTIONS ON A STONIAN SPACE

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1. Introduction. In this paper the authors continue the study (begun in [9] and carried on in [3] and [10]) of matrices with entries from the algebra $C(\mathfrak{X})$ of all continuous complex-valued functions on an extremely disconnected, compact Hausdorff space $\mathfrak{X}$. (Such spaces are sometimes called Stonian after M. H. Stone, who considered them in [14].) One of the authors has shown ([10], Theorem 3) that if A and B are $n \times n$ matrices over $C(\mathfrak{X})$ such that $A(x)$ is unitarily equivalent to $B(x)$ for each $x \in \mathfrak{X}$, then A and B are unitarily equivalent in the algebra $M_n(\mathfrak{X})$ of all $n \times n$ matrices over $C(\mathfrak{X})$. It is thus natural to ask whether the similarity of $A(x)$ and $B(x)$ for each $x \in \mathfrak{X}$ is sufficient to guarantee the similarity of A and B in $M_n(\mathfrak{X})$. We show by example in § 2 that the answer is no; however, we also show that if the hypothesis is strengthened by the addition of a uniform boundedness requirement, then the similarity of A and B in $M_n(\mathfrak{X})$ does indeed follow. As a by-product of the technique introduced to give this result, we obtain a new short proof of Theorem 3 of [10].

In § 3 we show that a certain class of entire functions maps $M_n(\mathfrak{X})$ onto itself; this is a generalization (with a different proof) of a result of Kurepa [8] for $n \times n$ matrices, and adds to the information obtained by Brown [1] on the question of which entire functions map which Banach algebras onto themselves. As a corollary, we learn that every invertible element of $M_n(\mathfrak{X})$ has a logarithm. Section 4 is devoted to proving that an element of $M_n(\mathfrak{X})$ has an identically vanishing trace if and only if it is a commutator in $M_n(\mathfrak{X})$. (See Remark 2, § 4, for a paraphrase of this result cast in the terminology of operator theory on Hilbert space.) Finally, in § 5 the authors give two examples which indicate that it is probably fruitless to pursue the structure theory of matrices over $C(\mathfrak{X})$ where $\mathfrak{X}$ is a more general topological space than a Stonian space.

2. Similarity in $M_n(\mathfrak{X})$. The most convenient definition of $M_n(\mathfrak{X})$ is as follows. Let M_n denote the full ring of $n \times n$ complex matrices under the operator norm, and let $\mathfrak{X}$ be any Stonian space. Denote by $M_n(\mathfrak{X})$ the *-algebra of continuous functions from $\mathfrak{X}$ to M_n, where the algebraic operations in $M_n(\mathfrak{X})$ are defined pointwise. Under the norm $\|A\| = \sup_{x \in \mathfrak{X}} \|A(x)\|$, $M_n(\mathfrak{X})$ is a C^*-algebra identifiable with the C^*-algebra of all $n \times n$ matrices over $C(\mathfrak{X})$. Moreover, $M_n(\mathfrak{X})$ is an

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AW^*-algebra [7], and this fact is used briefly in this section.

We first show that pointwise similarity of $A(x)$ and $B(x)$ on $\mathfrak{X}$ is not sufficient to ensure that A and B be similar in $M_n(\mathfrak{X})$. For this purpose, let $\mathscr{S}$ be the Stone-Czech compactification of the natural numbers $\mathscr{N}$. Then $\mathscr{S}$ is a Stonian space. (See, for example, the discussion on page 295 of [12].) Consider elements A and B of $M_2(\mathscr{S})$ defined by:

$$A(x) = \begin{pmatrix} 0 & 1/x^2 \\ 0 & 0 \end{pmatrix}, \qquad B(x) = \begin{pmatrix} 0 & 1/x \\ 0 & 0 \end{pmatrix}$$

for each natural number $x \in \mathscr{N}$. Then $A(x) = B(x) = 0$ for $x \in \mathscr{S} - \mathscr{N}$, and it is obvious that $A(x)$ and $B(x)$ are similar for each $x \in \mathscr{S}$. Suppose that $S = (s_{ij})$ is an invertible element in $M_2(\mathscr{S})$ satisfying $SA = BS$. Calculation yields $s_{21}(x) = 0$ for $x \in \mathscr{N}$ so that $s_{21} \equiv 0$. Furthermore, $s_{11}(x) = xs_{22}(x)$ for $x \in \mathscr{N}$, and the invertibility of S guarantees that s_{22} never vanishes. Thus s_{11} is unbounded, contradicting $s_{11} \in C(\mathscr{S})$, and it follows that A and B are not similar in $M_2(\mathscr{S})$.

The following theorem gives necessary and sufficient conditions for A and B to be similar in $M_n(\mathfrak{X})$.

THEOREM 1. *Let $\mathfrak{X}$ be any Stonian space, and let $A, B \in M_n(\mathfrak{X})$. Suppose that there is a dense subset $\mathscr{D} \subset \mathfrak{X}$ and a positive number M such that for $x \in \mathscr{D}$, there is an invertible matrix $S(x)$ satisfying $S(x)A(x)S^{-1}(x) = B(x)$, $\| S(x) \| < M$, and $\| S^{-1}(x) \| < M$. Then there is an invertible element $T \in M_n(\mathfrak{X})$ satisfying $TAT^{-1} = B$, $\| T \| \leqq M$, and $\| T^{-1} \| \leqq M$.*

Proof. We consider collections $\{\mathscr{U}_i\}$ of nonempty, disjoint, compact open sets $\mathscr{U}_i \subset \mathfrak{X}$ with the property that if $\mathscr{U}_i \in \{\mathscr{U}_i\}$, then there is an invertible element $T_i \in M_n(\mathscr{U}_i)$ satisfying $T_i(x)A(x)T_i^{-1}(x) = B(x)$, $\| T_i(x) \| < M$, and $\| T_i^{-1}(x) \| < M$ for each $x \in \mathscr{U}_i$. Let $\{\mathscr{U}_i\}_{i \in I}$ be a maximal such collection, and denote $\mathscr{U} = \overline{\bigcup_{i \in I} \mathscr{U}_i}$. Then $\mathscr{U}$ is compact open, and it follows from Lemma 2.1 of [3] that the function $\tilde{T}$ defined on $\bigcup_{i \in I} \mathscr{U}_i$ so as to extend each of the T_i can be extended to an element $T \in M_n(\mathscr{U})$. Similarly, there is a function $Z \in M_n(\mathscr{U})$ which extends each of the T_i^{-1}. It is clear from continuity considerations that $Z = T^{-1}$, and that T has all the desired properties on $\mathscr{U}$, so that it suffices to prove $\mathscr{U} = \mathfrak{X}$. Suppose, to the contrary, that $\mathfrak{X} - \mathscr{U} \neq \phi$. To obtain a contradiction, it suffices to find a compact open set $\mathscr{V} \subset \mathfrak{X} - \mathscr{U}$ and an invertible element $V \in M_n(\mathscr{V})$ such that for $x \in \mathscr{V}$, $V(x)A(x) = B(x)V(x)$, $\| V(x) \| < M$, and $\| V^{-1}(x) \| < M$. To do this, we regard the equation $VA = BV$ as a system of linear equations

$$
\text{(L)} \qquad \begin{aligned} c_{11}v_1 + c_{12}v_2 + \cdots + c_{1m}v_m &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots& \\ c_{m1}v_1 + c_{m2}v_2 + \cdots + c_{mm}v_m &= 0 \end{aligned}
$$

where

(1) the unknown functions v_i are the entries, in some prescribed order, of the matrix V

(2) the coefficients $c_{ij} \in C(\mathfrak{X} - \mathscr{U})$ are the appropriate combinations of the entries of the matrices A and B

(3) $m = n^2$.

For $x \in \mathfrak{X} - \mathscr{U}$, consider the corresponding system $(L(x))$ of linear equations, and let $x_0 \in \mathfrak{X} - \mathscr{U}$ be a point such that the rank $r(x)$ of the system $(L(x))$ assumes its maximum r_0 at x_0. (The case $r_0 = 0$ leads trivially to a contradiction of $\mathfrak{X} - \mathscr{U} \neq \phi$, and we ignore it. The case $r_0 = m$ cannot occur.) Then there is some $r_0 \times r_0$ minor N of the coefficient determinant of the system $(L(x_0))$ which is nonzero, and by continuity there exists a compact open neighborhood $\mathscr{V}_1 \subset \mathfrak{X} - \mathscr{U}$ of x_0 such that for $x \in \mathscr{V}_1$, the same minor N remains a nonzero minor of maximum size. According to the hypothesis, there is a point $x_1 \in \mathscr{V}_1$ and an invertible matrix $S(x_1)$ such that $S(x_1)A(x_1) = B(x_1)S(x_1)$, $\| S(x_1) \| < M$, and $\| S^{-1}(x_1) \| < M$. Let the corresponding nontrivial solution of the system $(L(x_1))$ be denoted by $(\mu_1, \mu_2, \cdots, \mu_m)$ (i.e., the μ_i are the entries of the matrix $S(x_1)$). We wish to define an m-tuple $(v_1(x), v_2(x), \cdots, v_m(x))$ at each point of $\mathscr{V}_1$ in such a way that

(1) the m-tuple is a solution of $(L(x))$ for each $x \in \mathscr{V}_1$,

(2) $v_i \in C(\mathscr{V}_1)$ for $1 \leqq i \leqq m$, and

(3) $v_i(x_1) = \mu_i$ for $1 \leqq i \leqq m$. This is accomplished as follows. Since for $x \in \mathscr{V}_1$, N is a nonzero minor of maximum size, it suffices to solve (continuously on $\mathscr{V}_1$) the r_0 equations affiliated with N. Thus for the appropriate $m - r_0$ values of i (the values not affiliated with N), define $v_i(x) \equiv \mu_i$ on $\mathscr{V}_1$; then for $x \in \mathscr{V}_1$ the other r_0 numbers $v_i(x)$ are determined by Cramer's rule, and since the functions c_{ij} are continuous it follows that (1), (2), and (3) above are satisfied. Next place the resulting functions $v_i \in C(\mathscr{V}_1)$ in their appropriate positions in the matrix V, and shrink the neighborhood $\mathscr{V}_1$ of x_1 to a compact open neighborhood $\mathscr{V} \subset \mathscr{V}_1$ of x_1 such that for $x \in \mathscr{V}$, the matrix $V(x)$ is invertible and the inequalities $\| V(x) \| < M$ and $\| V^{-1}(x) \| < M$ remain valid. The existence of the compact open set $\mathscr{V}$ contradicts the maximality of the collection $\{\mathscr{U}_i\}_{i \in I}$, and thus the proof is complete.

We can prove Theorem 3 of [10] in a similar fashion.

THEOREM 2. *If $\mathfrak{X}$ is Stonian and $A, B \in M_n(\mathfrak{X})$ are such that $A(x)$ and $B(x)$ are unitarily equivalent at each point of a dense subset of $\mathfrak{X}$, then A and B are unitarily equivalent in $M_n(\mathfrak{X})$.*

Proof. We consider collections $\{\mathscr{U}_i\}$ of nonempty, disjoint, compact open subsets $\mathscr{U}_i \subset \mathfrak{X}$ with the property that if $\mathscr{U}_i \in \{\mathscr{U}_i\}$, then there is a unitary element $U_i \in M_n(\mathscr{U}_i)$ satisfying $U_i(x)A(x)U_i^*(x) = B(x)$ for each $x \in \mathscr{U}_i$. As before, we choose a maximal collection $\{\mathscr{U}_i\}_{i \in I}$, and define $\mathscr{U} = \overline{\bigcup_{i \in I} \mathscr{U}_i}$. Again it suffices to prove $\mathscr{U} = \mathfrak{X}$. The argument then proceeds exactly as above, except that the system of linear equations to be considered is the system equivalent to the pair of equations $VA = BV$ and $VA^* = B^*V$. (Thus the system consists of $2n^2$ equations in n^2 unknowns, but it is clear that this has no effect on the argument.) Then, proceeding essentially as above, we obtain a compact open subset $\mathscr{V} \subset \mathfrak{X} - \mathscr{U}$ and an invertible (not necessarily unitary) element $V \in M_n(\mathscr{V})$ such that for $x \in \mathscr{V}$, $V(x)A(x) = B(x)V(x)$ and $V(x)A^*(x) = B^*(x)V(x)$. One knows from ([14], Lemma 2.1) that we can write V in polar form $V = UP$ where U is a unitary element of $M_n(\mathscr{V})$. A standard calculation shows that for $x \in \mathscr{V}$, $U(x)A(x)U^*(x) = B(x)$; thus the existence of $\mathscr{V}$ contradicts the maximality of the collection $\{\mathscr{U}_i\}_{i \in I}$, and the proof is complete.

REMARK. One would naturally like to have a collections of global objects to attach to an element $A \in M_n(\mathfrak{X})$ which would serve as a complete set of similarity invariants for A. In this connection, it is easy to see that one cannot always obtain an element $J \in M_n(\mathfrak{X})$ such that A is similar to J in $M_n(\mathfrak{X})$ and such that $J(x)$ is in Jordan form for each $x \in \mathfrak{X}$.

3. Entire functions on $M_n(\mathfrak{X})$. We say that an entire function f has property (K) if, for every complex number ζ, there is a complex number z satisfying $f(z) = \zeta$ and $f'(z) \neq 0$. In [8] Kurepa showed that an entire function f maps M_n onto itself if and only if f has property (K). The study was then taken up by Brown [1] who characterized the class of entire functions f which map the algebra $\mathscr{L}(\mathscr{H})$ of all bounded operators on an infinite dimensional Hilbert space $\mathscr{H}$ onto itself. Brown showed that such an f maps every Banach algebra onto itself, and we say that such an f has property (B). Since certain W^*-algebras of operators on Hilbert space have faithful C^*-representations as an $M_n(\mathfrak{X})$ (see [9]), one has, in a sense, $\mathscr{L}(\mathscr{H}) \supset M_n(\mathfrak{X}) \supset M_n$. Thus it is of interest to discover which entire functions map $M_n(\mathfrak{X})$ onto itself, and the answer is given by

THEOREM 3. *If f is an entire function and $\mathfrak{X}$ is a Stonian space, then f maps $M_n(\mathfrak{X})$ onto itself if and only if f has property (K).*

Proof. Since for each $x \in \mathfrak{X}$, $[p(A)](x) = p(A(x))$ for every polynomial $p(z)$, and since f is the uniform limit of polynomials on compact sets of the z-plane, $[f(A)](x) = f(A(x))$ for each $x \in \mathfrak{X}$. Thus, if f maps $M_n(\mathfrak{X})$ onto itself, then f must map M_n onto itself, so that by Kurepa's theorem [8], f has property (K). Now suppose that f has property (K), and let $A \in M_n(\mathfrak{X})$. We look for $B \in M_n(\mathfrak{X})$ such that $f(B)$ A. Let x_0 be an arbitrary point of $\mathfrak{X}$ and let $\zeta_1, \cdots, \zeta_p$ be the distinct eigenvalues of $A(x_0)$. Choose $z_1, \cdots, z_p$ to be complex numbers with the properties that $f(z_i) = \zeta_i$ and $f'(z_i) \neq 0$. For $i = 1, \cdots, p$, let $\mathscr{D}_i$ be a (non-degenerate) closed disc about z_i such that f is Schlicht on $\mathscr{D}_i$, and arrange it so that the sets $f(\mathscr{D}_i)$ are mutually disjoint. Let g denote the inverse of the restriction of f to $\bigcup_{i=1}^{p} \mathscr{D}_i$. Then g is defined and continuous on $\mathscr{D} = \bigcup_{i=1}^{p} f(\mathscr{D}_i)$ and is analytic at each interior point of $\mathscr{D}$. It follows from Lemma 2.2 of [3] that there exists a compact open neighborhood $\mathscr{N}_0 = \mathscr{N}(x_0)$ of x_0 such that for $x \in \mathscr{N}_0$, the spectrum of $A(x)$ (denoted hereafter $\Lambda[A(x)]$) is a subset of the interior of $\mathscr{D}$. If A_0 denotes the restriction of A to $\mathscr{N}_0$, then A_0 is an element of the C^*-algebra $M_n(\mathscr{N}_0)$, and it is clear that the spectrum of A_0 is $\bigcup_{x \in \mathscr{N}_0} \Lambda[A(x)]$. As usual, following Dunford [5], $g(A_0) \in M_n(\mathscr{N}_0)$ can be defined as the sum of the p integrals $1/2\pi i \int_{\Gamma_i} g(\lambda)(A_0 - \lambda I)^{-1} d\lambda$, where Γ_i is the boundary of the set $f(\mathscr{D}_i)$. If we denote $B_0 = g(A_0)$, it follows from Theorem 2.10 of [5] that $f(B_0) = A_0$. Since this construction was carried out about an arbitrary point $x_0 \in \mathfrak{X}$, we can apply the compactness of $\mathfrak{X}$ to obtain points $x_1, \cdots, x_r \in \mathfrak{X}$ and compact open neighborhoods $\mathscr{N}_i$ of the x_i such that $\bigcup_{i=1}^{r} \mathscr{N}_i = \mathfrak{X}$ and such that the above construction has been carried out to yield a corresponding B_i on each $\mathscr{N}_i$. Furthermore, we can assume that the $\mathscr{N}_i$ are pairwise disjoint. The element $B \in M_n(\mathfrak{X})$ defined by $B(x) = B_i(x)$ for $x \in \mathscr{N}_i$ is such that $f(B) = A$, and the proof is complete.

COROLLARY 3.1. *If $\mathfrak{X}$ is a totally disconnected, compact Hausdorff space, then each invertible element of $M_n(\mathfrak{X})$ has a logarithm in $M_n(\mathfrak{X})$, and thus has roots of all orders in $M_n(\mathfrak{X})$.*

Proof. Observe first that the proof of Theorem 3 above goes through word for word in the case that $\mathfrak{X}$ is only compact Hausdorff and totally disconnected. Then observe that if $A \in M_n(\mathfrak{X})$ and an entire function f are given, in order to carry out the construction in the above proof to obtain a B such that $f(B) = A$, it suffices to know that for each ζ in the spectrum of A, there is a complex number z such that $f(z) = \zeta$ and $f'(z) \neq 0$. These observations complete the proof.

It results easily from Theorem 3 that if

$$\mathfrak{A} = \sum_{k=0}^{k_0} \oplus M_{n_k}(\mathfrak{X}_k)$$

is any finite C^*-sum of algebras $M_{n_k}(\mathfrak{X}_k)$ where the $\mathfrak{X}_k$ are Stonian spaces, then the entire functions which map $\mathfrak{A}$ onto itself are exactly those with property (K). However, if one considers algebras

$$\mathfrak{B} = \sum_{k=1}^{\infty} \oplus M_{n_k}(\mathfrak{X}_k)$$

which are C^*-sums of infinitely many $M_{n_k}(\mathfrak{X}_k)$ where $n_k \to \infty$ and the $\mathfrak{X}_k$ are only assumed to be compact Hausdorff spaces, then the situation is different, as is demonstrated by the following theorem.

THEOREM 4. *If $\mathfrak{B}$ is any algebra of the form*

$$\mathfrak{B} = \sum_{k=1}^{\infty} \oplus M_{n_k}(\mathfrak{X}_k)$$

where $n_k \to \infty$ and each $\mathfrak{X}_k$ is a compact Hausdorff space, then the entire functions which map $\mathfrak{B}$ onto itself are exactly those with property (B)

The proof of this theorem is patterned after an argument of Brown [1], and depends on the following lemma.

LEMMA 3.2. *Let f be any entire function, let $g(z)$ be the polynomial*

$$g(z) = \sum_{i=0}^{n-1} a_i z^i ,$$

and let $A \in M_n$ be the "analytic Toeplitz" matrix

$$A = \begin{bmatrix} a_0 & & & & & \\ a_1 & a_0 & & & & \\ a_2 & a_1 & a_0 & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ a_{n-1} & \cdot & \cdot & \cdot & a_1 & a_0 \end{bmatrix}$$

Then $f(A)$ is an "analytic Toeplitz" matrix

$$f(A) = \begin{bmatrix} b_0 & & & & & \\ b_1 & b_0 & & & & \\ b_2 & b_1 & b_0 & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ b_{n-1} & \cdot & \cdot & \cdot & b_1 & b_0 \end{bmatrix}$$

and the entire function $h(z) = f(g(z))$ *has a power series expansion*

$$h(z) = \sum_{i=0}^{\infty} \beta_i z^i$$

where $\beta_i = b_i$ *for* $0 \leqq i \leqq n-1$.

Proof. If f is any positive integral power of z, or more generally any polynomial, an inductive computation shows that the result is valid. For an arbitrary entire function f, let $p_n(z)$ be a sequence of polynomials which converges uniformly to f on every compact subset of the z-plane. Then, since $p_n(g(z))$ converges uniformly to $h(z)$ on compact subsets of the plane, the coefficients in the power series expansions of the $p_n(g(z))$ must converge to the corresponding coefficients in the power series expansion of $h(z)$. (See, for example, ([2], § 211)) Furthermore, since $p_n(A)$ converges to $f(A)$ in the norm topology of M_n, the entries of $p_n(A)$ must converge to the corresponding entires of $f(A)$, and the result follows.

Proof of Theorem 4. For convenience we take $n_k = n$. It will be clear that this does not affect the argument. Let

$$B = \left(\sum_{n=1}^{\infty} \oplus B_n\right) \in \mathfrak{B}$$

be defined by setting

$$B_n \equiv \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 1 & 0 \end{bmatrix}$$

for each positive integer n. Let f be an entire function which maps onto $\mathfrak{B}$, and suppose that

$$A = \sum_n \oplus A_n$$

satisfies $f(A) = rB$ where r is some fixed positive real number. Since for any central projection $E \in \mathfrak{B}, f(EA) = Ef(A)$, it is clear that for each positive integer $n, f(A_n) = rB_n$. Now choose an arbitrary $x_n \in \mathfrak{X}_n$ for each integer n. The fact that $f[A_n(x_n)] = rB_n(x_n)$ follows just as in the proof of Theorem 3. Since $A_n(x_n)$ commutes with $B_n(x_n) = 1/r\, f[A_n(x_n)]$ and B_n is identically constant on $\mathfrak{X}_n$, a matrix calculation shows that for each positive integer n, the matrix $A_n(x_n)$ has the form

$$A_n(x_n) = \begin{bmatrix} a_0^n & & & & & \\ a_1^n & a_0^n & & & & \\ a_2^n & a_1^n & a_0^n & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ a_{n-1}^n & \cdot & \cdot & \cdot & a_1^n & a_0^n \end{bmatrix}$$

where the a_i^n are of course complex numbers. Define the sequence $g_n(z)$ of polynomials by

$$g_n(z) = \sum_{i=0}^{n-1} a_i^n z^i ,$$

and let $h_n(z) = f(g_n(z))$. Since $f[A_n(x_n)] = rB_n(x_n)$, it follows from Lemma 3.2 that for each positive integer n, $h_n(z)$ is an entire function having a power series expansion

$$h_n(z) = rz + \sum_{k=n}^{\infty} \beta_k^n z^k .$$

Since $A = \sum_n \oplus A_n$ is a bounded operator, it follows that there exists a positive number M such that

$$\sum_{i=0}^{n-1} | a_i^n |^2 < M$$

for each n. Let $\mathscr{D}$ denote the disc $\mathscr{D} = \{z: |z| \leqq 1/2\}$ and observe that it follows from the above inequality that the sequence $g_n(z)$ is uniformly bounded on $\mathscr{D}$ by the number $2\sqrt{M}$. It follows from Montel's theorem ([2], § 416) that one can extract a subsequence $g_{n_k}(z)$ which converges uniformly on $\mathscr{D}$ to a function $g(z)$ which is analytic on $\mathscr{D}$. It follows that $h_{n_k}(z) = f(g_{n_k}(z))$ converges uniformly to $f(g(z))$ on $\mathscr{D}$, and by virtue of the form of the power series expansion of each $h_{n_k}(z)$, we must have $f(g(z)) = rz$ on $\mathscr{D}$. It is now clear that $g(z)$ is a Schlicht mapping of the interior $\mathscr{D}^\circ$ of $\mathscr{D}$ onto some bounded domain $g(\mathscr{D}^\circ)$ and that f is a Schlicht mapping of $g(\mathscr{D}^\circ)$ onto the open disc $\{z: |z| < r/2\}$. Since r was arbitrary, it follows from ([1], Theorem 2) that f has property (B), and the proof is complete.

4. Commutators in $M_n(\mathfrak{X})$. We introduce the notation $\sigma(B)$ for the trace in the usual sense of an $n \times n$ complex matrix B. In this section, we generalize another result known for M_n, and thereby set forth a class of operators on Hilbert space which are commutators. (See Remark 2 at the end of this section.) More precisely, we establish

THEOREM 5. *If $\mathfrak{X}$ is a Stonian space and $A \in M_n(\mathfrak{X})$, then A*

satisfies $\sigma[A(x)] \equiv 0$ *if and only if there are elements* B *and* C *in* $M_n(\mathfrak{X})$ *such that* $A = BC - CB$.

One half of the theorem is trivial; to prove the other half we use an idea suggested by Halmos in [6]. The crucial lemma is the following.

LEMMA 4.1. *If* $\mathfrak{X}$ *is any Stonian space and* $A \in M_n(\mathfrak{X})$ *is such that* $\sigma[A(x)] \equiv 0$, *then there is an invertible* $S \in M_n(\mathfrak{X})$ *such that* $SAS^{-1} = D = (d_{ij})$ *satisfies* $d_{11} \equiv 0$.

Proof. We consider collections $\{\mathscr{U}_i\}$ of disjoint, nonempty, compact open sets $\mathscr{U}_i \in \mathfrak{X}$ with the property that if $\mathscr{U}_i \in \{\mathscr{U}_i\}$, then there is an invertible $S_i \in M_n(\mathscr{U}_i)$ such that $\|S_i\|, \|S_i^{-1}\| \leqq 6$ and such that for each $x \in \mathscr{U}_i$, the matrix $S_iAS_i^{-1}(x)$ has a zero in the upper left hand corner. Let $\{\mathscr{U}_i\}_{i\in I}$ be a maximal such collection, and define $\mathscr{U} = \overline{\bigcup_{i\in I}\mathscr{U}_i}$. It follows from Lemma 2.1 of [3] that to complete the proof, it suffices to establish $\mathscr{U} = \mathfrak{X}$. Thus, suppose to the contrary that $\mathfrak{X} - \mathscr{U} \neq \phi$. According to Theorem 1 of [3] there exist functions $\lambda_1, \cdots, \lambda_n \in C(\mathfrak{X} - \mathscr{U})$ such that for $x \in \mathfrak{X} - \mathscr{U}$, the numbers $\lambda_1(x), \cdots, \lambda_n(x)$ are exactly the eigenvalues of $A(x)$. Furthermore, there must be at least one point $x_0 \in \mathfrak{X} - \mathscr{U}$ such that some $\lambda_i(x_0) \neq 0$. (Otherwise, we could apply Theorem 2 of [3] to obtain a unitary $U \in M_n(\mathfrak{X} - \mathscr{U})$ such that $UAU^*(x)$ is in upper triangular form for each $x \in \mathfrak{X} - \mathscr{U}$. Then the diagonal entries of $UAU^*(x)$ would be identically zero, and the maximality of the collection $\{\mathscr{U}_i\}_{i\in I}$ would be contradicted.) Since we know from the hypothesis that

$$\sum_{i=1}^{n} \lambda_i \equiv 0 ,$$

there must be at least two distinct i such that $\lambda_i(x_0) \neq 0$. In fact, a little thought convinces one that there exist λ_j and λ_k $(j \neq k)$ such that

$$0 < |\lambda_j(x_0)| \leqq |\lambda_k(x_0)| < |\lambda_k(x_0) - \lambda_j(x_0)| .$$

It follows from the circle of ideas connected with the proof of Theorem 2 of [3] that there is a unitary element $U \in M_n(\mathfrak{X}) - \mathscr{U})$ such that $UAU^*(x) = (a_{ij}(x))$ is in upper triangular form for each $x \in \mathfrak{X} - \mathscr{U}$ and such that $a_{11} \equiv \lambda_k$ and $a_{22} \equiv \lambda_j$ on $\mathfrak{X} - \mathscr{U}$. Thus $0 < |a_{22}(x_0)| \leqq |a_{11}(x_0)| < |a_{11}(x_0) - a_{22}(x_0)|$, and by clever choice of U (i.e., by applying an additional rotation, and then changing notation) one can arrange things so that $|a_{11}(x_0) - a_{22}(x_0)| < |a_{12}(x_0) - [a_{11}(x_0) - a_{22}(x_0)]|$. It follows that for some δ, $0 < \delta < 1$, there is a compact open neighborhood $\mathscr{V} \subset \mathfrak{X} - \mathscr{U}$ of x_0 such that for $x \in \mathscr{V}$, $0 < |a_{22}(x)| \leqq (1 + \delta)|a_{11}(x)| < |a_{12}(x) - [a_{11}(x) - a_{22}(x)]|$. The argument now splits into two cases.

Case I. For every $x \in \mathscr{V}$, $|a_{12}(x)| \geqq |a_{11}(x)|$. In this case we define an invertible $S = (s_{ij}) \in M_n(\mathscr{V})$ to be the direct sum of the 2×2 matrix $(s_{ij}: i, j \leqq 2)$ and the identity element of $M_{n-2}(\mathscr{V})$, where for $x \in \mathscr{V}$, $s_{11}(x) = s_{22}(x) = 1$, $s_{12}(x) = 0$, and $s_{21}(x) = a_{11}(x)/a_{12}(x)$. An easy calculation shows that $\| S \|, \| S^{-1} \| \leqq 4$, and another calculation shows that for $x \in \mathscr{V}$, the matrix $SUAU^*S^{-1}(x)$ has a zero in the upper left hand corner. The existence of $\mathscr{V}$ thus contradicts the maximality of the collection $\{\mathscr{V}_i\}_{i \in I}$, and we proceed to

Case II. There is a compact open subset $\mathscr{W} \subset \mathscr{V}$ such that for $x \in \mathscr{W}$, $|a_{12}(x)| < |a_{11}(x)|$. As before we define an invertible $S = (s_{ij}) \in M_n(\mathscr{W})$ to be the direct sum of the 2×2 matrix $(s_{ij}: i, j \leqq 2)$ and the identity element of $M_{n-2}(\mathscr{W})$. This time for $x \in \mathscr{W}$ we take $s_{11}(x) = s_{12}(x) = s_{21}(x) = [a_{11}(x)/\{a_{12}(x) - [a_{11}(x) - a_{22}(x)]\}]^{1/2}$ and $s_{22}(x) = s_{11}(x)[\{a_{12}(x) + a_{22}(x)\}/a_{11}(x)]$, where the exponent $1/2$ denotes any square root taken in such a way that $s_{11} \in C(\mathscr{W})$. (Theorem 1 of [3] enables us to take continuous square roots.) As a result of the inequalities which are valid on $\mathscr{W}$, one has $|s_{11}(x)| < 1$ and $|s_{22}(x)| \leqq 2 + \delta$ for each $x \in \mathscr{W}$; furthermore, $s_{11}s_{22} - s_{12}s_{21} \equiv 1$ on $\mathscr{W}$, and it follows that $\| S \|, \| S^{-1} \| \leqq 6$. Calculation shows that for $x \in \mathscr{W}$, $SUAU^*S^{-1}(x)$ has a zero for its upper left hand entry, and thus the proof is complete.

The following corollary follows easily by induction on n, and we omit its proof.

COROLLARY 4.2. *If $A \in M_n(\mathfrak{X})$ is such that $\sigma[A(x)] \equiv 0$, then there is an invertible $S \in M_n(\mathfrak{X})$ such that $SAS^{-1} = (a_{ij})$ satisfies $a_{ii} \equiv 0$ for $1 \leqq i \leqq n$.*

Proof of Theorem 5. We are given that $\sigma[A(x)] \equiv 0$. Choose $S \in M_n(\mathfrak{X})$ according to Corollary 4.2 so that $SAS^{-1} = (a_{ij})$ satisfies $a_{ii} \equiv 0$ for $1 \leqq i \leqq n$. Define $B_1 = (b_{ij}) \in M_n(\mathfrak{X})$ by $b_{ii} \equiv i$ for $1 \leqq i \leqq n$ and $b_{ij} \equiv 0$ for $i \neq j$. Also define $C_1 = (c_{ij}) \in M_n(\mathfrak{X})$ by $c_{ij} \equiv a_{ij}/(b_{ii} - b_{jj})$ for $i \neq j$ and $c_{ij} \equiv 0$ for $i = j$. If B and C are defined by $B = S^{-1}C_1S$, then it is easy to see that $B_1C_1 - C_1B_1 = SAS^{-1}$, or, what is the same thing, $BC - CB = A$.

REMARKS.

(1) A stronger version of Lemma 4.1, obtained from the present version by requiring S to be unitary, actually holds. The proof, however, uses a completely different idea and is much longer than the above proof.

(2) A bounded operator B on Hilbert space is called n-normal [9] if the W^*-algebra which B generates satisfies a polynomial identity

of the form

$$\sum (sgn\, \pi) X_{\pi(1)} X_{\pi(2)} \cdots X_{\pi(2n)} = 0 \,,$$

where the sum is taken over all permutations π on $2n$ objects. It is known that such a W^*-algebra is a finite direct sum of algebras each of which has a faithful C^*-representation as some $M_k(\mathfrak{X}_k)$ with $\mathfrak{X}_k$ Stonian and $k \leqq n$. Furthermore such a W^*-algebra has a well-behaved center-valued trace function, so that Theorem 5 can be paraphrased: Any n-normal operator with trace zero is the commutator of a pair of n-normal operators.

(3) There are at least two classes of operators on Hilbert space which possess well-behaved numerical traces. These are operators in the trace-class [13], and operators in W^*-algebras which are factors of type II_1. Is it true that every operator with trace zero in one of these classes is a commutator?

5. **Two examples.** In this section we set forth two examples which show that Theorem 2 of [3] and Theorems 1 and 2 of the present paper cannot be extended to the setting in which $\mathfrak{X}$ is assumed only to be a compact Hausdorff, totally disconnected space. In these examples we take $\mathscr{T}$ to be the compact Hausdorff, totally disconnected space consisting of the set $\{a_1, a_2, \cdots, a_n, \cdots, 0\}$ with the relative topology, where the real sequence $\{a_n\}$ is strictly decreasing to zero and satisfies $\cos(1/a_n) = \sin(1/a_n) = 1/\sqrt{2}$ for n odd and $\cos(1/a_n) = 1$, $\sin(1/a_n) = 0$ for n even.

EXAMPLE 1. (This example is essentially due to Rellich [11].) Define $A \in M_2(\mathscr{T})$ by

$$A(a_n) = \begin{pmatrix} 1 - a_n \cos(2/a_n) & -a_n \sin(2/a_n) \\ -a_n \sin(2/a_n) & 1 + a_n \cos(2/a_n) \end{pmatrix};$$

$$A(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, even though A is Hermitian, *there exists no unitary $U \in M_2(\mathscr{T})$ such that $UAU^*(t)$ is in upper triangular form for each $t \in \mathscr{T}$.*

Proof. Assume that such a $U = (u_{ij})$ exists, and let $UAU^*(t) = (b_{ij}(t))$. Then the $b_{ij} \in C(\mathscr{T})$, and the vector $(\bar{u}_{11}(t), \bar{u}_{12}(t)) = V(t)$ has length one at each $t \in \mathscr{T}$ and has entries which are elements of $C(\mathscr{T})$. Futhermore, it is easy to see that $[A(t) - b_{11}(t)I]V(t) \equiv 0$. In other words, the vector $V(t)$ is a continuous eigenvector for $A(t)$ corresponding to the eigenvalue $b_{11}(t)$. An easy calculation shows that the eigenvalues of $A(a_n)$ are $1 - a_n$ and $1 + a_n$, so that for each n, $b_{11}(a_n) = 1 - a_n$ or $b_{11}(a_n) = 1 + a_n$. Furthermore, it is easy to see that the vector $(\cos(1/a_n), \sin(1/a_n))$ is an eigenvector for $A(a_n)$ corresponding to the eigenvalue $1 - a_n$, and the vector $(\sin(1/a_n), -\cos(1/a_n))$ is an eigenvector for $A(a_n)$ corresponding to the eigenvalue $1 + a_n$. It follows that for n odd, we must have $|\bar{u}_{11}(a_n)| = 1/\sqrt{2}$, and for n even, we must have $|\bar{u}_{11}(a_n)| = 0$ or 1. This contradicts $u_{11} \in C(\mathscr{T})$, and completes the proof.

EXAMPLE 2. Define $A, B \in M_2(\mathscr{T})$ by $A(0) = B(0) = 0$ and

$$A(a_n) = \begin{pmatrix} 0 & a_n \\ 0 & 0 \end{pmatrix}, \qquad B(a_n) = \begin{pmatrix} 0 & (-1)^n a_n \\ 0 & 0 \end{pmatrix}.$$

Then $A(t)$ is unitarily equivalent to $B(t)$ for each $t \in \mathscr{T}$, but *there exists no invertible* $S \in M_2(\mathscr{T})$ *such that* $SAS^{-1} = B$.

Proof. Suppose such an invertible $S = (s_{ij}) \in M_2(\mathscr{T})$ does exist. Then $SA = BS$, and calculation shows that $s_{21} \equiv 0$. Furthermore, $s_{11}(a_n) = (-1)^n s_{22}(a_n)$ for each n, and since S is invertible and $s_{21} \equiv 0$, s_{11} and s_{22} are bounded away from zero. It follows that s_{11} and s_{22} cannot both be continuous at zero, a contradiction.

REMARK. While the theory of elements $A \in M_n(\mathfrak{X})$ is not very satisfactory for $\mathfrak{X}$ only totally disconnected, it is nevertheless true that A has continuous eigenvalues [4].

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