KLEENE QUOTIENT THEOREMS

J. D. McKnight, Jr.

0. Introduction. The collection of *regular subsets* of a semigroup A is the smallest collection of subsets of A having among its members the finite subsets of A, the collectionwise products EF and unions $E \cup F$ of any members E and F of it, and the subsemigroups E^* of A generated by each of its members E. For convenience, we set $\varnothing^* = \varnothing$ for the empty set $\varnothing$ although a semigroup is required to have at least one element. By a *quotient* of a semigroup A we shall mean, as usual, the set of inverse images of elements of a homomorphic image of A with multiplication so defined that this partition of A is isomorphic to the homomorphic image in the natural way.

A theorem of S. C. Kleene [3], first proved a dozen years ago, not only may be regarded as the fundamental theorem of the traditional theory of finite automata (compare C. C. Elgot [2] and Section 4 below), but it may be considered a contribution to the theory of quotients of certain classes of semigroups. Kleene's results are often summarized [2] by a statement equivalent to the following: *The regular subsets of a finitely generated free semigroup are the unions of subsets of finite quotients of the semigroup.* This result has two evident parts. The first we call the *Kleene Quotient Theorem*: *Each element of a finite quotient of a finitely generated free semigroup is a regular subset of the semigroup.* The second is one of several theorems we call the *Converse Theorems*: *Every regular subset of a finitely generated free semigroup is a union of elements of some finite quotient of the semigroup.* This result may be decomposed into two main parts (Section 3), each of which is also a Converse Theorem:

(a) *The collection wise product of unions of elements of finite quotients of a finitely generated free semigroup is a union of elements of a finite quotient of the semigroup.*

(b) *The subsemigroup generated by a union of elements of a finite quotient of a finitely generated free semigroup is a union of elements of a finite quotient of the semigroup.*

Our main purposes have been to remove the word *free* from the Kleene Quotient Theorem and to remove the adjective *finitely generated* from all the Converse Theorems. We have indeed been able to give some general inductive formulas (Section 1) which may themselves be regarded as a generalization of the Kleene Quotient Theorem, and we

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have given a more incisive proposition (Theorem 5) than the Converse Theorems. But our main contribution is one of method. For example, we avoid transformation semigroups completely in our proofs of the Converse Theorems, introducing instead a class of refinements of quotients. It would be possible to mimic Kleene's proofs [3] to prove our major propositions, but the tools we introduce below are much more transparent.

Section 3 is independent of Sections 1 and 2. It is also easier to read because there are no long formulas to check and it is more like conventional algebra.

That part of our terminology which is not simply the language of general mathematics is found mainly in [1]. Unfortunately, collectionwise quotients, which are used in almost every branch of algebra, are not discussed there, but we have no use for any deep properties of them.

1. Collectionwise quotients of quotient classes. An operation often associated with the collectionwise product is the *collectionwise quotient* $\mathfrak{a}:\mathfrak{b}$ defined for each pair of subsets $\mathfrak{a}$ and $\mathfrak{b}$ of a semigroup A. The set $\mathfrak{a}:\mathfrak{b}$ consists of all those elements x of A for which $x\mathfrak{b}\subset\mathfrak{a}$. It follows trivially that $(\mathfrak{a}:\mathfrak{b})\mathfrak{b}\subset\mathfrak{a}$; in fact, $\mathfrak{a}:\mathfrak{b}$ is the largest of the subsets $\mathfrak{c}$ of A for which $\mathfrak{c}\mathfrak{b}\subset\mathfrak{a}$. The most familiar uses of collectionwise quotients require that A be the multiplicative semigroup of a ring and that $\mathfrak{a}$ and $\mathfrak{b}$ be left ideals of the ring. Then $\mathfrak{a}:\mathfrak{b}$ is an ideal called the quotient ideal of $\mathfrak{a}$ by $\mathfrak{b}$. We shall use $\mathfrak{a}:\mathfrak{b}$ only when $\mathfrak{a}$ and $\mathfrak{b}$ are elements of a quotient $\mathscr{A}$ of the semigroup A. In this case, $\mathfrak{a}:\mathfrak{b}$ contains every element of $\mathscr{A}$ which intersects it nonvacuously; therefore, $\mathfrak{a}:\mathfrak{b}$ is the union of all those quotient classes $\mathfrak{c}\in\mathscr{A}$ for which $\mathfrak{c}\mathfrak{b}\subset\mathfrak{a}$.

If B is a subsemigroup of A and $\mathscr{B}$ is the collection of all nonempty subsets of B of the form $\mathfrak{a}\cap B$ for $\mathfrak{a}$ in a quotient $\mathscr{A}$ of A, then $\mathscr{B}$ is a quotient of B, the quotient *induced by* $\mathscr{A}$ *on* B. If $\mathfrak{a},\mathfrak{b}\in\mathscr{A}$, then $(\mathfrak{a}:\mathfrak{b})\cap B$ is the union of all those $\mathfrak{c}\in\mathscr{B}$ for which $\mathfrak{c}\mathfrak{b}\subset\mathfrak{a}$. If also $\mathfrak{b}\cap B\neq\varnothing$, then $(\mathfrak{a}:\mathfrak{b})\cap B=((\mathfrak{a}\cap B):(\mathfrak{b}\cap B))\cap B$.

If $\mathfrak{B}$ is a collection of subsemigroups of A and $\mathfrak{a}\subset A$, we shall denote by $(\mathfrak{a})_{\mathfrak{B}}$ the set $\mathfrak{a}\cap(\bigcup\mathfrak{B})$. If $\mathfrak{a}\in\mathscr{A}$, where $\mathscr{A}$ is a quotient of A, then $(\mathfrak{a})_{\mathfrak{B}}$ is the union of the induced quotient classes $\mathfrak{a}\cap B$ for $B\in\mathfrak{B}$ and $\mathfrak{a}\cap B\neq\varnothing$. If $\mathfrak{a},\mathfrak{b}\in\mathscr{A}$, then $(\mathfrak{a}:\mathfrak{b})_{\mathfrak{B}}=\bigcup\{(\mathfrak{a}:\mathfrak{b})\cap B\colon B\in\mathfrak{B}\}$, which is the union of all the induced quotient classes $\mathfrak{c}$ in the quotients induced by $\mathscr{A}$ on elements of $\mathfrak{B}$ for which $\mathfrak{c}\mathfrak{b}\subset\mathfrak{a}$. In particular, if A is generated by $\mathfrak{B}$ (i.e., if A is the smallest subsemigroup of A containing every element of $\mathfrak{B}$), then for each $\mathfrak{a}\in\mathscr{A}$ we have $\mathfrak{a}=(\mathfrak{a})_{\mathfrak{B}}\cup\bigcup\{(\mathfrak{a}:\mathfrak{b})_{\mathfrak{B}}\mathfrak{b}:\mathfrak{b}\in\mathscr{A}\}$, for each element of $\mathfrak{a}$ either is an element of some element of $\mathfrak{B}$ or has a left factor which is an element of some element of $\mathfrak{B}$. More explicitly, if $x \in \mathfrak{a}$ and $x \notin (\mathfrak{a})_{\mathfrak{B}}$, then $x = yz$, where $y \in B$ for some $B \in \mathfrak{B}$. But then $z \in \mathfrak{b}$ for some $\mathfrak{b} \in \mathscr{A}$, so that $x \in (\mathfrak{a}:\mathfrak{b})_{\mathfrak{B}}\mathfrak{b}$. This observation may be strengthened by factoring each element of $\mathfrak{a}$ completely into factors lying in the elements of $\mathfrak{B}$. Thus, if $\mathfrak{B}$ generates A and $\mathfrak{a}$ is an element of the quotient $\mathscr{A}$ of A, then $\mathfrak{a}$ is the union of $(\mathfrak{a})_{\mathfrak{B}}$ and all those products of the form $(\mathfrak{a} : \mathfrak{a}_1)_{\mathfrak{B}}(\mathfrak{a}_1 : \mathfrak{a}_2)_{\mathfrak{B}} \cdots (\mathfrak{a}_{n-1} : \mathfrak{a}_n)_{\mathfrak{B}}(\mathfrak{a}_n)_{\mathfrak{B}}$, where $\mathfrak{a}_1, \cdots, \mathfrak{a}_n \in \mathscr{A}$. A factor $(\mathfrak{b} : \mathfrak{c})_{\mathfrak{B}}$ or $(\mathfrak{b})_{\mathfrak{B}}$ of some of these products may, of course, be the empty set, but then any such product is empty.

To simplify our computations we introduce the set $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b})$ for each $\mathfrak{a} \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{B} \subset \mathscr{A}$. We define $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b})$ to be the union of $(\mathfrak{a} : \mathfrak{b})_{\mathfrak{B}}$ and all sets of the form $(\mathfrak{a} : \mathfrak{a}_1)_{\mathfrak{B}}(\mathfrak{a}_1 : \mathfrak{a}_2)_{\mathfrak{B}} \cdots (\mathfrak{a}_n : \mathfrak{b})_{\mathfrak{B}}$, where $\mathfrak{a}_1, \cdots, \mathfrak{a}_n \in \mathscr{B}$. If $\mathfrak{B}$ generates A, then for any element $\mathfrak{a}$ of the quotient $\mathscr{A}$ we have

$$(1) \qquad \mathfrak{a} = (\mathfrak{a})_{\mathfrak{B}} \cup \bigcup \{\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{A}, \mathfrak{b})(\mathfrak{b})_{\mathfrak{B}} : \mathfrak{b} \in \mathscr{A}\} .$$

We intend to write $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b})$ in a convenient form for later use. To this end we introduce the sets $\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b})$ for $\mathfrak{a} \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{B} \subset \mathscr{A}$. We define $\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b})$ to be the union of all those sets of the form $(\mathfrak{a} : \mathfrak{a}_1)_{\mathfrak{B}}(\mathfrak{a}_1 : \mathfrak{a}_2)_{\mathfrak{B}} \cdots (\mathfrak{a}_n : \mathfrak{b})_{\mathfrak{B}}$ for which $\mathfrak{a}_1 \neq \mathfrak{a}$ and $\mathfrak{a}_1, \cdots, \mathfrak{a}_n \in \mathscr{B}$, where we count $(\mathfrak{a} : \mathfrak{b})_{\mathfrak{B}}$ as one of these products if $\mathfrak{a} \neq \mathfrak{b}$. We then have, if $\mathfrak{a} \in \mathscr{B}$, the formulas

$$\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{a}) = (\mathfrak{a} : \mathfrak{a})^*_{\mathfrak{B}} \cup \mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{a}) \cup (\mathfrak{a} : \mathfrak{a})^*_{\mathfrak{B}}\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{a})$$

(2) and

$$\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b}) = \mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b}) \cup (\mathfrak{a} : \mathfrak{a})^*_{\mathfrak{B}}\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b}) \text{ for } \mathfrak{a} \neq \mathfrak{b} .$$

The first of these formulas follows from the fact that the sets $(\mathfrak{a} : \mathfrak{a}_1)_{\mathfrak{B}}(\mathfrak{a}_1 : \mathfrak{a}_2)_{\mathfrak{B}} \cdots (\mathfrak{a}_n : \mathfrak{a})_{\mathfrak{B}}$ may be classified as those for which each $\mathfrak{a}_k$ is equal to $\mathfrak{a}$, those for which $\mathfrak{a}_1 \neq \mathfrak{a}$, and those for which $\mathfrak{a}_1 = \mathfrak{a}$ with some other $\mathfrak{a}_k \neq \mathfrak{a}$. The second follows from the fact that for $\mathfrak{b} \neq \mathfrak{a}$ the sets $(\mathfrak{a} : \mathfrak{a}_1)_{\mathfrak{B}}(\mathfrak{a}_1 : \mathfrak{a}_2)_{\mathfrak{B}} \cdots (\mathfrak{a}_n : \mathfrak{b})_{\mathfrak{B}}$ and $(\mathfrak{a} : \mathfrak{b})_{\mathfrak{B}}$, for $\mathfrak{a} \neq \mathfrak{b}$, may be classified as those, including $(\mathfrak{a} : \mathfrak{b})_{\mathfrak{B}}$, for which $\mathfrak{a}_1 \neq \mathfrak{a}$ and those for which $\mathfrak{a}_1 = \mathfrak{a}$.

We also have, if $\mathfrak{a} \in \mathscr{B}$, the formulas

$$\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{a}) = [\bigcup\{\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}\backslash\{\mathfrak{a}\}, \mathfrak{c})[(\mathfrak{c} : \mathfrak{a})_{\mathfrak{B}} \cup (\mathfrak{c} : \mathfrak{a})_{\mathfrak{B}}(\mathfrak{a} : \mathfrak{a})^*_{\mathfrak{B}}] : \mathfrak{c} \in \mathscr{B}\backslash\{\mathfrak{a}\}\}]^*$$

(3) and

$$\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b}) = \mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}\backslash\{\mathfrak{a}\}, \mathfrak{b}) \cup \mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{a})\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}\backslash\{\mathfrak{a}\}, \mathfrak{b}) \text{ for } \mathfrak{a} \neq \mathfrak{b} .$$

To show that the first of these equations is valid, we observe that its right-hand member is the union of all sets of the form

$\mathfrak{q}_1\mathfrak{q}_2\cdots\mathfrak{q}_{2n}$, where each $\mathfrak{q}_{2k-1}$ has the form $(\mathfrak{a}:\mathfrak{c})_{\mathfrak{B}}$ or $(\mathfrak{a}:\mathfrak{a}_1)_{\mathfrak{B}}(\mathfrak{a}_1:\mathfrak{a}_2)_{\mathfrak{B}}\cdots(\mathfrak{a}_m:\mathfrak{c})_{\mathfrak{B}}$ for some $\mathfrak{c}, \mathfrak{a}_1, \cdots, \mathfrak{a}_m \in \mathscr{B}\backslash\{\mathfrak{a}\}$ and $\mathfrak{q}_{2k}$ is either $(\mathfrak{c}:\mathfrak{a})_{\mathfrak{B}}$ or $(\mathfrak{c}:\mathfrak{a})_{\mathfrak{B}}(\mathfrak{a}:\mathfrak{a})^p_{\mathfrak{B}}$ for some p. Because of the forms of $\mathfrak{q}_1, \cdots, \mathfrak{q}_{2n}$, each such $\mathfrak{q}_1\cdots\mathfrak{q}_{2n}$ is contained in $\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{a})$ according to its definition. Conversely, if $(\mathfrak{a}:\mathfrak{a}_1)_{\mathfrak{B}}\cdots(\mathfrak{a}_q:\mathfrak{a})_{\mathfrak{B}}$ is one of the sets which, according to our definition, makes up $\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{a})$, then $\mathfrak{a}_1 \neq \mathfrak{a}$. Either $\mathfrak{a}_k \neq \mathfrak{a}$ for each k, in which case $(\mathfrak{a}:\mathfrak{a}_1)_{\mathfrak{B}}\cdots(\mathfrak{a}_q:\mathfrak{a})_{\mathfrak{B}}$ is of the form $\mathfrak{q}_1\mathfrak{q}_2$ above (i.e., $(\mathfrak{a}:\mathfrak{a}_1)_{\mathfrak{B}}\cdots(\mathfrak{a}_q:\mathfrak{a})_{\mathfrak{B}} \subset \mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}\backslash\{\mathfrak{a}\}, \mathfrak{a}_q)(\mathfrak{a}_q:\mathfrak{a})_{\mathfrak{B}}$) or there is a smallest k for which $\mathfrak{a}_{k+1} = \mathfrak{a}$. In the latter case, there is some $p \geqq 1$ for which $\mathfrak{a} = \mathfrak{a}_{k+1} = \cdots = \mathfrak{a}_{k+p}$ and for which either $k+p = q$ or $\mathfrak{a}_{k+p+1} \neq \mathfrak{a}$. If $k+p=q$, then again $(\mathfrak{a}:\mathfrak{a}_1)_{\mathfrak{B}}\cdots(\mathfrak{a}_q:\mathfrak{a})_{\mathfrak{B}}$ is of the form $\mathfrak{q}_1\mathfrak{q}_2$ above (i.e., $(\mathfrak{a}:\mathfrak{a}_1)_{\mathfrak{B}}\cdots(\mathfrak{a}_q:\mathfrak{a})_{\mathfrak{B}} \subset \mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}\backslash\{\mathfrak{a}\}, \mathfrak{a}_k)(\mathfrak{a}_k:\mathfrak{a})_{\mathfrak{B}}(\mathfrak{a}:\mathfrak{a})^p_{\mathfrak{B}}$). If $k+p<q$, then $(\mathfrak{a}:\mathfrak{a}_1)_{\mathfrak{B}}\cdots(\mathfrak{a}_q:\mathfrak{a})_{\mathfrak{B}}$ is of the form $\mathfrak{q}_1\mathfrak{q}_2(\mathfrak{a}:\mathfrak{a}_{k+p+1})_{\mathfrak{B}}\cdots(\mathfrak{a}_q:\mathfrak{a})_{\mathfrak{B}}$, where $\mathfrak{q}_1$ and $\mathfrak{q}_2$ have the form specified above. The shorter expression $(\mathfrak{a}:\mathfrak{a}_{k+p+1})_{\mathfrak{B}}\cdots(\mathfrak{a}_q:\mathfrak{a})_{\mathfrak{B}}$ may be treated exactly as $(\mathfrak{a}:\mathfrak{a}_1)_{\mathfrak{B}}\cdots(\mathfrak{a}_q:\mathfrak{a})_{\mathfrak{B}}$ to produce factors $\mathfrak{q}_3$ and $\mathfrak{q}_4$, and by repetition of the above procedure we finally have, for some n, $(\mathfrak{a}:\mathfrak{a}_1)_{\mathfrak{B}}\cdots(\mathfrak{a}_q:\mathfrak{a})_{\mathfrak{B}} = \mathfrak{q}_1\mathfrak{q}_2\cdots\mathfrak{q}_{2n}$, where the $\mathfrak{q}_k$ are as specified above. Therefore, $\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{a})$ is contained in set expressed in the right-hand member of the first equation (3).

The second formula of (3) is more easily verified. If $\mathfrak{a} \neq \mathfrak{b} \in \mathscr{B}$, the sets $(\mathfrak{a}:\mathfrak{b})_{\mathfrak{B}}$ and $(\mathfrak{a}:\mathfrak{a}_1)_{\mathfrak{B}}(\mathfrak{a}_1:\mathfrak{a}_2)_{\mathfrak{B}}\cdots(\mathfrak{a}_n:\mathfrak{b})_{\mathfrak{B}}$ for which $\mathfrak{a}_1 \neq \mathfrak{a}$ and $\mathfrak{a}_1, \cdots, \mathfrak{a}_n \in \mathscr{B}$ may be classified as those, including $(\mathfrak{a}:\mathfrak{b})_{\mathfrak{B}}$, for which $\mathfrak{a}_k \neq \mathfrak{a}$ for every k and those for which there is a greatest k satisfying $\mathfrak{a}_k = \mathfrak{a}$. The union of those of the former class is simply $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}\backslash\{\mathfrak{a}\}, \mathfrak{b})$; the union of those of the latter is $\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{a})\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}\backslash\{\mathfrak{a}\}, \mathfrak{b})$.

The pairs (2) and (3) of formulas apply only to the case $\mathfrak{a} \in \mathscr{B}$. For $\mathfrak{a} \notin \mathscr{B}$ we have the obvious relationships

$$(4)\quad \mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b}) = \mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b}) = (\mathfrak{a}:\mathfrak{b})_{\mathfrak{B}} \cup \bigcup \{(\mathfrak{a}:\mathfrak{c})_{\mathfrak{B}}\mathfrak{r}_{\mathfrak{B}}(\mathfrak{c}, \mathscr{B}, \mathfrak{b}) : \mathfrak{c} \in \mathscr{B}\}\,.$$

The formulas (1)–(4) are valid quite generally. Our reasons for deriving them are apparent only when the quotient $\mathscr{A}$, the subset $\mathscr{B}$ of $\mathscr{A}$, and the collection $\mathfrak{B}$ of subsemigroups of A suffer special joint conditions. Our first lemma, its corollary (Theorem 1), and the results of the next section make our reasons clear.

LEMMA 1. *If $\mathscr{A}$ is a quotient of the semigroup A, if $\mathscr{B}$ is a finite subset of $\mathscr{A}$, if $\mathfrak{B}$ is a collection of subsemigroups of A, and if $(\mathfrak{a}:\mathfrak{b})_{\mathfrak{B}}$ is a regular subset of A for each $\mathfrak{a}, \mathfrak{b} \in \mathscr{B}$, then $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b})$ is a regular subset of A for each $\mathfrak{a}, \mathfrak{b} \in \mathscr{B}$. If, moreover, a quotient class $\mathfrak{a} \in \mathscr{A}$ has the property that $(\mathfrak{a}:\mathfrak{b})_{\mathfrak{B}}$ is a regular subset of A for each $\mathfrak{b} \in \mathscr{B}$, then $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b})$ is a regular subset of A for each $\mathfrak{b} \in \mathscr{B}$.*

Proof. The second statement follows easily from the first and formula (4). We prove the first statement by induction on the number of elements of $\mathscr{B}$. The first of formulas (2) reduces to $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \{\mathfrak{a}\}, \mathfrak{a}) = (\mathfrak{a} : \mathfrak{a})^*_{\mathfrak{B}}$ for $\mathscr{B} = \{\mathfrak{a}\}$. If $\mathfrak{a} \in \mathscr{B}$, then $\mathscr{B}\backslash\{\mathfrak{a}\}$ has fewer elements than has $\mathscr{B}$, and, according to formulas (3), $\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b})$ is regular if each set $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}\backslash\{\mathfrak{a}\}, \mathfrak{c})$ is regular for $\mathfrak{c} \in \mathscr{B}\backslash\{\mathfrak{a}\}$. Writing $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}\backslash\{\mathfrak{a}\}, \mathfrak{c})$ in the form prescribed by (4) and invoking the induction hypothesis, we conclude that $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}\backslash\{\mathfrak{a}\}, \mathfrak{c})$ is regular; hence, $\mathfrak{p}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b})$ is regular. According to formulas (2), therefore, $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{B}, \mathfrak{b})$ is regular.

THEOREM 1. *If $\mathscr{A}$ is a finite quotient of the semigroup A, if $\mathfrak{B}$ is a collection of subsemigroups of A generating A, and if $\mathfrak{a} \cap (\bigcup \mathfrak{B})$ is a regular subset of A for each $\mathfrak{a} \in \mathscr{A}$, then each element of $\mathscr{A}$ is a regular subset of A.*

Proof. According to Lemma 1, $\mathfrak{r}_{\mathfrak{B}}(\mathfrak{a}, \mathscr{A}, \mathfrak{b})$ is regular for $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}$, Formula (1) then leads to our conclusion. Here, and in Lemma 2 below, the referee has saved the author from the error of adjoining an "obviously redundant" additional hypothesis that such sets as $(\mathfrak{a} : \mathfrak{b}) \cap (\bigcup \mathfrak{B})$ be regular.

2. Quotients of finitely generated semigroups. The last section has provided us with the main tools for carrying out a proof that the elements of a finite quotient of a finitely generated semigroup are regular subsets of the semigroup. After one more bit of preparation, we shall use an argument involving the number of generators of the semigroup.

LEMMA 2. *If $\mathscr{A}$ is a finite quotient of the finitely generated semigroup A, if $\mathfrak{B}$ is a collection of subsemigroups of A generating A, and if $\mathfrak{a} \cap B$ is a regular subset of A for each $\mathfrak{a} \in \mathscr{A}$ and $B \in \mathfrak{B}$, then each element of $\mathscr{A}$ is a regular subset of A.*

Proof. Each element of a finite set of generators of A is in the subsemigroup of A generated by some finite subcollection of $\mathfrak{B}$. Therefore, there is a finite subset $\mathfrak{B}'$ of $\mathfrak{B}$ generating A. Since $\mathfrak{a} \cap (\bigcup \mathfrak{B}') = \bigcup \{\mathfrak{a} \cap B : B \in \mathfrak{B}'\}$ the set $\mathfrak{a} \cap (\bigcup \mathfrak{B}')$ is regular for $\mathfrak{a} \in \mathscr{A}$. The conclusion then follows from Theorem 1.

LEMMA 3. *If $\mathscr{A}$ is a quotient of the semigroup A and if A is generated by one element, then every element of $\mathscr{A}$ is a regular subset of A.*

Proof. A is either free or finite. That is, every semigroup with a single generator is a homomorphic image of the additive semigroup of positive integers N. If A is finite, the proposition is trivial. If A is free, it is isomorphic to N. Since the form of quotient classes of N is well known, we may regard the proposition as well known for the free case.

THEOREM 2. *If $\mathscr{A}$ is a finite quotient of the finitely generated semigroup A, then every element of $\mathscr{A}$ is a regular subset of A.*

Proof. If A has several generators, it is generated by a finite collection $\mathfrak{B}$ of subsemigroups, each of which has fewer generators than A. For example, $\mathfrak{B}$ may be the cellection of subsemigroups of A generated by the single elements of a finite set of generators of A. The result then follows from Lemmas 2 and 3.

COROLLARY. *Every element of any restriction of a finite quotient of a semigroup A to a finitely generated subsemigroup of A is a regular subset of A.*

3. **Finite quotients of free semigroups.** We shall use some special refinements of quotients to study regular subsets of free semigroups. We shall first define them as refinements in the set-theoretic sense, proving afterward that they are quotients if the semigroups are appropriate.

The *joint refinement* of quotients $\mathscr{A}$ and $\mathscr{B}$ of A, written $\mathscr{A} \wedge \mathscr{B}$, is the coarsest covering of A which is a refinement of both $\mathscr{A}$ and $\mathscr{B}$; i.e., $\mathscr{A} \wedge \mathscr{B}$ consists of all nonempty sets of the form $\mathfrak{a} \cap \mathfrak{b}$, where $\mathfrak{a} \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{B}$. The *composite refinement* of the quotient $\mathscr{A}$ of A, written $\pi\mathscr{A}$, is the coarsest covering of A by disjoint subsets which refines $\mathscr{A}$ and which is such that each $\mathfrak{a}\mathfrak{b}$ is a union of its elements for each $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}$; i.e., if $x, y \in A$, then x and y are in the same element of $\pi\mathscr{A}$ if and only if they belong to the same element of $\mathscr{A}$ and the same products $\mathfrak{a}\mathfrak{b}$ of elements $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathscr{A}$. More generally, if $\mathscr{B}$ is a subset of the quotient $\mathscr{A}$ of A, we define the *composite refinement* of $\mathscr{A}$ *modulo* $\mathscr{B}$, written $\pi_{\mathscr{B}}\mathscr{A}$: Elements x and y of A are in the same element of $\pi_{\mathscr{B}}\mathscr{A}$ if and only if they are in the same element of $\mathscr{A}$ and the same products $\mathfrak{a}\mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}$, and, for $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}$, x is in $\mathfrak{a}\mathfrak{b}$ or $\mathfrak{a}\mathfrak{c}_1 \cdots \mathfrak{c}_m\mathfrak{b}$ for some $\mathfrak{c}_1, \cdots, \mathfrak{c}_m \in \mathscr{B}$ if and only if y is in $\mathfrak{a}\mathfrak{b}$ or $\mathfrak{a}\mathfrak{d}_1 \cdots \mathfrak{d}_n\mathfrak{b}$ for some $\mathfrak{d}_1, \cdots, \mathfrak{d}_n \in \mathscr{B}$. Obviously, $\pi = \pi_{\varnothing}$.

We regard the next result as well known and easily proved.

LEMMA 4. *If $\mathscr{A}$ and $\mathscr{B}$ are quotients of a semigroup A, the joint refinement $\mathscr{A} \wedge \mathscr{B}$ is a quotient of A.*

LEMMA 5. *If $\mathscr{B}$ is a subset of the quotient $\mathscr{A}$ of a free semigroup A, the composite refinement of $\mathscr{A}$ modulo $\mathscr{B}$ is a quotient of A.*

Proof. We suppose that x and y are in the same element of $\pi_{\mathscr{B}}\mathscr{A}$ and that x' and y' are in the same element of $\pi_{\mathscr{B}}\mathscr{A}$. Since xx' and yy' are obviously in the same element of $\mathscr{A}$, we only have to show that $xx' \in \mathfrak{a}\mathfrak{b}$ implies $yy' \in \mathfrak{a}\mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}$ and that $xx' \in \mathfrak{a}\mathfrak{c}_1 \cdots \mathfrak{c}_m\mathfrak{b}$ for some $\mathfrak{c}_1, \cdots, \mathfrak{c}_m \in \mathscr{B}$ implies $yy' \in \mathfrak{a}\mathfrak{b}$ or $yy' \in \mathfrak{a}\mathfrak{d}_1 \cdots \mathfrak{d}_n\mathfrak{b}$ for some $\mathfrak{d}_1, \cdots, \mathfrak{d}_n \in \mathscr{B}$, where $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}$,

If $xx' \in \mathfrak{a}\mathfrak{b}$, then $xx' = ab$ for some $a \in \mathfrak{a}$, $b \in \mathfrak{b}$. If $x = a$ and $x' = b$, then $y \in \mathfrak{a}$ and $y' \in \mathfrak{b}$ so that $yy' \in \mathfrak{a}\mathfrak{b}$. If $x \neq a$, then, because A is free, either x is a left factor of a or x' is a right factor of b. In the former case we have $a = xa_1$ and $x' = a_1b$. There are elements $\mathfrak{a}_0, \mathfrak{a}_1 \in \mathscr{A}$ for which $x \in \mathfrak{a}_0$ and $a_1 \in \mathfrak{a}_1$. Then $\mathfrak{a}_0\mathfrak{a}_1 \subset \mathfrak{a}$ and $y' \in \mathfrak{a}_1\mathfrak{b}$. Since $y \in \mathfrak{a}_0$, $yy' \in \mathfrak{a}_0\mathfrak{a}_1\mathfrak{b} \subset \mathfrak{a}\mathfrak{b}$. The latter case is analogous.

If $xx' \in \mathfrak{a}\mathfrak{c}_1 \cdots \mathfrak{c}_m\mathfrak{b}$, then $xx' = ac_1 \cdots c_mb$, where $a \in \mathfrak{a}$, $c_k \in \mathfrak{c}_k$ and $b \in \mathfrak{b}$. There are six mutually exclusive cases to consider:

(1) $a = xa_1$,

(2) $a = x$,

(3) $b = b_1x'$,

(4) $b = x'$,

(5) $x = ac_1 \cdots c_k$ for some $k < m$, and

(6) $x = ac_1 \cdots c_{k-1}c'_k$ for some k, $1 \leqq k \leqq m$, where $c_k = c'_kc''_k$. In case (1) we have $x \in \mathfrak{a}_0$, $a_1 \in \mathfrak{a}_1$ for some $\mathfrak{a}_0, \mathfrak{a}_1 \in \mathscr{A}$, with $\mathfrak{a}_0\mathfrak{a}_1 \subset \mathfrak{a}$ and $x' \in \mathfrak{a}_1\mathfrak{c}_1 \cdots \mathfrak{c}_m\mathfrak{b}$. Then either $y' \in \mathfrak{a}_1\mathfrak{b}$ or there exist $\mathfrak{d}_1, \cdots, \mathfrak{d}_n \in \mathscr{B}$ with $y' \in \mathfrak{a}_1\mathfrak{d}_1 \cdots \mathfrak{d}_n\mathfrak{b}$, so that $yy' \in \mathfrak{a}_0\mathfrak{a}_1\mathfrak{b} \subset \mathfrak{a}\mathfrak{b}$ or $yy' \in \mathfrak{a}_0\mathfrak{a}_1\mathfrak{d}_1 \cdots \mathfrak{d}_n\mathfrak{b} \subset \mathfrak{a}\mathfrak{d}_1 \cdots \mathfrak{d}_n\mathfrak{b}$. In case (2) we have $x \in \mathfrak{a}$ and $x' \in \mathfrak{c}_1 \cdots \mathfrak{c}_m\mathfrak{b}$. Then either $y' \in \mathfrak{c}_1\mathfrak{b}$ or $y' \in \mathfrak{c}_1\mathfrak{d}_1 \cdots \mathfrak{d}_n\mathfrak{b}$ for $\mathfrak{d}_1, \cdots, \mathfrak{d}_n \in \mathscr{B}$, so that $yy' \in \mathfrak{a}\mathfrak{c}_1\mathfrak{b}$ or $yy' \in \mathfrak{a}\mathfrak{c}_1\mathfrak{d}_1 \cdots \mathfrak{d}_n\mathfrak{b}$. Our condition is fulfilled since $\mathfrak{c}_1 \in \mathscr{B}$. Cases (3) and (4) are analogous to cases (1) and (2), respectively. In case (5) we can conclude that $y \in \mathfrak{a}\mathfrak{c}_k$ or $y \in \mathfrak{a}\mathfrak{d}_1 \cdots \mathfrak{d}_n\mathfrak{c}_k$ for some $\mathfrak{d}_1, \cdots, \mathfrak{d}_n \in \mathscr{B}$ and that $y' \in \mathfrak{c}_{k+1}\mathfrak{b}$ or $y' \in \mathfrak{c}_{k+1}\mathfrak{e}_1 \cdots \mathfrak{e}_p\mathfrak{b}$ for some $\mathfrak{e}_1, \cdots, \mathfrak{e}_p \in \mathscr{B}$, and the desired conclusion follows easily since $\mathfrak{c}_k, \mathfrak{c}_{k+1} \in \mathscr{B}$. In case (6) $y \in \mathfrak{a}\mathfrak{c}'_k$ or $y \in \mathfrak{a}\mathfrak{d}_1 \cdots \mathfrak{d}_n\mathfrak{c}'_k$ for some $\mathfrak{d}_1, \cdots, \mathfrak{d}_n \in \mathscr{B}$, where $c'_k \in \mathfrak{c}'_k$, and $y' \in \mathfrak{c}''_k\mathfrak{b}$ or $y' \in \mathfrak{c}''_k\mathfrak{e}_1 \cdots \mathfrak{e}_p\mathfrak{b}$ for some $\mathfrak{e}_1, \cdots, \mathfrak{e}_p \in \mathscr{B}$, where $c''_k \in \mathfrak{c}''_k$. The desired conclusion follows easily since $\mathfrak{c}'_k\mathfrak{c}''_k \subset \mathfrak{c}_k \in \mathscr{B}$.

COROLLARY. *If $\mathscr{A}$ is a quotient of a free semigroup A, the composite refinement of $\mathscr{A}$ is a quotient of A.*

THEOREM 3. *If $\mathscr{A}$ and $\mathscr{B}$ are finite quotients of the free semigroup A, then there is a finite quotient $\mathscr{C}$ of A so that if $E = \bigcup \mathscr{E}$ and $F = \bigcup \mathscr{F}$, where $\mathscr{E} \subset \mathscr{A}$, $\mathscr{F} \subset \mathscr{B}$, there is some $\mathscr{G} \subset \mathscr{C}$ so that $EF = \bigcup \mathscr{G}$.*

Proof. There exists $\mathscr{E}', \mathscr{F}' \subset \mathscr{A} \wedge \mathscr{B}$ for which $E = \bigcup \mathscr{E}'$ and $F = \bigcup \mathscr{F}'$. If $\mathscr{A}$ and $\mathscr{B}$ are finite, so is $\mathscr{A} \wedge \mathscr{B}$. Therefore, it is enough to prove the theorem for $\mathscr{A} = \mathscr{B}$, so that $\mathscr{A} = \mathscr{A} \wedge \mathscr{B}$ and the notation is simplified. Then if $\mathfrak{a} \in \mathscr{E}$ and $\mathfrak{b} \in \mathscr{F}$, $\mathfrak{a}\mathfrak{b}$ is a union of elements of $\pi\mathscr{A}$, as is any product of quotient classes, and EF is the union of all the products $\mathfrak{a}\mathfrak{b}$ for $\mathfrak{a} \in \mathscr{E}, \mathfrak{b} \in \mathscr{F}$. If $\mathscr{A}$ is finite, so is $\pi\mathscr{A}$ because there are then finitely many sets of products $\mathfrak{a}\mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}$.

THEOREM 4. *If $\mathscr{A}$ is a finite quotient of the free semigroup A and $E = \bigcup \mathscr{E}$ for some $\mathscr{E} \subset \mathscr{A}$, then there is a finite quotient $\mathscr{C}$ of A and some $\mathscr{G} \subset \mathscr{C}$ so that $E^* = \bigcup \mathscr{G}$.*

Proof. For a given subset $\mathscr{E}$ of $\mathscr{A}$, the elements of $\pi_{\mathscr{E}}\mathscr{A}$ contained in some element of $\mathscr{A}$ are determined by quadruples of sets of pairs $(\mathfrak{a}, \mathfrak{b})$ of elements of $\mathscr{A}$. That is, the element of $\pi_{\mathscr{E}}\mathscr{A}$ with element x is the subset of the element of $\mathscr{A}$ with element x consisting of all those elements y of that element of $\mathscr{A}$ satisfying:

(1) y is in the intersection of all those products $\mathfrak{a}\mathfrak{b}$, $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}$, for which $x \in \mathfrak{a}\mathfrak{b}$, (1′)$y$ is in the intersection of the complements of all those products $\mathfrak{a}\mathfrak{b}$, $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}$, for which $x \notin \mathfrak{a}\mathfrak{b}$,

(2) if $x \notin \mathfrak{a}\mathfrak{b}$ for some $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}$, but x is in the union of those sets of the form $\mathfrak{a}\mathfrak{c}_1, \cdots, \mathfrak{c}_m\mathfrak{b}$ for $\mathfrak{c}_1, \cdots, \mathfrak{c}_m \in \mathscr{E}$, then y is in the same union, and (2′) if $x \notin \mathfrak{a}\mathfrak{b}$ for some $\mathfrak{a}, \mathfrak{b} \in \mathscr{A}$ and x is not in the union of those sets of the form $\mathfrak{a}\mathfrak{c}_1, \cdots, \mathfrak{c}_m\mathfrak{b}$ for $\mathfrak{c}_1, \cdots, \mathfrak{c}_m \in \mathscr{E}$, then y is not in that union. Therefore, if $\mathscr{A}$ is finite, so is $\pi_{\mathscr{E}}\mathscr{A}$.

If $x \in E^*$, then $x \in E^k$ for some $k \geqq 1$. If $k = 1$, then $x \in \mathfrak{a}$ for some $\mathfrak{a} \in \mathscr{E}$. Then the whole element of $\pi_{\mathscr{E}}\mathscr{A}$ of which x is an element is contained in $\mathfrak{a} \subset E \subset E^*$. If $k = 2$, then $x \in \mathfrak{a}\mathfrak{b}$ for some $\mathfrak{a}, \mathfrak{b} \in \mathscr{E}$. Then the element of $\pi_{\mathscr{E}}\mathscr{A}$ of which x is an element is contained in $\mathfrak{a}\mathfrak{b} \subset E^2 \subset E^*$. If $k > 2$, then $x \in \mathfrak{a}\mathfrak{c}_1 \cdots \mathfrak{c}_{k-2}\mathfrak{b}$ for some $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}_1, \cdots, \mathfrak{c}_{k-2} \in \mathscr{E}$. If y is an element of the element of $\pi_{\mathscr{E}}\mathscr{A}$ of which x is an element, then $y \in \mathfrak{a}\mathfrak{b}$ or $y \in \mathfrak{a}\mathfrak{d}_1 \cdots \mathfrak{d}_n\mathfrak{b}$ for some $\mathfrak{d}_1, \cdots, \mathfrak{d}_n \in \mathscr{E}$; i.e., $y \in E^2 \subset E^*$ or $y \in E^{n+2} \subset E^*$ for some $n \geqq 1$. Therefore, E^* is a union of elements of $\pi_{\mathscr{E}}\mathscr{A}$.

THEOREM 5. *If $\mathscr{A}$ is a finite quotient of the free semigroup A, then there is a finite quotient $\mathscr{B}$ of A with the following properties:*

(i) *$\mathscr{B}$ refines $\mathscr{A}$.*

(ii) *Products of elements of $\mathscr{A}$ are unions of elements of $\mathscr{B}$; hence, products of unions of elements of $\mathscr{A}$ are unions of elements of $\mathscr{B}$.*

(iii) *The subsemigroup of A generated by any union of elements of $\mathscr{A}$ is a union of elements of $\mathscr{B}$.*

Proof. Since there are finitely many subsets of $\mathscr{A}$, then according to Theorems 3 and 4 we have only to form a finite sequence of joint refinements. In fact, according to the proofs of Theorems 3 and 4 we may use the quotient $\pi_{\mathscr{E}}\mathscr{A} \wedge \pi_{\mathscr{F}}\mathscr{A} \wedge \cdots \wedge \pi_{\mathscr{G}}\mathscr{A}$ if $\mathscr{E}, \mathscr{F}, \cdots, \mathscr{G}$ are all the subsets of $\mathscr{A}$.

LEMMA 6. *Any finite subset of a free semigroup A is a union of elements of a finite quotient of A.*

Proof. We let E be the smallest subset of A generating A, F be a finite subset of A, and E_0 a finite subset of E so that $F \subset E_0^*$. Then $F \subset E_0 \cup E_0^2 \cup \cdots \cup E_0^n$ for some $n > 0$. The partition of A consisting of the singleton subsets of $E_0 \cup E_0^2 \cup \cdots \cup E_0^n$ and the set $A \backslash (E_0 \cup \cdots \cup E_0^n)$ is a finite quotient of A, and F is a union of some of its elements

THEOREM 6. *Each regular subset of a free semigroup A is a union of elements of a finite quotient of A.*

Proof. By Lemma 6 the finite subsets of A are unions of elements of a finite quotient of A. By Theorems 3 and 4 collectionwise products of such unions and the subsemigroups of A generated by such unions are also such unions. The conclusion follows directly from the definition of regular subsets.

4. Transformation semigroups and automata. If h is a homomorphism from the semigroup A to the semigroup (under composition) of transformations on a set X (i.e., if (A, h) is a transformation semigroup) and if $x_0 \in X$, then we may define the subset A_x of A to consist of all those elements $a \in A$ for which $h(a)(x_0) = x$. Each such A_x is a union of elements of the quotient A/h, and the nonempty sets A_x constitute a partition of A which we denote by $A/(h, x_0)$. Moreover, if $\mathscr{A}$ is a quotient of the semigroup A, then A acts on the semigroup $\mathscr{A}^+$ consisting of $\mathscr{A}$ and an identity $1 \notin \mathscr{A}$ as follows: The homomorphism h^+ from A to the semigroup of transformations on $\mathscr{A}^+$ is defined by setting $h^+(a)(\mathfrak{a})$ equal to the element of $\mathscr{A}$ containing $h(a)\mathfrak{a}$ for $a \in A$, $\mathfrak{a} \in \mathscr{A}^+$, where h is the natural homomorphism from

A to $\mathscr{A}$. Then $\mathscr{A} = A/(h^+, 1)$. Thus, all quotients of A are of the form $A/(h, x_0)$ for some transformation semigroup (A, h). In particular, finite quotients may be obtained from finite "phase sets" X.

According to Theorem 6 and these last remarks, if E is a regular subset of the free semigroup A then there is a finite set X, a homomorphism h from A to the semigroup of transformations on X, and an element $x_0 \in X$ so that E is a union of elements of $A/(h, x_0)$; that is, there is a set $X_0 \subset X$ so that $E = \bigcup \{A_x : x \in X_0\}$. According to Theorem 2 and the remarks, if X is a finite set, x_0 an element of X, and h a homomorphism from the finitely generated semigroup A to the semigroup of transformations on X, then each element of $A/(h, x_0)$ is regular; that is, each A_x is regular.

A finite *automaton* (cf. [2]) is a homomorphism h from a finitely generated free semigroup A to the semigroup of transformations on a finite set X together with a subset X_0 of X. Each element x_0 of X then determines a regular subset of A consisting of the union of those elements A_x of $A/(h, x_0)$ for which $x \in X_0$. On the other hand, if E is a regular subset of the finitely generated free semigroup A, there exist appropriate X, X_0 and h defining a finite automaton so that E is the union of all A_x in $A/(h, x_0)$ for which $x \in X_0$, where x_0 is some element of X. These last statements are obvious consequences of the remarks above.

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University of Miami