ON THE DEFINING RELATIONS OF A FREE PRODUCT

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If $F_n = F(x_1, \cdots, x_n)$ is the free group generated by the symbols $x_1, \cdots, x_n$, and $R_i = R_i(x_1, \cdots, x_n)$, i: $1, \cdots, k$ are element in it, let

$$R = \{R_1, \cdots, R_k\}$$

be their normal closure in F_n, and let

$$F/R = (x_1, \cdots, x_n; R_1, \cdots, R_k)$$

be the factor group of F_n by R, with n and k assumed finite.

My object is to prove the following theorem and corollaries.

THEOREM. *If*

$$F/R = (x_1, \cdots, x_n; R_1, \cdots, R_k)$$

and

$$F^*/R^* = (x_1^*, \cdots, x_{n^*}^*; R_1^*, \cdots, R_{k^*}^*)$$

are isomorphic groups and $n^* + k < n + k^*$, *then in the free group* F_{n+n^*} *generated by* $x_1^*, \cdots, x_{n^*}^*, z_1, \cdots, z_n$ *the normal subgroup* $\{R_1^*(x_1^*, \cdots, x_{n^*}^*), \cdots, R_{k^*}^*(x_1^*, \cdots, x_{n^*}^*), z_1, \cdots, z_n\}$ *is normal closure of* $n^* + k < n + k^*$ *elements.*

The two corollaries concern the cases $k \leqq 1$ and $k^* \leqq 1$ — that is, free groups and groups possessing presentations on a single defining relation. (Deficiency is defined in Remark 1. below.)

COROLLARY 1. *If a group possesses the two presentations of the theorem, then the one with the lesser deficiency has at least two defining relations* ($k^* > 1$).

COROLLARY 2. *If* $G = F/R$ *and* $k \leqq 1$, *then* $n - k = d$ *is maximal for all presentations of* G.

The theorem is trivial if for example certain of the k^* defining relations are redundant. It becomes interesting when the number k^* is least possible for the subgroup R^*. For suppose that k^* is minimal for R^*:

$$R^* = \{R_1^*, \cdots, R_{k^*}^*\} \neq \{S_1, \cdots, S_{k^*-1}\}$$

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for any elements $S_1, \cdots, S_{k^*-1}$ of F_{n^*}. Then if $z_1, \cdots, z_n$ are new symbols and F_{n+n^*} the free group on the symbols $x_1^*, \cdots, x_{n^*}^*, z_1, \cdots, z_n$, the normal closure of $R_1^*, \cdots, R_{k^*}^*$ in F_{n+n^*} still requires k^* elements to generate it. Consider however the normal subgroup

$$T = \{R_1^*, \cdots, R_{k^*}^*, z_1, \cdots, z_n\}$$

of F_{n+n^*}. The R_j^* do not involve the symbols z_i; k^* is least possible for R^*, and, of course, n is least possible for the group

$$(z_1, \cdots, z_n ; z_1, \cdots, z_n) = 1.$$

Now the group

$$(x_1^*, \cdots, x_{n^*}^*, z_1, \cdots, z_n ; R_1^*, \cdots, R_{k^*}^*, z_1, \cdots, z_n)$$

is the free product $G_1 {}^* G_2$ of $G_1 = F^*/R^*$ and G_2 the trivial group $(z_1, \cdots, z_n ; z_1, \cdots, z_n)$. The theorem claims that the sum of the (minimal) numbers of defining relations for the G_i is not always minimal for $G_1 {}^* G_2$.

Compare this with Grushko's theorem, which implies that the number of generators of a free product is the sum of those for the factors.

REMARK 1. If one takes the number k in the presentation F/R (of any group) to be least possible for R in F, then this presentation is said to have the deficiency

$$d = n - k .$$

Thus, setting $d^* = n^* - k^*$, the inequality $n^* + k < n + k^*$ is the same as

$$d^* < d ,$$

provided that k is minimal for R and k^* is minimal for R^* in their respective free groups. The deficiency of a presentation is not a group invariant [1].

REMARK 2. The group with noninvariant deficiency given in [1] happens to be non-Hopfian (it is isomorphic to a proper factorgroup of itself: $G \simeq G/N$, $N \neq 1$). It is not known whether such groups must be non-Hopfian; however, two presentations, G and G/N of a non-Hopfian group may have identical deficiencies even if N is not trivial. That is, if a group is given by the presentation

$$G_1 = F/R = (x_1, \cdots, x_n ; R_1, \cdots, R_k)$$

and is isomorphic to its own proper factorgroup given by

$$G_2 = F/R^* = (G_1\,;\, R_{k+1})\ ,$$

it need not follow that $k+1$ is minimal for R^* whenever k is minimal for R. For example, the first known pair of presentations of a non-Hopfian group, due to Higman and quoted in [2], is

$$G_1 = (a, b, c\,;\, a^{-1}bab^{-2}, bcb^{-1}c^{-1})$$
$$G_2 = (G_1\,;\, ab^{-1}a^{-1}c^{-1}aba^{-1}c)\ ,$$

but $R^* = \{R_1, R_2, R_3\} = \{R_1, R_3\}$ (the R_i being the three defining words above in the order given).

Proof of the theorem. Set

$$x = (x_1, \cdots, x_n),\ x^* = (x_1^*, \cdots, x_{n^*}^*),\ f(x^*) = (f_1(x^*), \cdots, f_n(x^*))\ ,$$

etc., and let

$$G_1 = F/R\ ,$$
$$G_2 = IG_1 = F^*/R^*\ ,$$

with the isomorphism I given by

$$I(x_i) = f_i(x^*),\ \ i: 1, \cdots, n\ ,$$
$$I(g_j(x)) = x_j^*,\ \ j: 1, \cdots, n^*\ .$$

Form the groups

$$H_1 = \mathring{F}/\mathring{R} = (x_1, \cdots, x_n, y_1, \cdots, y_{n^*}\,;\, R_1, \cdots, R_k, y_j g_j^{-1}(x),\ j: 1, \cdots, n^*)$$

and

$$H_2 = \mathring{F}*/\mathring{R}* = (x_1^*, \cdots, x_{n^*}^*, y_1^*, \cdots, y_n^*\,;\, R_1^*, \cdots, R_{k^*}^*, y_i^* f_i^{-1}(x^*),\ i: 1, \cdots, n)\ .$$

These are new presentations of the same group and the following mapping, J_1, defines, and isomorphism between H_1 and H_2 [cf. 4]:

$$J_1(x_i) = y_i^*,\ \ i: 1, \cdots, n$$
$$J_1(y_j) = x_j^*,\ \ j: 1, \cdots, n^*\ .$$

Since

$$x_j^* = I(g_j(x)) = g_j(I(x)) = g_j(f(x^*)) \text{ modulo } R^*$$

and

$$R(f(x^*)) = 1 \text{ modulo } R^*$$

in G_2, and hence in H_2, one gets the following identities in H_2:

$$J_1(R) = R(y^*) = R(f(x^*)) = 1 \; ,$$
$$J_1(y_j g_j^{-1}(x)) = x_j^* g_j^{-1}(y^*) = x_j^* g_j^{-1}(f(x^*)) = 1 \; .$$

Clearly, J_1 maps not only H_1 on H_2, but also $\mathring{F}$ on $\mathring{F}*$, and $\mathring{R}$ on $\mathring{R}*$, isomorphically.

Finally let

$$H_3 = F'/R' = (x_1^*, \cdots, x_{n^*}^*, z_1, \cdots, z_n \,; R_1^*, \cdots, R_{k^*}^*, z_1, \cdots, z_n) \; .$$

Under the transformation J_2 given by

$$J_2(y_i^*) = z_i f_i(x^*), \;\; i : 1, \cdots, n \; ,$$
$$J_2(x_j^*) = x_j^*, \;\; j : 1, \cdots, n^* \; ,$$

H_2 is mapped isomorphically on H_3 and J_2 is also an isomorphism between the free groups involved:

$$J_2(\mathring{F}*) = F' \qquad J_2(\mathring{R}*) = R' \; .$$

Writing $J = J_2 J_1$ gives $JH_1 = H_3$, $J\mathring{F} = F'$, $J\mathring{R} = R'$, since J maps two free groups of equal rank (namely $n + n^*$) onto each other. Hence

$$J\mathring{R} = \{JR_1, \cdots, JR_k, J(y_1 g_1^{-1}(x)), \cdots, J(y_{n^*} g_{n^*}^{-1}(x))\}$$
$$= R' = \{R_1^*, \cdots, R_{k^*}^*, z_1, \cdots, z_n\} \; ,$$

showing that R' is the normal closure of $n^* + k$ elements in spite of the possibility that k^* is minimal for R^* and that the R_i^* contain no z-symbols.

This concludes the proof.

Proof of Corollary 1. If $k^* = 0$ then R' becomes $\{z_1, \cdots, z_n\}$, the normal closure of a free factor of F', and so requires $n = n + k^* > n^* + k$ defining elements. This contradicts the theorem, so $k^* \neq 0$.

If $k^* = 1$, then $R' = \{R_1^*, z_1, \cdots, z_n\} = \{s_1, \cdots, s_n\}$ obtains. Now R' contains the free factor $F(z_1, \cdots, z_n)$ of F' and is the closure of n elements in F'. Therefore, by [3], $R' = \{z_1, \cdots, z_n\}$; and since the R_j^* contain no z-symbols, R_1^* is empty, so $k^* = 0$. Hence $k^* > 1$.

Proof of Corollary 2. Suppose, on the contrary, that there is a presentation with deficiency $d^{**} > d$. Then k can play the role of k^* above and Corollary 1 is contradicted.

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