FAITHFUL *-REPRESENTATIONS OF NORMED ALGEBRAS II

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1. Introduction. Let A be a complex Banach algebra with an involution $x \to x^*$. By the positive cone P of A is meant the closure, in the set H of self-adjoint elements of A, of the set of all finite sums of elements of the form x^*x. Kelley and Vaught [5] have shown that, if A has an identity,[1] A has a faithful *-representation (as bounded linear operators on a Hilbert space) if and only if (1) $x \to x^*$ is continuous and (2) $P \cap (-P) = (0)$. Consider the (incomplete) normed algebra case. Examples exist with a faithful*-representation and both conditions false, with (1) true and (2) false, and with (1) false and (2) true. Moreover, even if (1) holds so that $x \to x^*$ extends to the completion A_c of A, one can have a continuous faithful *-representation for A when none exists for A_c. It follows that the results which we now describe, even for the normed algebra case, can *not* be deduced from the theory of Banach algebras.

These facts led us to consider the development of a theory of *-representations of a complex algebra A with involution (with or without an identity) under minimal assumptions on A but with results sufficiently definitive to illuminate the counter-examples mentioned above. We suppose that the real linear space H has a norm in terms of which it is a real normed linear space such that

(a) the real subalgebra generated by each $h \in H$ is a normed algebra and

(b) the Jordan product $x \cdot h = xh + hx$ is a continuous function on H for each fixed $h \in H$.

It is shown that A has a faithful *-representation continuous on H if and only if A is semi-simple and $P \cap (-P) = (0)$. If A is a normed *-Q-algebra, any *-representation is automatically continuous on H so that these conditions are necessary and sufficient there for a faithful *-representation. As already noted, this can fail if the Q-algebra hypothesis is dropped.

For previous work on *-representations we refer to [5], [7], [8], and [10].

2. Preliminaries. Let A be an algebra over the complex field

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[1] As pointed out in [**10**, p. 352] this statement is incorrect if A has no identity. For a version covering that case see [**10**, Theorem 3.4]. Theorem 4.3 below shows that A has a faithful *-representation if and only if A is semi-simple and $P \cap (-P) = (0)$.

with an involution $x \to x^*$. The set of self-adjoint (s.a.) elements of A is denoted by H. By a *-representation of A we mean a homomorphism $x \to T_x$ of A into the algebra of bounded linear operators on a Hilbert space where, for each x, T_{x^*} is the adjoint of T_x. A *-representation which is one to one is called *faithful*. A general representation procedure of Gelfand and Naimark [7] which we adapt to our needs leads to *-representations via positive linear functionals.

A complex linear functional f on A is called *positive* if $f(x^*x) \geqq 0$ for all $x \in A$. We call f *hermitian* if $f(x^*) = \overline{f(x)}$ for all $x \in A$ or equivalently if f is a real linear functional when restricted to the real linear space H. As in [8, p. 200] we define $L_f = \{x : f(zx) = 0$ for all $z \in A\} = \{x : f(x^*x) = 0\}$; L_f is a left ideal of A. Let X_f be the linear space $A - L_f$ and π be the natural homomorphism of A onto X_f. Then, [8, p. 212], $(\pi(x), \pi(y)) = f(y^*x)$ defines an inner product on X_f in terms of which X_f is a pre-Hilbert space. Let H_f be the completion of X_f in the pre-Hilbert space norm. As in [7, p. 120] we associate with $y \in A$ a linear operator T^f_y defined on X_f by the rule $T^f_y[\pi(x)] = \pi(yx)$. In order that every T^f_y, $y \in A$, be extendable to a bounded linear operator U^f_y on H_f it is necessary and sufficient [8, p. 213] that f be *admissable*, that is, to each $x \in A$ there corresponds a number $K(x) < \infty$ such that $f(y^*x^*xy) \leqq K(x)f(y^*y)$ for all $y \in A$. If f is admissable, the mapping $x \to U^f_x$ is a *-representation of A.

For any positive linear functional f and any $y \in A$ we define the positive linear functional $f_y(x) = f(y^*xy)$.

2.1. LEMMA. *Let f be a positive linear functional on A. Then f is admissable if and only if*

$$\sup_n [f_y(h^{2^n})]^{2^{-n}} < \infty , \tag{2.1}$$

for each $y \in A$, $h \in H$, where the sup is taken over the set of positive integers.

Suppose that f is admissable. Then, for $h \in H$, U^f_h is a bounded s.a. operator on the Hilbert space H_f. For convenience, let U^f_z where $z = h^{2^n}$ be denoted by V_n. For each $y \in A$,

$$f_y(h^{2^{n+1}}) = \| V_n\pi(y) \|^2 \leqq \| U^f_h \|^{2^{n+1}} f(y^*y)$$

for $n = 0, 1, 2, \cdots$. This implies (2·1).

For the converse we make use of an inequality due to Kaplansky [4, p. 55] concerning a positive linear functional f which asserts that

$$f_y(x^*x) \leqq f(y^*y)^{1-2^{-n}}[f_y((x^*x)^{2^n})]^{2^{-n}} \tag{2.2}$$

for all $x, y \in A$ and all positive integers n. Assume (2.1). It is clearly sufficient to show that T_h^f is a bounded operator on X_f for each h s.a. Using (2.2) we have

$$\| T_h^f \pi(y) \|^2 = f_y(h^2) \leqq f(y^*y)^{1-2^{-n}}[f_y(h^{2^{n+1}})]^{2^{-n}}$$

so that $\| T_h^f \|$ cannot exceed the sup of (2.1).

2.2. Lemma. *Suppose H is given a topology in which it is a real linear topological space. Then the mapping $p \to (1 + h)\, x(1+h)$ is continuous on H, for each $h \in H$, if and only if the Jordan product $x \cdot h = hx + xh$ is continuous on H for each $h \in H$.*

Let $a(x, h) = x + xh + hx + hxh$. Then $x \cdot h = [a(x, h) - a(x, -h)]/2$ and $a(x, h) = x + x \cdot h + [(x \cdot h) \cdot h - x \cdot h^2]/2$ from which the lemma is immediate.

We now state metric requirements which we put on the algebra A with involution. We suppose given a norm $\| h \|$ on H in terms of which H is a real normed linear space and, for each $h \in H$, the real subalgebra generated by h is a normed algebra. No assumptions are made about the elements not in H nor are there any requirements of completeness or identity element. We assume that the Jordan product $x \cdot h$ is continuous on H for each $h \in H$. We call A a *normed *-algebra* if, A is a normed algebra. Following [3] we say that the normed *-algebra A is a *normed Q-*-algebra* if the set of quasi-regular elements of A is open. If A is a Banach algebra it has this property [3, p. 155].

For $h \in H$, $\lim \| h^n \|^{1/n} = \nu(h)$ exists. Clearly $\nu(h) \leqq \| h \|$ and $\nu(h^2) = [\nu(h)]^2$ (see [8, p. 10]).

2.3. Lemma. *Let f be a positive linear functional on A. The following statements are equivalent.*

(a) *Each f_y is continuous on H.*

(b) *$f_y(x^*x) \leqq \nu(x^*x) f(y^*y)$ for all $x, y \in A$.*

(c) *f is admissable and the mapping $x \to T_x^f$ is continuous on H.*

Suppose (a) holds. From the inequality (2.2) we obtain

$$f_y(x^*x) \leqq f(y^*y)^{1-2^{-n}}(\| f_y \|_H \| (x^*x)^{2^n} \|)^{2^{-n}} .$$

If we let $n \to \infty$ we obtain (b).

Suppose (b) holds. Clearly f is admissable. For $h \in H$ we have $f_y(h^2) \leqq \nu(h)^2 f(y^*y)$ so that $\| T_h^f \pi(y) \| \leqq \| h \| \| \pi(y) \|$ and $\| T_h^f \| \leqq \| h \|$.

Suppose (c) with $\| T_h^f \| \leqq k \| h \|$, $h \in H$. Then, by the Cauchy-Schwarz inequality,

$$|f_y(h)|^2 \leqq f(y^*y)f_y(h^2) = f(y^*y)\,||\,T_h^f\pi(y)\,||^2 \leqq k^2\,||\,h\,||^2[f(y^*y)]^2$$

so that f_y is continuous on H.

We note that, under these conditions, the norm of the mapping $x \to T_x^f$ on H does not exceed one.

2.4. LEMMA. *Any *-representation of a normed *-Q-algebra A is continuous on H.*

Let $x \to T_x$ be a *-representation of A. Let $\rho(u)$ denote the spectral radius of u [8, p. 30]. For $h \in H$ we have $||\,T_h\,|| = \rho(T_h) \leqq \rho(h) \leqq ||\,h\,||$ by [9, p. 373]. Thus in the Q-algebra case the admissable positive linear functionals are those satisfying (b) of Lemma 2.3; if also A has an identity the admissable positive linear functionals are those continuous on H.

2.5. LEMMA. *Suppose f is positive linear functional on A which is continuous on H. Then f_y is continuous on H for each $y \in A$.*

It follows from Lemma 2.2 that the mapping $x \to hxh$ is continuous on H for each $h \in H$. Therefore the functional f_h is continuous on H for each $h \in H$. Now, if $y = u + iv$, $u, v \in H$ we have $f_y(x) = f_u(x) + f_v(x) + if(uxv - vxu)$. But, by the Cauchy-Schwarz inequality, for any $x \in H$, $|f(uxv)|^2 \leqq f(u^2)f_v(x^2) \leqq f(u^2)\,||\,f_v\,||\,||\,x\,||^2$ where $||\,f_v\,||$ is the norm of f_v considered as a linear functional on H. This makes $||\,f_y(x)\,|| \leqq K\,||\,x\,||$, $x \in H$, where

$$K = ||\,f_u\,|| + ||\,f_v\,|| + 2[f(u^2)\,||\,f_v\,|| + f(v^2)\,||\,f_u\,||]^{1/2}\,.$$

In view of Lemma 2.3, f is admissable.

We give an example of a normed *-algebra A whose involution is continuous with the following properties.

(1) A has a faithful *-representation.

(2) Every *-representation of A other than the zero representation is discontinuous on H.

(3) The completion A_c of A has only the zero *-representation.

Let A be the set of all polynomials in the complex variable z which vanish at the origin. For $p(z) = \Sigma\alpha_k z^k$ we define $p^*(z) = \Sigma\bar{\alpha}_k z^k$ and $||\,p(z)\,|| = \Sigma\,|\,\alpha_k\,|/k!$. Then (see [3, p. 158]) A is a normed *-algebra. That (1) holds will be pointed out in § 4. Let $p \to T_p$ be a *-representation of A continuous on H. The polynomial z is s.a.. For each real scalar λ, $||\,\lambda^n z^n\,|| \to 0$. Therefore $||\,\lambda^n T_z^n\,|| = |\,\lambda\,|^n||\,T_z\,||^n \to 0$. This makes $T_z = 0$ so that $T_p = 0$ on A. Now the involution on A, being bicontinuous, extends to an involution on A_c. Any *-representation $x \to V_x$ of the Banach algebra A_c must be continuous by [8, Theorem

4.1.20]. Therefore, by the above, $V_x = 0$ for all $x \in A_c$.

Let F be a set of admissable positive linear functionals on A. We call F a *compatible set* if for each $x \in A$ there exists a real number $K(x)$ such that $\| U_x^f \| \leqq K(x)$ for all $f \in F$. This is equivalent to requiring that, for each $x \in A$, there exists $C(x) < \infty$, such that $f_y(x^*x) \leqq C(x) f(y^*y)$ for all $y \in A$ and all $f \in F$. By Lemmas 2.3 and 2.4 the set of all admissable positive linear functionals on a normed *-Q-algebra is a compatible set.

For each f in the compatible set F consider the Hilbert space H_f and the corresponding *-representation $x \rightarrow U_x^f$. Let H be the Hilbert space direct sum of the Hilbert spaces H_f. Since $\| U_x^f \| \leqq K(x)$ for all $f \in F$ we can take [7, p. 113] the direct sum $x \rightarrow U_x$ of the *-representations $x \rightarrow U_x^f$, $f \in F$ where U_x is a bounded operator on H and $\| U_x \| \leqq K(x)$. We call this *-representation the *canonical *-representation of A induced by F*. For a left ideal L of A we use the notation $(L : A)$ as in [8, p. 53] to denote the set of all $x \in A$ such that $xA \subset L$. The kernel of the canonical *-representation induced by F is given by $\bigcap (L_f : A)$ where the intersection is taken over all $f \in F$.

3. On *-representations. For our purposes we wish to define the **-radical* $\mathfrak{R}^*$ of A as the intersection of the kernels of all *-representations of A which are continuous on H. Let $A^\sharp$ denote the set of all positive linear functionals on A. At the outset we consider three subsets of $A^\sharp$. Let $\mathfrak{B} = \{f \in A^\sharp : f_y(x^*x) \leqq \nu(x^*x) f(y^*y),$ for all $x, y \in A\}$. Let $\mathfrak{D}$ be the set of *dual functionals* by which we mean $\{f \in A^\sharp : f$ is hermitian and f is continuous on $H\}$. Let $\mathfrak{E} = \{f \in D : | f(x)|^2 \leqq f(x^*x)$ for all $x \in A\}$. By Lemmas 2.3 and 2.5 we see that $\mathfrak{B} \supset \mathfrak{D} \supset \mathfrak{E}$ and that these are compatible sets. Let $\mathfrak{B}_0$, $\mathfrak{D}_0$, and $\mathfrak{E}_0$ be the kernels of the canonical *-representations of A induced by $\mathfrak{B}$, $\mathfrak{D}$, and $\mathfrak{E}$ respectively. Then $\mathfrak{E}_0 \supset \mathfrak{D}_0 \supset \mathfrak{B}_0$.

3.1. LEMMA. $\mathfrak{R}^* = \mathfrak{E}_0 = \mathfrak{D}_0 = \mathfrak{B}_0$. *$A/\mathfrak{R}^*$ is semi-simple.*

For any $f \in \mathfrak{B}$, and $x, y \in A$, $\| T_x^f \pi(y) \|^2 \leqq \nu(x^*x) \| \pi(y) \|^2$ so that $\| T_x^f \| \leqq \nu(x^*x)^{1/2}$. Consequently $\| T_h^f \| \leqq \nu(h) \leqq \| h \|$, $h \in H$. Therefore if $x \rightarrow T_x$ is any of the canonical *-representations in question, $\| T_h \| \leqq \| h \|$, $h \in H$, and the *-representation is continuous on H. This proves that $\mathfrak{R}^* \subset \mathfrak{B}_0 \subset \mathfrak{D}_0 \subset \mathfrak{E}_0$. We show that $\mathfrak{E}_0 \subset \mathfrak{R}^*$.

Let $x \rightarrow V_x$ be any *-representation of A continuous on H, say as operators on the Helbert space M. For each $\alpha \in M$ the functional $g^\alpha(x) = (V_x(\alpha), \alpha)$ is continuous on H and is a dual functional. For α in the unit ball Σ of M, $| g^\alpha(x) |^2 \leqq \| V_x(\alpha) \|^2 = g^\alpha(x^*x)$ so that $g^\alpha \in \mathfrak{E}$. We have

$$
\begin{aligned}
\mathfrak{E}_0 &= \bigcap_{f \in E} (L_f : A) \subset \bigcap_{\alpha \in \Sigma} (L_{g^\alpha} : A) \\
&= \bigcap_{\alpha \in \Sigma} \{z \in A : g^\alpha((zy)^*(zy)) = 0 \text{ for all } y \in A\} \\
&= \bigcap_{\alpha \in \Sigma} \{z \in A : V_{zy}(\alpha) = 0 \text{ for all } y \in A\} \\
&= \{z \in A : V_{zy} = 0 \text{ for all } y \in A\} \\
&\quad \subset \{z \in A : \| V_{zz^*} \| = \| V_z \|^2 = 0\} .
\end{aligned}
$$

Therefore $\mathfrak{E}_0 \subset \mathfrak{R}^*$.

Since $A/\mathfrak{R}^* = A/\mathfrak{E}_0$ is algebraically *-isomorphic to a *-subalgebra of the algebra of all bounded linear operators on a Hilbert space, we see from [8, Theorem 4.1.19] that $A/\mathfrak{R}^*$ is semi-simple. From this we see also that the radical of A is contained in $\mathfrak{R}^*$.

3.2. LEMMA. *A normed *-algebra A has a faithful *-representation continuous on H if and only if* $\mathfrak{R}^* = (0)$.

Suppose $\mathfrak{R}^* = (0)$. The preceding Lemma 3.1 then asserts the canonical *-representations induced by $\mathfrak{B}$, $\mathfrak{D}$, or $\mathfrak{E}$ are faithful. As noted above, these *-representations are continuous on H. We naturally seek conditions on A which force $\mathfrak{R}^* = (0)$.

We set forth notation which will be used below. Let R_0 be the collection of all finite sums of elements of A of the form x^*x and let P be the closure of R_0 in H. The set P will be considered as a closed cone in the real normed linear space H. Let A_1 be the algebra obtained by adjoining an identity e to A. As usual the involution on A is extended to A_1 by $(\lambda e + x)^* = \bar{\lambda} e + x^*$ where λ is a scalar and $x \in A$. We shall have occasion to consider the sets H, $\mathfrak{R}^*$, $\mathfrak{D}$, R_0, and P in A simultaneously with the corresponding sets defined for A_1. When we do so, we denote the latter sets by H_1, $\mathfrak{R}_1^*$, $\mathfrak{D}_1$, R_{01}, and P_1 respectively. The given norm on H leads to a norm on H_1 via $\| \lambda e + h \| = | \lambda | + \| h \|$, λ real, $h \in H$. A_1 satisfies the requirements of our theory.

We set $Z(\mathfrak{D}) = \bigcap f^{-1}(0)$, $f \in \mathfrak{D}$ and $Z(\mathfrak{E}) = \bigcap f^{-1}(0)$, $f \in E$. We define two versions of the *reducing ideal* [7, p. 130] suitable for this setting. Let $\bigcap L_f$ where f runs over $\mathfrak{D}(\mathfrak{E})$ be denoted by $RI(\mathfrak{D})$ and $RI(\mathfrak{E})$ respectively.

Let g be a continuous real linear functional on H, $g(P) \geqq 0$. If we extend g to A by the rule $g(x) = g(h_1) + ig(h_2)$ for $x = h_1 + ih_2$, h_1, h_2 s.a., we obtain an element of $\mathfrak{D}$. Conversely the restriction to H of any $f \in D$ has the property that $g(P) \geqq 0$. From the theory of closed cones in a normed linear space [5, Lemma 1.2] it follows that $P \cap (-P)$ is the s.a. part of $Z(\mathfrak{D})$ so that $Z(\mathfrak{D}) = P \cap (-P) + iP \cap (-P)$. In a more restrictive context, this was pointed out and used in [1].

It will appear that $Z(\mathfrak{E})$ can differ from $Z(\mathfrak{D})$; $Z(\mathfrak{E})$ does not seem to have as neat an interpretation as $Z(\mathfrak{D})$. For that reason the results of § 4 involving $\mathfrak{D}$ are more interesting than the theory for $\mathfrak{E}$.

3.3. LEMMA. $\mathfrak{R}^* = (RI(\mathfrak{D}) : A) = (RI(\mathfrak{E}) : A)$

We see, by Lemma 3.1, that $\mathfrak{R}^*$ is the kernel of the canonical *-representation induced by $\mathfrak{D}$. Thus

$$\mathfrak{R}^* = \bigcap_{f \ni \mathfrak{D}} (L_f : A) = \bigcap_{f \in \mathfrak{D}} \{x : xy \in L_f, \text{ for all } y \in A\}$$
$$= \left\{x : xy \in \bigcap_{f \in \mathfrak{D}} L_f, \text{ for all } y \in A\right\} = (RI(\mathfrak{D}) : A) .$$

Let $x \to V_x$ be the canonical *-representation induced by $\mathfrak{E}$. We show, by direct computation, (see [7, p. 132]) that

$$\| V_x \|^2 = \sup_{f \in \mathfrak{E}} f(x^*x),\ x \in A . \tag{3.1}$$

Let $\beta(x)$ denote the right hand side of (3.1). Take $f \in \mathfrak{E}$. Then $| f_y(x) |^2 = | f(y^*xy) |^2 \leqq f(y^*y) f_y(x^*x)$ by the Cauchy-Schwarz inequality. Therefore $f_y \in \mathfrak{E}$ whenever $f(y^*y) \leqq 1$. Now $\| T_x^f \pi(y) \|^2 = f_y(x^*x)$ so that $\| T_x^f \|^2 \leqq \beta(x)$ from which we see that $\| V_x \|^2 \leqq \beta(x)$. On the other hand, for $f \in \mathfrak{E}$,

$$[f(x^*x)]^2 \leqq f(x^*xx^*x) = \| T_{x^*}^f \pi(x) \|^2 \leqq \| T_x^f \|^2 f(x^*x)$$

which shows that $\beta(x) \leqq \| V_x \|^2$.

From Lemma 3.1 we observe that $\mathfrak{R}^* = \{x : f(x^*x) = 0, \text{ for all } f \in \mathfrak{E}\} = RI(\mathfrak{E})$. This formula, as we shall see in § 4, can be invalid if $\mathfrak{E}$ is replaced by $\mathfrak{D}$.

We consider next a version of Kelley and Vaught's result [5, Theorem 4.4].

3.4. THEOREM. *Let $x \to V_x$ be the conical *-representation of A induced by $\mathfrak{E}$. Then* $\| V_x \|^2 = \operatorname{dist}(-x^*x, P_1)$.

Let $h \in H$, $\| h \| \leqq 1$. In the algebra A_1 let B be the real subalgebra generated by e and h and let B_c be its completion. For $m = 1, 2, \cdots$ let

$$w_m = \sum_{k=1}^{m} \binom{1/2}{k} (-1)^k h^k . \tag{3.2}$$

Clearly $w_m \in H$. In B_c we have $(e - h) = [\lim (e + w_m)]^2$ so that, in A_1, we get

$$e - h = \lim_m (e + w_m)^2 . \tag{3.3}$$

This shows that, in H_1, e is an interior point of the cone P_1.

The discussion in [6, p. 96] shows that any $f \in \mathfrak{E}$ is extendable to A_1 so as to belong to $\mathfrak{D}_1$ where $f(e) \leqq 1$. On the other hand if $g \in D_1$, $g(e) \leqq 1$ then its restriction to A lies in $\mathfrak{E}$ by the Cauchy-Schwarz inequality. Since $\| e \| = 1$ and e is an interior point of P_1 we see, from Lemma 1.3 of [5], that, for each $x \in A$,

$$\operatorname{dist}(-x^*x, P_1) = \sup_{\substack{f \in \mathfrak{D}_1 \\ f(e) \leqq 1}} f(x^*x) = \sup_{f \in \mathfrak{E}} f(x^*x) .$$

An application of formula (3.1) completes the proof.

4. Faithful *-representations.

4.1. THEOREM.

(a) *$Z(\mathfrak{D})$ is a two-sided ideal of A.*

(b) *$Z(\mathfrak{D}) \subset RI(\mathfrak{D}) \subset \mathfrak{R}^*$ and the inclusions can be proper.*

(c) *If A has an identity, $Z(\mathfrak{D}) = RI(\mathfrak{D}) = \mathfrak{R}^*$.*

(d) *If $x \in \mathfrak{R}^*$ then $x^3 \in Z(\mathfrak{D})$.*

(e) *$\mathfrak{R}^*$ is the complete inverse image of the radical of $A/Z(\mathfrak{D})$ under the natural homomorphism of A onto $A/Z(\mathfrak{D})$.*

We refer to formula (3.2) for notation. For each $m = 1, 2, \cdots$ we define the operator α_m on H by the rule $\alpha_m(x) = (e + w_m)x(e + w_m)$. Since $\alpha_m(x^*x) = (x + xw_m)^*(x + xw_m)$ we see that $\alpha_m(R_0) \subset R_0$. Because α_m is continuous on H by Lemma 2.2, we also get $\alpha_m(P) \subset P$.

Suppose next that also $h \in P$. Then $(e+w_m)h(e+w_m)=h(e+w_m)^2 \in P$. Passing to the limit as $m \to \infty$ we see from (3.3) that $h - h^2 \in P$. We have established that, for any $h \in P$ whatever its norm, $f(h) \| h \| \geqq f(h^2) \geqq 0$, $f \in \mathfrak{D}$. By the Cauchy-Schwarz inequality $|f(hx)|^2 \leqq f(h^2)f(x^*x)$ and $|f(xh)|^2 \leqq f(xx^*)f(h^2)$, f $\in \mathfrak{D}$. Now $P \cap (-P) = \{y \in H : f(y) = 0, f \in \mathfrak{D}\}$, so that $f(yx) = 0 = f(xy)$ for all $x \in A$, $f \in \mathfrak{D}$. Next let $w \in Z(\mathfrak{D})$. We can write $w = y_1 + iy_2$ where each $y_k \in P \cap (-P)$. We then see that $f(wx) = 0 = f(xw)$ for all $f \in \mathfrak{D}$, $x \in A$, so that wx and xw lie in $Z(\mathfrak{D})$. This establishes (a).

Let $x \in Z(\mathfrak{D})$. By (a) we see that $x^*x \in Z(D)$ so that $f(x^*x) = 0$ for all $f \in \mathfrak{D}$. Thus $Z(\mathfrak{D}) \subset RI(\mathfrak{D})$. Next let $x \in RI(\mathfrak{D})$, $y \in A$. Then $xy \in RI(\mathfrak{D})$ so that $x \in (RI(\mathfrak{D}) : A) = \mathfrak{R}^*$ by Lemma 3.3.

We now produce an example for which $Z(\mathfrak{D}) \neq RI(\mathfrak{D})$. Let $A = C([0, 1])$ with the usual norm and involution but considered as a zero algebra. Then all linear functionals on A are positive. This implies that $RI(\mathfrak{D}) = A$. On the other hand it is trivial that $Z(\mathfrak{D}) = (0)$.

We now provide an instance where $RI(\mathfrak{D}) \neq \mathfrak{R}^*$. Let $q(w)$ be the function $q(w) = w$ on $[0, 1]$. Again we take $A = C([0, 1])$ with the usual norm and involution but define the product by the rule $xy = x(0)y(0)q$. Under these definitions A is a Banach algebra and $A^3 = (0)$. Since the radical of A is contained in $\mathfrak{R}^*$ by Lemma 3.1, we see that $\mathfrak{R}^* = A$. Now for any linear functional f on A, $f(x^*x) = |x(0)|^2 f(q)$. By the Hahn-Banach theorem, there exists a continuous real linear functional g on H such that $g(q) = 1$. We extend g to A by the rule $g(h_1 + ih_2) = g(h_1) + ig(h_2)$ where $h_1, h_2 \in H$. Then $g \in \mathfrak{D}$. If $x \in RI(\mathfrak{D})$, $g(x^*x) = |x(0)|^2 = 0$. Thus $RI(\mathfrak{D}) = \{x \in A : x(0) = 0\}$. This completes the proof of (b).

Suppose that A has an identity e. For any $x \in \mathfrak{R}^*$, $x = xe \subset RI(\mathfrak{D})$ by Lemma 3.3. Next take $x \in RI(\mathfrak{D})$. Since $|f(x)|^2 \leqq f(e)f(x^*x) = 0$, for all $f \in \mathfrak{D}$, we see that $x \in Z(\mathfrak{D})$. Combining this information with the set inequalities of (b) we obtain (c).

By Lemma 3.1 there exists a *-representation of A_1 continuous on H_1 with kernel $\mathfrak{R}_1^*$. By restricting this *-representation to A we see that

$$\mathfrak{R}^* \subset A \cap \mathfrak{R}_1^* . \tag{4.1}$$

Let λ be a scalar and $x, y \in A$. Then $y^*(\lambda e + x)^*(\lambda e + x)y = (\lambda y + xy)^*(\lambda y + xy) \in R_0$. Thus $y^*R_{01}y \subset R_0$ for each $y \in A$. From Lemma 2.2 it can be seen that, for $h \in H$, the mapping $x \longrightarrow hxh$ is continuous on H. It is easily shown that $x \longrightarrow hxh$ is also continuous on H_1. It then follows that $hP_1h \subset P$. This shows that $h[P_1 \cap (-P_1) + iP_1 \cap (-P)]h \subset P \cap (-P) + iP \cap (-P)$. By (c) this gives

$$h\mathfrak{R}_1^*h \subset Z(\mathfrak{D}),\ h \in H . \tag{4.2}$$

From (4.1) and (4.2) we have $h\mathfrak{R}^*h \subset Z(\mathfrak{D})$. It follows readily that $uzw + wzu \in Z(\mathfrak{D})$ for all $u, w \in H$ and $z \in \mathfrak{R}^*$. Let $x = u + iv \in \mathfrak{R}^*$, $u, v \in H$ and note that $u, v \in \mathfrak{R}^*$. Writing $x^3 = u^3 - iv^3 + i(u^2v + vu^2) + iuvu - vuv - (v^2u + uv^2)$ we see that the individual terms of the expansion lie in $Z(\mathfrak{D})$.

We turn to (e). Let γ be the natural homomorphism of A onto $A/Z(\mathfrak{D})$. For $x \in R^*$, $[\gamma(x)]^3 = 0$ by (d) so that $\gamma(\mathfrak{R}^*) \subset W$, the radical of $A/Z(\mathfrak{D})$. Inasmuch as A/R^* is semi-simple by Lemma 3.1, so is $[A/Z(\mathfrak{D})]/[\mathfrak{R}^*/Z(\mathfrak{D})]$. Therefore $\mathfrak{R}^*/Z(\mathfrak{D}) \supset W$.

4.2. THEOREM. *The following statements are equivalent.*

(a) *There exists a faithful *-representation of A continuous on H.*

(b) *A is semi-simple and $P \cap (-P) = (0)$.*

(c) *A is semi-simple and $RI(\mathfrak{D}) = (0)$.*

Suppose (a). A is semi-simple by [8, Theorem 4.1.19]. Lemma 3.2 gives $\mathfrak{R}^* = (0)$ so that $P \cap (-P) = (0)$ from Theorem 4.1 (b).

Suppose (b). Then $Z(\mathfrak{D}) = (0)$ so that, by Theorem 4.1 (d), $x^3 = 0$ for each $x \in \mathfrak{N}^*$. Since A is semi-simple, $\mathfrak{N}^* = (0)$. Then Theorem 4.1 (b) shows that $RI(\mathfrak{D}) = (0)$.

Suppose (c). Again $Z(\mathfrak{D}) = (0)$ by Theorem 4.1 (b). As just seen this implies that $\mathfrak{N}^* = (0)$ so that (a) follows from Lemma 3.2.

4.3. COROLLARY. *Let A be a normed *-Q-algebra. Then A has faithful *-representation if and only if A is semi-simple and* $P \cap (-P) = (0)$.

This follows immediately from Theorem 4.2 and Lemma 2.4.

We now exhibit a normed *-algebra with a faithful *-representation but for which $P \cap (-P) \neq (0)$. Let A be the algebra of all polynomials in the complex variable z. F or $p(z) = \Sigma \alpha_n z^n$ set $p^*(z) = \Sigma \bar{\alpha}_z z^n$. First consider A in the norm

$$\|p\| = \sup_{0 \leqq t \leqq 1} |p(t)| .$$

Here, for each t, $0 \leqq t \leqq 1$ the functional $f_t(p) = p(t)$ is a positive linear functional continuous on A and real-valued on H. Thus $Z(\mathfrak{D}) = (0)$. By Theorem 4.2 we see that A has a faithful *-representation. This also justifies a remark following Lemma 2.5.

Next consider A in the norm $\|p\| = \Sigma |\alpha_k|/k!$ (see § 2). For $p(z) = \alpha_0 + \cdots + \alpha_n z^n$ let $f(p) = \alpha_0$. This gives us a continuous *-representation of A as operators on one-dimensional Hilbert space with kernel $M = \{p : p(0) = 0\}$ so that $M \supset \mathfrak{N}^*$. The arguments of § 2 following Lemma 2.5 show that any *-representation of A continuous on H must vanish on M. Therefore $M = \mathfrak{N}^*$. Via Theorem 4.1 we see that $P \cap (-P)$ is the set of all polynomials with real coefficients vanishing at the origin. We investigate the commutative case more closely in § 5.

4.4. LEMMA. $\mathfrak{N}^* = A \cap \mathfrak{N}_1^*$.

We already have $\mathfrak{N}^* \subset A \cap \mathfrak{N}_1^*$ by (4.1). Let $\mathfrak{K} = A \cap \mathfrak{N}_1^*$. By (4.2), $h\mathfrak{K}h \subset Z(\mathfrak{D})$ for each $h \in H$. Reasoning exactly as in the proof of Theorem 4.1 (d) we obtain $x^3 \in Z(\mathfrak{D})$ for each $x \in \mathfrak{K}$. Let β be the natural homomorphism of A onto $A/\mathfrak{N}^*$. Since $Z(\mathfrak{D}) \subset \mathfrak{N}^*$ by Theorem 4.1 (b), we see that $[\beta(x)]^3 = 0$ for each $x \in \mathfrak{K}$. From Lemma 3.1 we obtain $\beta(\mathfrak{K}) = (0)$. We now derive another formula for $\mathfrak{N}^*$.

4.5. THEOREM. $\mathfrak{N}^* = Z(\mathfrak{E})$.

As noted in the proof of Theorem 3.4, $\mathfrak{E}$ is the set of positive linear functionals on A which are extendable to positive linear functionals f on A_1, lying in $\mathfrak{D}_1$, with $f(e) \leqq 1$. Now any $g \in \mathfrak{D}_1$ is a multiple of such a functional. Therefore $Z(\mathfrak{E}) = A \cap Z(\mathfrak{D}_1) = A \cap \mathfrak{R}_1^* = \mathfrak{R}^*$ by way of Theorem 4.1 and Lemma 4.4.

We have $\mathfrak{R}^* = RI(\mathfrak{E}) = Z(\mathfrak{E})$, a situation which differs from what can happen for $\mathfrak{D}$. In particular $Z(\mathfrak{E}) \neq Z(\mathfrak{D})$ can occur.

5. **The commutative case.** Let A be a commutative algebra with an involution. By commutativity, H is a real subalgebra of A. We suppose in § 5 that H has a norm in terms of which it is a real normed algebra. Let $\mathfrak{M}$ be the set of modular maximal ideals of A_1. We call $M \in \mathfrak{M}$ *symmetric* if $M = M^*$ and single out for special attention the set of symmetric M for which $M \cap H$ is closed in H.

5.1. LEMMA. *Let μ be a homomorphism of H into the reals. Define, for each $x = h + ik$, $h, k \in H$ the functional μ_a by the rule $\mu_a(x) = \mu(h) + i\mu(k)$. Then μ_a is a multiplicative (complex) linear functional on A.*

This can be verified in a straight forward way.

5.2. LEMMA. *Let M be a symmetric modular maximal ideal of A where $M \cap H$ is closed in H. Then there exists a continuous homomorphism μ of H onto the reals such that $\mu_a^{-1}(0) = M$.*

Let j be an identity for A modulo M. Then so is $(j + j^*)/2$ so without loss of generality we can take j s.a. Then $ju - u \in M \cap H$ for all $u \in H$ and therefore $M \cap H$ is a modular ideal of H. Since $M = M \cap H \oplus i(M \cap H)$ it is clear the $M \cap H \neq H$. We claim that $M \cap H$ is a modular maximal ideal of H. For otherwise there exists a modular maximal ideal K of H containing $M \cap H$, $K \neq M \cap H$. An easy computation shows that $K \oplus iK$ is an ideal of A containing M. Then $K \oplus iK = A$ which is impossible as $j \notin K$ (otherwise $K = H$). Inasmuch as $M \cap H$ is closed in H, $H/M \cap H$ is a normed field in the quotient algebra norm. By Mazur's theorem, $H/M \cap H$ is a copy of the real or complex field. We rule out the latter possibility If $H/M \cap H$ were a copy of the complexes then it would be two-dimensional over the real field and there would be a two-dimensional real subspace L of H such that $H = M \cap H \oplus L$. Then $A = H \oplus iH = M \oplus L \oplus iL$ which compels A/M to be four-dimensional over the reals. But surely A/M is a division algebra over the reals. Thus a well-known theorem of Frobenius makes A/M a copy of the quaternions. This is impossible in view of commutativity. Consequently there is a continuous homomorphism μ of H onto the reals with kernel $M \cap H$.

5.3. THEOREM. *$\mathfrak{R}^*$ is the intersection of those symmetric modular maximal ideals M of A such that $M \cap H$ is closed in H.*

Take $x = u + iv$, $u, v \in H$. By commutativity, $x^*x = (u^2 + v^2)/2$. Thus P is the closure in H of finite sums of squares of elements of H. Suppose first that A has an identity e. The proof of Theorem 3.4 shows that $e \in \mathrm{Int}(P)$. Let Σ represent the set of all continuous real linear functionals g on H where $g(P) \geqq 0$ and $g(e) \leqq 1$. The arguments of [5, Theorem 2.1] show that the set Σ_e of extreme points of Σ is the set the continuous homomorphisms of H into the reals. As in [5, Remark 2.3] at follows that $P \cap (-P) = \bigcap f^{-1}(0)$ where f ranges over Σ_e. Let S be the intersection of the symmetric modular maximal ideals M of A with $M \cap H$ closed in H. Lemmas 5.1 and 5.2 show that $H \cap S = \bigcap f^{-1}(0) = P \cap (-P)$ and Lemma 4.1 (b) shows that $S = \mathfrak{R}^*$.

Now suppose that A has no identity. Each multiplicative linear functional on A which is real and continuous on H extends, as is easily verified, to a multiplicative linear functional on A_1 which is real and continuous on H_1. Applying the result for the case with the identity we get $S = A \cap \mathfrak{R}_1^* = \mathfrak{R}^*$ with the aid of Lemma 4.4.

6. **An example.** We give an example of a normed *-algebra A which has a *continuous* faithful *-representation and a continuous involution but for which the ccmpletion[2] A_c has no faithful *-representation. This demonstrates conclusively that our results in the case of a normed *-algebra (e.g. Theorem 4.2 and Corollary 4.3) cannot possibly deduced from the theory of Banach algebras.

The algebra A which we use is a subalgebra of an algebra devised for other purposes by C. Feldman [2]. His algebra is the commutative algebra B which is the completion of the algebra of all finite sums

$$\sum_{i=1}^{n} \alpha_i e_i + \beta r$$

where α_1 and β are complex, the e_i are mutually orthogonal idempotents, $r^2 = 0 = e_i r = r e_i$ for all i and

$$\| \Sigma \alpha_i e_i + \beta r \| = \max \{ (\Sigma | \alpha_i |^2)^{1/2}, | \beta - \Sigma \alpha_i | \} .$$

Consider the subalgebra A consisting of all finite sums $\Sigma \alpha_i e_i$. The involution

$$(\Sigma \alpha_i e_i)^* = \Sigma \bar{\alpha}_i e_i$$

on A is an isometry. For each integer $n > 1$ let $s(n)$ be the smallest

[2] The involution on A extends to an involution on A_c.

integer of the form $n + k$ such that $\sum_{j=0}^{k}(n+j)^{-1} > 1$. Let $z_n = n^{-1}e_n + (n+1)^{-1}e_{n+1} + \cdots + [s(n)]^{-1}e_{s(n)}$. It is readily verified that $\| r - z_n \| \to 0$. Therefore B is the completion of A. For each $n = 1, 2, \cdots$ the functional f_n defined on A by the rule $f_n(\Sigma\alpha_i e_i) = \alpha_n$ is a continuous multiplicative linear functional on A. Moreover $\bigcap f_n^{-1}(0) = (0)$ so that A is semi-simple and, by Theorem 5.3 and Lemma 3.2, A has a faithful *-representation continuous on H. The continuity of the involution allows us to assert that this *-representation is continuous on A. However, the completion B of A is not semi-simple [2] and so has no faithful *-representation [8, Theorem 4.1.19].

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