DERIVATIONS ON B^* ALGEBRAS

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1. A *derivation* D of a B^* algebra A is a linear map of A into itself satisfying the multiplicative rule

$$D(xy) = (Dx)y + x(Dy) .$$

The obvious examples are the inner derivations D_x (x in A) defined by

$$D_x(y) = [x, y] = xy - yx .$$

All other derivations are called outer. For future use, we call a derivation D self-adjoint if

$$D(x^*) = -(Dx)^*$$

for all x in A. Thus inner derivation by a self-adjoint element is a self-adjoint derivation. Every derivation can be written in the form $D = D_1 + iD_2$ where D_1 and D_2 are self-adjoint; indeed, we may take

$$D_1(x) = \frac{1}{2}\{Dx - (Dx^*)^*\}$$

$$D_2(x) = \frac{1}{2i}\{Dx + (Dx^*)^*\} .$$

The central fact about derivations of B^* algebras is that they are bounded; this is proved by Sakai [6, Theorem 11.1]. Somewhat more may be said when A is weakly closed. In particular, Kaplansky [5] has shown that a derivation of an AW^* algebra of type I is necessarily inner. (It seems to be an open question whether or not this is true of weakly closed algebras of types II and III).

Our purpose is to state a weak sense in which every derivation of a B^* algebra is inner. This cannot be true in a strict sense, as is shown by the following typical example: Let A be all compact operators on some Hilbert space H, with an identity adjoined if desired. Then for any x in $\mathscr{B}(H)$, D_x is a derivation on A. If, for some y in $\mathscr{B}(H)$, $D_x = D_y$ on A, then D_{x-y} is zero on A, so $x - y$ commutes with all elements of A, and so $x - y$ is a scalar multiple of the identity e. Thus if x is chosen so that $x - \lambda e$ is not in A for any scalar λ (e.g., if x is a shift), D_x is an outer derivation on A. The reason for calling this example typical is made clear by the following theorem:

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THEOREM. *Let A be a B^* algebra, D a derivation on A. Then there exist a Hilbert space H, a faithful representation φ of A in $\mathscr{B}(H)$, and an operator S in the weak closure of $\varphi(A)$ such that*

$$\varphi(Dx) = D_S\varphi(x)$$

for all x in A.

As a sample consequence, we give two generalizations of Wielandt's result that if K is a self-adjoint element of $\mathscr{B}(H)$, there is no X in $\mathscr{B}(H)$ such that $KX - XK = iI$; we view this as saying that D_K does not take on the value iI.

COROLLARY. (i) (Generalized Putnam's Theorem) *If D is a self-adjoint derivation on a B^* algebra A, and if x is an element of A such that $D^2(x) = 0$, then $Dx = 0$.*

(ii) *If D is a derivation on the B^* algebra A, then $D(x)$ is not in the interior of the positive cone for any x in A.*

2. Proof of the theorem. The following fact is implicit in much of the literature on derivations.

PROPOSITION. Let A be a B^* algebra, D a derivation on A, I a closed, two-sided ideal in A. Then $D(I) \subseteqq I$, so D is a derivation on I. If $\varphi: A \to B$ is a $*$-homomorphism of A into a B^* algebra B, then the operator D_φ defined on $\varphi(A)$ by

$$D_\varphi(\varphi(x)) = \varphi(Dx)$$

is a derivation on $\varphi(A)$.

One sees this by noticing that any x in I may be written in the form

$$x = h_1^2 - h_2^2 + i(h_3^2 - h_4^2)$$

where the h_i are self-adjoint elements of I. The multiplicative rule for D and the fact that I is a two-sided ideal yield the result that Dx is in I. For φ as above, the kernel of φ is a closed, two-sided ideal, and so $\varphi(x) = 0$ implies $\varphi(Dx) = 0$. It follows that D_φ is well defined, and the obvious verifications show it a derivation.

The Gelfand-Naimark representation referred to in the following lemma is standard; it is described in some detail immediately following the proof of the lemma.

LEMMA 1. *Let A be a B^* algebra, D a derivation on A. Let $\tilde{A}$ be the weak closure of (the image of) A in the Gelfand-Naimark* *representation formed by using all states of* A. *Then there is a derivation* $\tilde{D}$ *on* $\tilde{A}$ *which agrees with* D *on* (*the image of*) A.

Proof. Since D is necessarily bounded, the transformation D^* defined on A^* by

$$(D^*f)(x) = f(Dx)$$

is a bounded transformation of A^* into itself. Likewise the transformation D^{**} defined on A^{**} by

$$(D^{**}\xi)(f) = \xi(D^*f)$$

is a bounded transformation of A^{**} into itself. But A^{**} can be identified with $\tilde{A}$ so that Arens multiplication on A^{**} corresponds to ordinary operator multiplication on $\tilde{A}$ (and so that the linear and norm structures of the two spaces coincide) [1, p. 869]. A straightforward verification via the definition of Arens multiplication shows that D^{**} is a derivation on A^{**}, which we identify with the derivation $\tilde{D}$ on $\tilde{A}$.

To fix notation, we review the construction of the Gelfand-Naimark representation of a B^* algebra A.

Given a state f on A, we form the left ideal

$$I_f = \{x \in A : f(x^*x) = 0\}$$

and the difference space

$$X_f = A \ominus I_f .$$

We denote by x_f the image of x in X_f. X_f has an inner product

$$(x_f, y_f) = f(y^*x)$$

and the completion of X_f under the norm induced by this inner product is a Hilbert space, denoted by H_f.

Given x in A, the operator $\varphi_f(x)$ defined on X_f by

$$\varphi_f(x)y_f = (xy)_f$$

is bounded, and so has a bounded extension to H_f, also denoted by $\varphi_f(x)$. To obtain the Gelfand-Naimark representation, we form the direct sum of the H_f, extended over all states f; this Hilbert space we call H. We think of its elements ξ as "sequences,"

$$\xi = \{\xi^f\}$$

where ξ^f is the component of ξ in H_f. The Gelfand-Naimark representation φ is then the direct sum of the φ_f:

$$\varphi(x)\{\xi^f\} = \{\varphi_f(x)\xi^f\} .$$

Given a pure state f_0 on A, let $\omega = \{\omega^f\}$ be the element of H defined by

$$\omega^f = \begin{cases} e_{f_0} & f = f_0 \\ 0 & f \neq f_0 . \end{cases}$$

Define the vector state f_ω on $\tilde{A}$ by

$$f_\omega(T) = (T\omega, \omega) .$$

As above, let $I_\omega = \{S \in \tilde{A} : f_\omega(S^*S) = 0\}$, let $X_\omega = \tilde{A} \ominus I_\omega$, let S_ω be the image of S in X_ω, and let H_ω be the completion of X_ω in the norm induced by f_ω.

LEMMA 2. *The map* $U: X_{f_0} \rightarrow X_\omega$ *defined by*

$$U(x_{f_0}) = x_\omega$$

is in fact an isometry of H_{f_0} *onto* H_ω (For simplicity, we have identified A with its image in $\tilde{A}$).

Proof. Throughout the proof we replace "f_0" by "0" in sub- and superscripts.

Identifying A with its image in $\tilde{A}$, we have $f_0 = f_\omega$ on A. Therefore

$$(U_{x_0}, U_{y_0}) = (x_\omega, y_\omega) = f_\omega(y^*x) = f_0(y^*x) = (x_0, y_0)$$

and U is an isometry on X_0.

But since f_0 is a pure state, $\varphi_0(A)$ acts irreducibly on H_0. It follows from the theorem of Kadison [4, Theorem 1] that irreducibility may be taken in a purely algebraic sense: thus, given any ξ in H_0, there is an x in A such that

$$\xi = \varphi_0(x)e_0 = x_0 .$$

Therefore, $X_0 = H_0$. Since H_0 is complete and U an isometry, UH_0 is complete, and so closed in H_ω. Thus any η in H_ω may be written uniquely in the form

$$\eta = \eta_1 + \eta_2 , \quad \eta_1 \varepsilon UH_0 , \quad \eta_2 \varepsilon (UH_0)^\perp .$$

If η is in X_ω then, since $\eta_1 \varepsilon UH_0 \subseteqq X_\omega$, η_2 is also in X_ω, and so there is some S in $\tilde{A}$ with $\eta_2 = S_\omega$. Since $\eta_2 \varepsilon (UH_0)^\perp$,

$$0 = (\eta_2, Ux_0) = (S_\omega, x_\omega) = f_\omega(x^*S) = (S\omega, x\omega)$$

for all x in A. On the other hand, since S is in $\tilde{A}$, we can find x in A making

$$|(S\omega, (x - S)\omega)|$$

arbitrarily small. It follows that $(S\omega, S\omega) = 0$, so $S \varepsilon I_\omega$, $S_\omega = 0$.

Thus $X_\omega \subseteqq UH_0$. Since X_ω is dense, and UH_0 closed, in H_ω, we have $UH_0 = H_\omega$.

LEMMA 3. $$\varphi_\omega(\tilde{A}) = \mathscr{B}(H_\omega)\,.$$

Proof. Evidently the map ψ: $\mathscr{B}(H_0) \to \mathscr{B}(H_\omega)$ given by $\psi(S) = USU^*$ is a $*$-isomorphism of $\mathscr{B}(H_0)$ onto $\mathscr{B}(H_\omega)$, bi-continuous with respect to the weak operator topologies. Thus

$$\begin{aligned}\psi(\text{weak closure } \varphi_0(A)) &= \text{weak closure } \psi(\varphi_0(A)) \\ &= \text{weak closure } \varphi_\omega(A)\,.\end{aligned}$$

Since $\varphi_0(A)$ acts irreducibly on H_0, weak closure $\varphi_0(A) = \mathscr{B}(H_0)$. On the other hand, f_ω is a vector state on $\tilde{A}$, and so normal [2, p. 54]. Consequently, $\varphi_\omega(\tilde{A})$ is a weakly closed subalgebra of $\mathscr{B}(H_\omega)$ [2, p. 57]. Thus

$$\text{weak closure } \varphi_\omega(A) \subseteqq \text{weak closure } \varphi_\omega(\tilde{A}) = \varphi_\omega(\tilde{A})\,.$$

$\mathscr{B}(H_\omega) = \psi(\text{weak closure } \varphi_0(A)) = \text{weak closure } \varphi_\omega(A) \subseteqq \varphi_\omega(\tilde{A})$.

We now get at the proof of the theorem. By Lemma 1, the derivation D on A extends to a derivation $\tilde{D}$ on $\tilde{A}$. Since φ_ω is a $*$-homomorphism, $\tilde{D}$ induces a derivation D_ω on $\varphi_\omega(\tilde{A})$ by

$$D_\omega(\varphi_\omega(T)) = \varphi_\omega(\tilde{D}(T))\,.$$

As we have just seen, $\varphi_\omega(\tilde{A})$ is very much a type I weakly closed algebra, so we may appeal to Kaplansky's result to find an S in $\mathscr{B}(H_\omega)$ such that

$$D_\omega(\varphi_\omega(T)) = [S, \varphi_\omega(T)]$$

for all T in $\tilde{A}$.

Consequently,

$$\begin{aligned}\varphi_0(Dx) = U^*\varphi_\omega(Dx)U &= U^*D_\omega(\varphi_\omega(x))U \\ &= (U^*SU)(U^*\varphi_\omega(x)U) - (U^*\varphi_\omega(x)U)(U^*SU)\,.\end{aligned}$$

Letting $S_0 = U^*SU$, we thus have

$$(*) \qquad \varphi_0(Dx) = S_0\varphi_0(x) - \varphi_0(x)S_0\,.$$

Assume for the moment that D is self-adjoint; it follows that

$$\varphi_0(D(x^*)) = -(\varphi_0(Dx))^*$$

and so

$$S_0\varphi_0(x)^* - \varphi_0(x)^*S_0 = S_0^*\varphi_0(x)^* - \varphi_0(x)^*S_0^*$$

for all x in A. In other words, $S_0 - S_0^*$ commutes with $\varphi_0(A)$, and so is a scalar multiple of the identity. Now altering S_0 by adding a scalar multiple of the identity does not affect any of the Lie products $[S_0, T]$. Consequently we may choose S_0 so as to satisfy $(*)$ and to be self-adjoint.

By further addition of a real scalar multiple of the identity, we may assure that the spectrum $\sigma(S_0)$ is centered at the origin. We assert that when this has been done, we have

$$\| S_0 \| \leqq \| \tilde{D} \| = \| D \| ,$$

the norm on the left being the norm in $\mathscr{B}(H_0)$ and the two on the right (whose equality is easily verified via the identification $\tilde{D} = D^{**}$) the norms $\tilde{D}$ and D have as operators on $\tilde{A}$ and A respectively.

For, given any $\varepsilon > 0$, the spectral theorem applied to the self-adjoint S_0 supplies us with vectors ξ and η in H_0 such that

$$\| \xi \| = \| \eta \| = 1 , \quad \xi \perp \eta$$
$$\left\| S_0\xi + \frac{1}{2} \| S_0 \| \xi \right\| < \varepsilon$$
$$\left\| S_0\eta - \frac{1}{2} \| S \| \eta \right\| < \varepsilon .$$

Since ξ and η are orthogonal, there is a unitary element of $\mathscr{B}(H_0)$ which interchanges them. Appealing again to Kadison's theorem [4, Theorem 1], we have a unitary v in A such that $\varphi_0(v)$ interchanges ξ and η.

We thus have

$$\left\| S_0\varphi_0(v)\xi - \frac{1}{2} \| S_0 \| \eta \right\| = \left\| S_0\eta - \frac{1}{2} \| S_0 \| \eta \right\| < \varepsilon$$
$$\left\| \varphi_0(v)S_0\xi + \frac{1}{2} \| S_0 \| \eta \right\| = \left\| \varphi_0(v)\Big(S_0\xi + \frac{1}{2} \| S_0 \| \xi\Big) \right\|$$
$$\leqq \| \varphi_0(v) \| \cdot \left\| S_0\xi + \frac{1}{2} \| S_0 \| \xi \right\| < \varepsilon .$$

Therefore

$$\left\| [S_0, \varphi_0(v)]\xi - \| S_0 \| \eta \right\| < 2\varepsilon$$

and so

$$\|[S_0, \varphi_0(v)]\xi\| \geqq \|S_0\| \cdot \|\eta\| - 2\varepsilon = \|S_0\| - 2\varepsilon .$$

On the other hand,

$$\|[S_0, \varphi_0(v)]\xi\| = \|\varphi_0(Dv)\xi\| \leqq \|\varphi_0\| \cdot \|D\| \cdot \|v\| \cdot \|\xi\| = \|D\| .$$

Combining these inequalities, we obtain $\|D\| \geqq \|S_0\| - 2\varepsilon$ for any positive ε, which proves our assertion.

To obtain the promised representation, let $\mathscr{F}$ be any family of pure states maximal with respect to the property that the representations induced by any two distinct members of $\mathscr{F}$ shall not be unitarily equivalent. Let H be the direct sum of the H_f, extended over all f in $\mathscr{F}$, and φ the direct sum of the φ_f, also extended over $\mathscr{F}$. Since the direct sum representation extended over all pure states is faithful, φ must also be faithful. By the argument just finished, there exists for each f in $\mathscr{F}$ an element S^f in $\mathscr{B}(H_f)$ satisfying

$$\varphi_f(Dx) = S^f\varphi_f(x) - \varphi_f(x)S^f , \quad \text{all } x\varepsilon A \quad \|S^f\| \leqq \|D\|.$$

Thus the operator S defined on H by

$$S\{\xi^f\} = \{S^f\xi^f\}$$

is in $\mathscr{B}(H)$, and indeed $\|S\| \leqq \|D\|$. It is at once verified that for any x in A,

$$\varphi(Dx) = [S, \varphi(x)] .$$

That S is in the weak closure of $\varphi(A)$ is a consequence of the fact [3, Cor. 4] that our choice of $\mathscr{F}$ causes the weak closure of $\varphi(A)$ to be the C^* direct sum $\Sigma \oplus (H_f)$ extended over $\mathscr{F}$.

We have been operating for some time under the assumption that D was self-adjoint. Since any derivation is a linear combination of self-adjoint ones, and since the representation φ did not depend on the derivation, it is clear that the theorem has in fact been proved for any derivation D.

The relation of $\|S\|$ and $\|D\|$ when D is arbitrary remains a loose end.

3. Proof of the corollary. (i) Given the self-adjoint derivation D on the B^* algebra A, we take a faithful representation φ of A in some $\mathscr{B}(H)$ and a self-adjoint S in $\mathscr{B}(H)$ such that

$$\varphi(Dx) = S\varphi(x) - \varphi(x)S$$

for all x in A. If $D^2(x) = 0$, then

$$0 = \varphi(D^2x) = \varphi(D(Dx)) = [S, [S, \varphi(x)]] .$$

We can now apply the well known theorem of Putnam to conclude that $[S, \varphi(x)] = 0$, and so that $Dx = 0$.

(ii) If D is self-adjoint and $D(x)$ is self-adjoint, then $x = ik$ for some self-adjoint k. Let φ, S, H be as above: We may also take $\varphi(e)$ to be the identity I on H. If $iD(k)$ is in the interior of the positive cone of A, then $iD(k) \geqq \delta e$ for some $\delta > 0$, and consequently $i\varphi(Dk) \geqq \delta I$.

Given any state f on $\mathscr{B}(H)$, let $f(S\varphi(k)) = \alpha + i\beta$. Then

$$f(\varphi(k)S) = \alpha - i\beta$$

Thus

$$if(\varphi(Dk)) = if([S, \varphi(k)]) = -2\beta \geqq \delta f(I) = \delta .$$

Consequently

$$f(\varphi(k)^2)f(S^2) \geqq |f(S\varphi(k)|^2 \geqq \alpha^2 + \beta^2 \geqq \delta^2/4 .$$

Thus $f(\varphi(k)^2)$ is not zero for any state f. Since all multiplicative functionals on the closed (commutative) algebra generated by $\varphi(k)$ and I extend to states of $\mathscr{B}(H)$, this implies $\varphi(k)$ regular.

Now for any scalar λ, $D(k + \lambda e) = D(k)$. We may therefore repeat the argument above with k replaced by $k + \lambda e$, coming to the conclusion that $k + \lambda e$ is regular for all scalars λ, an impossibility. Thus our original assumption was false, and (ii) is proved.

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