A DECISION PROCEDURE FOR A CLASS OF FORMULAS OF FIRST ORDER PREDICATE CALCULUS

M. R. KROM

1. Introduction. In [4] (c.f. also case V' page 256 of [1]) J. Herbrand provides a decision procedure which is equivalent to a decision procedure for determining for a fixed contradiction C and for any first order prenex formula Γ whose matrix is a conjunction of signed atomic formulas, whether $\Gamma \rightarrow C$ is valid. In this paper we define a class a of first order formulas and then provide a decision procedure for determining for any first order prenex formula Γ whose matrix is a conjunction of signed atomic formulas and and member Δ of the class, whether $\Gamma \rightarrow \Delta$ is valid. Although the class of formulas Δ that we consider is rather large, it is clear that some restriction is necessary since a decision procedure for the class itself is obtained by using for Γ a single propositional parameter that does not occur in Δ.

The formulas we consider are those of any system of pure first order predicate calculus without equality and without function symbols. We use $\vee$, $\wedge$, $\neg$, and $\rightarrow$ for the propositional connectives disjunction, conjunction, denial, and the conditional, respectively. The symbols $\Gamma, \Delta, \Gamma_0, \Delta_0, \Gamma_1, \Delta_1, \cdots$ shall range over arbitrary formulas, $P, Q, P_1, Q_1, \cdots$ over prefixes, and $M, N, M_1, N_1, \cdots$ over matrices. A propositional parameter or predicate parameter together with its attached individual variables or individual parameters will be called an *atomic formula*. An occurrence of an atomic formula in a formula Γ is called an *atomic part* of Γ. Two prenex formulas are *similar* if their matrices differ only in the symbols occupying individual variable places of the atomic formulas. Two prenex formulas are *congruent* if they differ only by equivalent replacements of bound variables. We indicate that Δ is a logical consequence of Γ by writing $\Gamma \models \Delta$. If $\Gamma \models \Delta$ then there exists a symmetric L-deduction of Δ from Γ as described in [2]. For any formulas Γ and Δ an L-deduction of Δ from Γ is an ordered $(n+1)$-tuple $\langle \Gamma_0, \cdots, \Gamma_n \rangle$ where $\Gamma_0 = \Gamma$ and $\Gamma_n = \Delta$, together with a specification of how, for any $m < n$, Γ_{m+1} results from Γ_m by an application of an L-rule. The reader is referred to pages 252 and 253 of [2] for the definitions of the eleven L-rules. An L-deduction is *symmetric* if and only if the order in which the different kinds of L-rules are applied satisfies conditions (iii) through (vi) on page 257 of [2]. In addition, for convenience, we require that a symmetric L-deduction have exactly one application of the operation matrix change.

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Our method for deciding whether $\Gamma \rightarrow \Delta$ is valid will be to determine whether there exists an L-deduction of Δ from Γ. We begin by describing the class of the formulas Δ for which our method applies.

2. Negation distinguished formulas. A *Negation Distinguished* formula (we will write *ND* formula) is a prenex formula Δ whose matrix is in disjunctive normal form such that for any two occurrences $\bar{A}$, $\bar{B}$ in Δ, one positive and the other negative, of atomic formulas A, B containing the same predicate parameter: (1) If $A = B$, then quantifier occurrences in Δ, if any, binding a variable in $\bar{A}$, $\bar{B}$ the rightmost is universal; and (2) If $A \neq B$, then some place in $\bar{A}$ and the corresponding place in $\bar{B}$ contain occurrences of different individual symbols such that neither occurrence is existentially bound in Δ. In order to establish some properties of *ND* formulas we make some observations about Linear Reasoning. Let $\mathscr{D}$ be a symmetric L-deduction of a prenex disjunctive formula Δ from a prenex formula Γ. For any formula[1] Δ_1 of $\mathscr{D}$ to which the operation simplication is applied the occurrence of the connective $\vee$ between the subformulas of Δ_1 that are combined in the simplication will be called the *center* of the formula Δ_1. For any atomic part of any formula occurring after the matrix change in $\mathscr{D}$ we will define exactly one *successor* atomic part in any subsequent formula of $\mathscr{D}$. Let Δ_1 and Δ_2 be two formulas occurring after the matrix change in $\mathscr{D}$ such that Δ_2 occurs immediately after Δ_1. The successor in Δ_2 of an atomic part of Δ_1 is the atomic part of Δ_2 in the same relative position, where in the case of an application of simplication the position is determined by counting from the left for an atomic part left of the center and by counting from the right for an atomic part right of the center. The successor in any later formula of an atomic part is defined as required so that the relation of successor is the smallest transitive relation including the members described above. Similarly, for any atomic part of a formula occurring before the matrix change in $\mathscr{D}$ there is exactly one *predecessor* atomic part of any earlier formula of $\mathscr{D}$.

Let Δ_1 and Δ_2 be two formulas occurring after the matrix change in $\mathscr{D}$ such that Δ_2 occurs immediately after Δ_1. If Δ_2 is obtained from Δ_1 by an application of simplication then the relation of successor between the atomic parts of Δ_1 and the atomic parts of Δ_2 is two-to-one for atomic parts of the subformulas of Δ_1 that are combined by the simplication. For any other atomic parts of Δ_1 the relation of successor is one-to-one. Also if Δ_2 is obtained from Δ_1 by an L-operation other

[1] We use the expression "formula of $\mathscr{D}$" rather than the more cumbersome "occurrence of a formula of $\mathscr{D}$" for convenience. Actually, we could assume that all formulas of an L-deduction are distinct.

than simplication then the relation of successor between the atomic parts of Δ_1 and the atomic parts of Δ_2 is one-to-one.

Let Δ_1 be a formula occurring after the matrix change in $\mathscr{D}$, let Δ_2 be any later formula of $\mathscr{D}$, and let $\bar{A}_1$ and $\bar{B}_1$ be atomic parts of Δ_1 such that no individual variables in their successors $\bar{A}_2$ and $\bar{B}_2$, respectively, in Δ_2 are bound by a quantifier occurrence that is not in the main prefix of Δ_2 (c.f. page 254 of [2]). It follows from the properties of the L-rules that could be applied in the subdeduction of $\mathscr{D}$ of Δ_2 from Δ_1 that if $\bar{A}_2$ and $\bar{B}_2$ are occurrences of the same atomic formula then so are $\bar{A}_1$ and $\bar{B}_1$. Because of this property of symmetric L-deductions we may assume that for any application of the operation simplication the subformulas combined in the simplication include at least one quantifier. That is, simplications combining quantifier free subformulas (we will call these *trivial* simplications) can be avoided by deleting one or more of some identical disjuncts in the matrix of the formula resulting from matrix change in $\mathscr{D}$. Let $\bar{\Delta}_1^1$ be a subformula of Δ_1 consisting of an occurrence of a prenex formula Δ_1^1 in Δ_1 such that there is a quantifier occurrence in $\bar{\Delta}_1^1$ which is not a quantifier occurrence in the main prefix of Δ_1 and such that any quantifier occurrence of Δ_1 which is not in $\bar{\Delta}_1^1$ and with $\bar{\Delta}_1^1$ in its scope is in the main prefix of Δ_1. We will say that $\bar{\Delta}_1^1$ is a *proper prenex subformula* of Δ_1. It follows from properties of the disassembling operations that Δ_1^1 is similar to Δ.

We assumed that the concluding formula Δ of $\mathscr{D}$ is prenex. We now assume further that $\mathscr{D}$ includes no trivial simplications. In this case each formula of $\mathscr{D}$ occurring after the matrix change consists of a main prefix followed by a disjunction of what we will call *main disjuncts*. Any proper prenex subformula of any formula of $\mathscr{D}$ occurring after matrix change will be called a *main disjunct* of that formula. If $\bar{A}$ is an atomic part of some formula Δ_1 occurring after the matrix change of $\mathscr{D}$ such that $\bar{A}$ is not part of a proper prenex subformula of Δ_1, then we define the *main disjunct* in which $\bar{A}$ occurs as follows. Let Δ_2 the first subsequent formula of $\mathscr{D}$ in which the successor of $\bar{A}$ occurs in a proper prenex subformula. The main disjunct in which $\bar{A}$ occurs is that disjunct of the main alternation of Δ_1 in which occur exactly those atomic parts of Δ_1 whose successors in Δ_2 occur in the proper prenex subformula of Δ_2 in which the successor of $\bar{A}$ occurs. Here we assume Δ has at least one quantifier. Otherwise there are no applications of disassembling operations and the entire matrix of any formula of $\mathscr{D}$ after the matrix change is the only main disjunct of the formula. Any main disjunct of a formula of $\mathscr{D}$ is similar to Δ. Suppose that Δ has no vacuous quantifier occurrences and that Δ_1 is a formula occurring after all applications of existential vacuous removal in $\mathscr{D}$. Then $\varDelta_1$ has no vacuous quantifier occurrences and any main disjunct of $\varDelta_1$ together with all quantifier occurrences of $\varDelta_1$ that apply nonvacuously to it is congruent to $\varDelta$.

LEMMA 1. *Let $\varGamma$ be a prenex formula and $\varDelta$ be a ND formula such that $\varDelta$ is obtained from $\varGamma$ by an application of existential generalization, an application of existential vacuous removal, or by a series of applications of disassembling operations. Then $\varGamma$ is a ND formula.*

Proof. By properties of the L-rules considered here it follows that $\varGamma$ is a prenex formula whose matrix is in disjunctive normal form. To show that $\varGamma$ is a ND formula we consider together the cases of an application of existential generalization and an application of existential vacuous removal. For these cases $\varGamma$ and $\varDelta$ are similar. If any two atomic parts of $\varGamma$, one positive and one negative, determine that $\varGamma$ is not a ND formula, then their successors in $\varDelta$ determine that $\varDelta$ is not a ND formula.

Next we assume that $\varDelta$ is obtained from $\varGamma$ by a series of applications of disassembling operations. Suppose $\bar{A}_1$ and $\bar{B}_2$ are atomic parts, one positive and one negative, occurring in main disjuncts $\bar{\varGamma}_1$ and $\bar{\varGamma}_2$, respectively, of $\varGamma$. The formulas $\varGamma_1$ and $\varGamma_2$ of $\bar{\varGamma}_1$ and $\bar{\varGamma}_2$, respectively, are similar since they are both similar to $\varDelta$. We let $\bar{B}_1$ be the atomic part of $\varGamma$ in $\bar{\varGamma}_1$ which corresponds to $\bar{B}_2$ in $\bar{\varGamma}_2$. We claim that if $\bar{A}_1$ and $\bar{B}_2$ determine that $\varGamma$ is not a ND formula then so do $\bar{A}_1$ and $\bar{B}_1$. If $\bar{A}_1$ and $\bar{B}_1$ do not determine that $\varGamma$ is not a ND formula and their atomic formulas A_1 and B_1, respectively, are the same atomic formula then the right most quantifier applying nonvacuously to this atomic formula is a universal quantifier. In this case either A_1 and B_2 are the same atomic formula (so that $\bar{A}_1$ and $\bar{B}_2$ would not determine that $\varGamma$ is not a ND formula) or else they are atomic formulas which differ in the symbol occupying the individual variable place which contains the right most quantified individual variable (so that $\bar{A}_1$ and $\bar{B}_2$ again not determine that $\varGamma$ is not a ND formula). That is, if B_1 and B_2 have this right most quantified individual variable in common then none of the quantifier occurrences applying nonvacuously to $\bar{B}_1$ and $\bar{B}_2$ can be imported before main disjuncts corresponding to $\bar{\varGamma}_1$ and $\bar{\varGamma}_2$ are simplicated together since this right most quantifier occurrence is to be imported first and, by restrictions imposed on importations, it would have the main disjuncts corresponding to both $\bar{\varGamma}_1$ and $\bar{\varGamma}_2$ in its scope after its importation. If $\bar{A}_1$ and $\bar{B}_1$ do not determine that $\varGamma$ is not a ND formula and A_1 and

B_1 are not the same atomic formula then A_1 and B_2 differ at least in the individual variables occupying the places at which A_1 and B_1 differ and therefore $\bar{A}_1$ and $\bar{B}_2$ do not determine that Γ is not a *ND* formula. Thus if there are two atomic parts of Γ which determine that Γ is not a *ND* formula then there are two such atomic parts occurring in one occurrence of a main disjunct of Γ. But a main disjunct together with its nonvacuous quantifiers is congruent to Δ with its vacuous quantifiers deleted. It follows that Γ is a *ND* formula.

LEMMA 2. *Let QN be a ND formula that is obtained from a prenex formula PM by an application of existential generalization (by an application of existential vacuous removal). Let C be a conjunction of signed atomic formulas the atomic formulas of which occur in M such that* $C \models M$. *Then there exists a conjunction D of signed atomic formulas the atomic formulas of which occur in N such that* $D \models N$ *and QD can be obtained from PC by applications of duplication and an existential generalization (by an application of existential vacuous removal).*

Proof. By Lemma 1 it follows that *PM* is a *ND* formula. By the definition of the *L*-rule existential generalization (existential vacuous removal) *PM* is similar to *QN*. For the formulation in terms of an application of existential vacuous removal the conclusion of the lemma follows easily by letting $D = C$. Suppose that *QN* is obtained from *PM* by an application of existential generalization. If an atomic formula that occurs in *C* has both a positive and negative occurrence in *PM* then either each occurrence of the atomic formula in *PM* is identical with its successor in *QN* or all occurrences of the atomic formula differ from their successors in *QN* and all of the successors in *QN* of these occurrences are occurrences of the same atomic formula. Otherwise there would be positive occurrence and a negative occurrence of a predicate parameter in *QN* such that the occurrences are occurrences of atomic formulas which differ only in individual variables occupying individual variable places for which in at least one of the atomic parts the individual variable is existentially bound. This contradicts the hypothesis that *QN* is a *ND* formula. The required formula *D* is a conjunction (in the right order so necessary duplications can be applied) of signed atomic formulas which are the atomic formulas of successors of the atomic parts in *PM* the atomic formulas of which occur in *C*.

We now present an example to show that the conditions given in the definition of a Negation Distinguished formula are necessary for our method. Let

$$\Gamma_1 = \exists a \exists b \exists c[[Gb \wedge Fa] \vee [Hb \wedge \neg Fa] \\ \vee [Gc \vee Fa] \vee [Hc \wedge \neg Fa]] ,$$

$$\Gamma_2 = \exists a \exists b \exists f \exists c[[Gb \wedge Fa] \vee [Hb \wedge \neg Fa] \\ \vee [Gc \wedge Ff] \vee [Hc \wedge \neg Ff]] ,$$

$$\Gamma_3 = \exists a \exists b[[Gb \wedge Fa] \vee [Hb \wedge \neg Fa]] \\ \vee \exists a \exists b[[Gb \wedge Fa] \vee [Hb \wedge \neg Fa]] ,$$

and

$$\Delta = \mathrm{E} a \mathrm{E} b[[Gb \wedge Fa] \vee [Hb \wedge \neg Fa]] .$$

Let $P, Q, M,$ and N be such that $PM = \Gamma_1$ and $QN = \Gamma_2$ and let $C = Gb \wedge Hc$. Here Γ_2 is obtained from Γ_1 by an application of existential generalization and $C \models M$. However there is no conjunction D of signed atomic formulas the atomic formulas of which occur in N such that $D \models N$ and $PC \models QD$. Thus to obtain a conclusion like that in Lemma 2 above we need to require that QN has no occurrences of subformulas like Fa and $\neg Ff$ here formed with atomic formulas that differ only in the individual variable occupying an individual variable place containing an existentially bound individual variable in at least one of the atomic parts. In our applications we will want to assume that a concluding formula Δ of a symmetric L-deduction is a ND formula and from this conclude by Lemma 1 that formulas to which existential generalizations are applied in the L-deduction are ND formulas so that a conclusion like that of Lemma 2 can be obtained for the results of these existential generalizations. Our example also shows that in order to prove Lemma 1 we need the condition in case (1) of the definition of a Negation Distinguished formula requiring that a right most quantifier be a universal quantifier. Here Γ_3 is obtained from Γ_2 by applications of existential importation and Δ is obtained from Γ_3 by a simplication. Δ fails to be a ND formula only because of the requirement that certain quantifiers are to be universal quantifiers. However Γ_2 fails to be a ND formula because of other conditions in the definition of a ND formula which are important in obtaining the conclusion of Lemma 2.

LEMMA 3[1]. *Let Γ be a prenex formula with a valid matrix and let Δ be a ND formula such that Δ is obtained from Γ by an application of existential generalization, an application of existential vacuous removal, or by a series of application of disassembling operations. Then the matrix of Δ is valid.*

[1] For Lemma 3 and Theorem 1 condition (1) of the definition of ND formula is unnecessary.

Proof. If Δ is obtained from Γ by an application of existential vacuous removal then the matrix of Δ is the same as that of Γ and so it is valid. Suppose Δ is obtained from Γ by an application of existential generalization. Since Δ is a *ND* formula it follows that if an atomic formula has both a positive and a negative occurrence in Γ then all successors in Δ of any occurrences of the atomic formula in Γ are occurrences of the same atomic formula. The matrix of Δ is obtained from the matrix of Γ by replacing occurrences of atomic formulas with other atomic formulas in such a way that an atomic formula having both a positive and negative occurrence in Γ is replaced in all of its occurrences by the same atomic formula. It follows that the matrix of Δ is valid. Finally suppose Δ is obtained from Γ by a series of applications of disassembling operations. In this case some of the intermediate formulas may not be prenex, so we let the *matrix* of any formula be the quantifier free formula obtained by deleting all of its quantifiers. We see that each of the disassembling operations preserves the property of a formula of having a valid matrix. Thus for all three cases we conclude that the matrix of Δ is valid.

THEOREM 1. *A ND formula is valid if and only if its matrix is valid.*

Proof. Suppose Δ is a *ND* formula that is valid and let A be any propositional parameter that does not occur in Δ. Then $A \models \Delta$, so there exists a symmetric L-deduction, say $\mathscr{D}$, of Δ from A. By properties of L-rules that can be applied before the matrix change in a symmetric L-deduction it follows that the matrix of the formula to which the operation matrix change is applied in $\mathscr{D}$ is a conjunction of occurrences of A. By properties of L-rules that can be applied after the matrix change in a symmetric L-deduction it follows that the predicate parameter A that does not occur in Δ also does not occur in the formula resulting from the application of matrix change. It follows that the matrix of the formula resulting from the matrix change in $\mathscr{D}$ is valid. By Lemma 3 it follows that the matrix of Δ is valid. Conversely, if we assume that the matrix of a *ND* formula is valid then it follows that the formula is valid.

LEMMA 4. *Let Γ and Δ be prenex formulas such that Δ is a ND formula, the matrix of Δ is not valid, the matrix of Γ is a conjunction of signed atomic fomulas, and $\Gamma \rightarrow \Delta$ is valid. Then there exists an L-deduction $\langle \Gamma, \cdots, Q'C', Q'N', \cdots, \Delta\rangle$ such that $\langle \Gamma, \cdots, Q', C'\rangle$ is a symmetric L-deduction in which no disassembling operations occur, $\langle Q'N', \cdots, \Delta\rangle$ is an L-deduction in which only dis-*

assembling operations occur, and C' is a conjunction of signed atomic formulas the atomic formulas of which occur in N'.

Proof. Let Γ and Δ be as prescribed. Then there exists a symmetric L-deduction, say $\mathscr{D}$, of Δ from Γ. Let PM be the formula resulting from the application of matrix change in $\mathscr{D}$. By successive applications of Lemma 1 it follows that PM and all subsequent prenex formulas of $\mathscr{D}$ are ND formulas. Let $Q'N'$ be the formula occurring just before disassembling operations in $\mathscr{D}$. The existence of the conjunction C' of signed atomic formulas the atomic formulas of which occur in N' such that $C' \models N'$ is established by successive applications of Lemma 2 to the two line subdeductions of $\mathscr{D}$ consisting of successive formulas occurring between PM and $Q'N'$. For the first application of Lemma 2 the matrix M of PM is taken as the conjunction of signed atomic formulas required in the hypothesis of Lemma 2. Let $\mathscr{E}$ be the L-deduction obtained by continuing on from the subdeduction of $\mathscr{D}$ of PM from Γ by piecing together two line deductions obtained from the conclusions of the applications of Lemma 2. The symmetric L-deduction $\langle \Gamma, \cdots, Q'C' \rangle$ required in this lemma is obtained from $\mathscr{E}$ by replacing any occurrences of duplication after the matrix change with an obvious modification of the matrix change. Then $\langle \Gamma, \cdots, Q'C', Q'N', \cdots, \Delta \rangle$ is obtained from $\langle \Gamma, \cdots, Q'C' \rangle$ by continuing on with the subdeduction of $\mathscr{D}$ of Δ from $Q'N'$.

3. Modified symmetric L-deductions. Now we consider a particular kind of L-deduction that arises from applications of Lemma 4 above. For any prenex formulas Γ and Δ whose matrices are conjunctions of signed atomic formulas, a *modified symmetric* L-deduction of Δ from Γ is an L-deduction $\mathscr{D}$ of Δ from Γ satisfying the following conditions:

(1) There is a prenex formula, say Δ_1, whose matrix is a conjunction of signed atomic formulas and such that the subdeduction of $\mathscr{D}$ of Δ_1 from Γ satisfies all of the conditions of being a symmetric L-deduction in which no disassembling operations occur except that vacuous existential generalizations may occur immediately before universal instantiations.

(2) The subdeduction of $\mathscr{D}$ of Δ from Δ_1 consists of, first an application of matrix change the effect of which is the deletion of zero or more but not all of the conjuncts of the matrix of Δ_1, then zero or more applications of existential vacuous removal, and finally zero or more applications of universal instantiation for which the universal quantifier occurrences to be instantiated are vacuous,

Let $\mathscr{D}$ be a modified symmetric L-deduction of Δ from Γ where Δ and Γ are prenex formulas whose matrices are conjunctions of signed atomic formulas. Each formula of $\mathscr{D}$ occurring before the first application of matrix change consists of a prefix followed by a conjunction, each conjunct of which is called a *main conjunct* and is an occurrence of a prenex formula similar to Γ. Each formula occurring after the first application of matrix change in $\mathscr{D}$ is a prenex formula the matrix of which is a conjunction of signed atomic formula. We will call each of these conjuncts a *main conjunct* of the formula in which it occurs. If Γ_i and Γ_{i+1} are two successive formulas of $\mathscr{D}$ and $\{\bar{C}_1, \cdots, \bar{C}_h\}$ is any set of main conjuncts of Γ_i then we define the *corresponding* set of main conjuncts of Γ_{i+1} as follows. If Γ_i and Γ_{i+1} occur before the first application of matrix change in $\mathscr{D}$ then the corresponding set of main conjuncts of Γ_{i+1} includes those main conjuncts of Γ_{i+1} in which occur an atomic part of Γ_{i+1} the predecessor of which in Γ_i occurs in some $\bar{C}_i$, $i = 1, \cdots, h$. If Γ_{i+1} is obtained from Γ_i by an application of matrix change in $\mathscr{D}$ then Γ_i and Γ_{i+1} are prenex formulas the matrices of which are conjunctions of signed atomic formulas and the corresponding set of main conjuncts of Γ_{i+1} includes any conjuncts of the matrix of Γ_{i+1} that must be deleted to obtain a formula that is a consequence of the formula obtained from the matrix of Γ_i by deleting $\bar{C}_1, \cdots, \bar{C}_h$. If Γ_{i+1} occurs after the first application of matrix change in $\mathscr{D}$ and is obtained from Γ_i by an application of a quantifier rule then Γ_i and Γ_{i+1} are similar prenex formulas and the corresponding set of main conjuncts of Γ_{i+1} includes those in the same relative position as $\bar{C}_1, \cdots, \bar{C}_h$. Let Γ_i and Γ_{i+1} be successive formulas of $\mathscr{D}$, let Γ_i^1 be a formula obtained by deletion of a set of main conjuncts of Γ_i and let Γ_{i+1}^1 be the formula obtained from Γ_{i+1} by deleting the corresponding set of main conjuncts in Γ_{i+1} and also by deleting the quantifier occurrence just introduced into the main prefix of Γ_{i+1} in case Γ_{i+1} is obtained from Γ_i by an exportation of a quantifier from a main conjunct occurrence of Γ_i that was deleted to obtain Γ_i^1. By the definitions of the L-rules it follows that Γ_{i+1}^1 is identical with Γ_i^1 or Γ_{i+1}^1 is obtained from Γ_i^1 by an application of the same L-rule which is applied in $\mathscr{D}$ to obtain Γ_{i+1} from Γ_i. A modified symmetric L-deduction $\mathscr{D}'$ will be said to be obtained from $\mathscr{D}$ by *duplicate deletion* in case $\mathscr{D}'$ is obtained from $\mathscr{D}$ in the following way. There is a formula, say Γ_i, to which duplication is applied in $\mathscr{D}$. One of the two identical main conjuncts introduced into Γ_{i+1} by the application of duplication is deleted and the corresponding sets of main conjuncts of the successive formulas of $\mathscr{D}$ are deleted. Quantifier occurrences that were introduced into the main prefix of formulas of $\mathscr{D}$ by exportation from a main conjunct that is deleted are also deleted except that any which later apply nonvacuously to other atomic parts of the formula because of universal instantiations are introduced by universal vacuous introduction or existential generalization, which happens to be vacuous, at the appropriate place in the prefix.

LEMMA 5. *There exists an effective procedure for deciding, for any prenex formulas Γ and Δ whose matrices are conjunctions of signed atomic formulas, whether there exists a modified symmetric L-deduction of Δ from Γ.*

Proof. Let n be the number of conjuncts in the matrix of Δ. If we have a modified symmetric L-deduction of Δ from Γ, then by a succession of applications of duplicate deletion we can obtain one in which no more than $n-1$ duplications occur. Here we may delete a main conjunct introduced in an application of duplication unless the final formula of the resulting deduction does not have a main conjunct corresponding to one of the n main conjuncts of Δ. Let h be the number of occurrences of universal quantifiers and k the number of occurrences of existential quantifiers in Δ. If we have a modified symmetric L-deduction of Δ from Γ, then we may delete from it all but $\leqq h$ applications of universal vacuous introduction together with the corresponding applications of universal (vacuous) instantiation and all but $\leqq k$ applications of existential generalization together with the corresponding applications of existential vacuous removal. To determine whether there exists a modified symmetric L-deduction of Δ from Γ we consider each member of a maximal set of noncongruent modified symmetric L-deductions from Γ which include no more than n applications of duplication, h applications of universal vacuous introduction, k applications of existential generalization and in which the matrix of the formula resulting from the last application of matrix change is similar to Δ.

4. Disassembling operations. In this section we provide a procedure for determining the existence of L-deductions in which only disassembling operations occur and then we combine the procedure with earlier results to obtain our major theorem.

For each Q and N let $|\overline{N}Q$ be the results of deleting from Q all quantifier occurrences which in QN would be vacuous. For prenex formulas QN and $Q'N' = Q'[N_1 \vee \cdots \vee N_e]$ in which there are no vacuous quantifier occurrences we say that QN is a D-consequent of $Q'N'$ and write $Q'N' \vDash_D QN$ in case each $|\overline{N_i}\ Q'N_i$ is congruent to QN and for any $i \neq j$ there are Q_1, Q_2, Q_3 with Q_1 possibly empty such that $|\overline{N_i}Q' = Q_1Q_2$, $|\overline{N_j}Q' = Q_1Q_3$ and (Q_1), Q_2, and Q_3 have no variable in common. In this case we call the occurrences of the disjuncts $N_1, \cdots, N_e$ *main disjuncts* of $Q'N'$ and of N'. The following Lemma is an immediate consequence of the definitions of the disassembling operations.

LEMMA 6. *The following two conditions are equivalent:*

(1) *QN and $Q'N'$ are prenex formulas such that QN has no vacuous quantifier occurrences and is obtained from $Q'N'$ by a sequence of applications of disassembling operations in which no trivial simplications occur;*

(2) *$Q'N'$ has the form $Q'N' = Q'[N_1 \vee \cdots \vee N_e]$ with no vacuous quantifier occurrences and $Q'N' \models_D QN$.*

LEMMA 7. *Let θ, σ, ϕ be distinct propositional variables, let $H(\theta, \sigma)$ and $J(\sigma)$ be propositional formulas containing only the variables indicated, and let $H(\phi, \sigma)$ be the result of substituting ϕ for θ in $H(\theta, \sigma)$. Then $H(\theta, \sigma) \wedge H(\phi, \sigma) \models J(\sigma)$ implies that $H(\theta, \sigma) \models J(\sigma)$.*

Proof. Suppose the conclusion is false. Then it is easy to describe an assignment of truth values to all propositional variables in such a way as to demonstrate that the assumed logical implication does not hold.

Let $p(n)$ be the number of partitions of n for any positive integer n (vid. page 273 of [3]). For any nonnegative integer h let $m_h(i)$ be defined for positive integers i by the equations $m_h(1) = 1$ and

$$m_h(i + 1) = m_h(i)p(m_h(i))^{im_h(i)}\ 3^{m_h(i)h}\ .$$

For any prenex formula QN with k bound individual variables and h atomic formulas in which bound individual variables occur we let $g(QN) = m_h(k + 1)$. We observe that g is an effectively calculable function.

LEMMA 8. *Let QN and $Q'N' = Q'[N_1 \vee \cdots \vee N_e]$ be prenex formulas with no vacuous quantifier occurrences such that $Q'N' \models_D QN$. Let C' be any conjunction of signed atomic formulas such that all atomic formulas of C' occur in N' and such that $C' \models N'$. Then there is an N'' obtained from N' by deleting all but $\leqq g(QN)$ main disjuncts (of N') such that $C' \models N''$.*

Proof. Let QN, $Q'N'$ and C' be as prescribed and assume QN has k bound individual variables and h atomic formulas in which bound individual variables occur. Let N'' be any matrix obtained from N' by deleting zero or more main disjuncts such that $C' \models N''$ and such that if any additional main disjuncts are deleted the result is either the empty matrix or is not a consequence of C'. Thus $Q'N''$ has a minimum number of the main disjuncts of $Q'N'$. We will show that this minimum number of main disjuncts of $Q'N'$ that remain in $Q'N''$ is $\leqq g(QN)$.

For any bound individual variable in $Q'N'$ we define its *order* to be a positive integer determined as follows. Select any main disjunct of $Q'N'$ in which the individual variable occurs, delete from $Q'N'$ all quantifier occurrences that bind no individual variable occurrences in this main disjunct, and then assign as the *order* of the individual variable the number of the position of its binding quantifier occurrence counting from right to left among the remaining quantifier occurrences. From the definition of $\models_D$ it follows that this definition is independent of the main disjunct selected to determine the order for bound individual variables that occur in more than one main disjunct. For any atomic formula occurring in $Q'N'$ in which occurs a bound individual variable we define the *order* of the atomic formula to be the order of the bound individual variable of least order that occurs in it. Let h_i be the number of atomic formulas of order i that occur in any one main disjunct of $Q'N'$ for $i = 1, \cdots, k$. Since each $\overline{|N_i}$ $Q'N_i$ is congruent to QN it follows that $h = h_1 + \cdots + h_k$. Let $\rho_i(i = 1, \cdots, k)$ be the equivalence relation defined on the set of main disjuncts of $Q'N'$ as follows. For main disjuncts N_u and N_v, N_u is ρ_i related to N_v if and only if the bound individual variables of order i are the same in N_u and N_v. We let ρ_{k+1} be the universal relation on the set of main disjuncts. By properties of $\models_D$ it follows that if two main disjuncts have a bound individual variable in common then any bound individual variable of greater order in one of the two main disjuncts is identical with the bound individual variable of the same greater order in the other main disjunct. Thus if $1 \leqq i \leqq j \leqq k + 1$ then $\rho_i \subseteqq \rho_j$. Also, if $1 \leqq i \leqq k$ and if N_u and N_v are two main disjuncts that are not ρ_i related, then any atomic formula occurring in N_u that is of order $\leqq i$ does not occur in N_v. Let $\bar{\rho}_i$ be the restriction of ρ_i to the set of main disjuncts of N' that are not deleted to obtained N''. Let $\bar{m}(i)$ be the maximum number of main disjuncts that occur in any $\bar{\rho}_i$ class for $i = 1, \cdots, k + 1$. Using the fact that $Q'N''$ has a minimum number of main disjuncts we apply a generalization of Lemma 7 to obtain an upper bound for $\bar{m}(i + 1)$ in terms of $\bar{m}(i)$ for $i = 1, \cdots, k$. We first assume that we know $\bar{m}(i)$ and from this determine an upper bound $b(i)$ for the number of $\bar{\rho}_i$ classes included in any one $\bar{\rho}_{i+1}$ class for $1 \leqq i \leqq k$. For a given $\bar{\rho}_{i+1}$ class $\mathscr{C}_{i+1}$ let $\mathscr{T}$ be the set of ordered pairs $\langle C, F \rangle$ where F is the disjunction of all main disjuncts in a $\bar{\rho}_i$ class which is included in $\mathscr{C}_{i+1}$ and C is the subconjunction of C' the conjuncts of which are formed with atomic formulas that occur in F. Here we identify two disjunctions F that differ only in the order of their disjuncts so that $\mathscr{T}$ contains exactly one member for each $\bar{\rho}_i$ subclass of $\mathscr{C}_{i+1}$. We will show that no two members of $\mathscr{T}$ are congruent. Otherwise, suppose $\langle C_1, F_1\rangle$ and $\langle C_2, F_2\rangle$ are two members of $\mathscr{T}$ such that (after perhaps permutations of disjuncts and conjuncts) C_1 and C_2 are similar, F_1 and F_2 are similar, and such that $\langle C_1, F_1\rangle$ can be obtained from $\langle C_2, F_2\rangle$ by alphabetic changes of individual variables in such a way that the same individual variable occurs at two places in $\langle C_1, F_1\rangle$ if and only if the individual variable occurring at the two corresponding places in $\langle C_2, F_2\rangle$ are the same. Let D be the subconjunction of C' including all conjuncts the atomic formulas of which do not occur in C_1 or C_2 and let G be the disjunction of all main disjuncts of N'' that do not occur in F_1 or F_2. Here the empty conjunct is taken as the logical constant truth and the empty disjunction is taken as the logical constant falsity. The assumption that $C' \models N''$ is equivalent to the condition that $C_1 \wedge \neg F_1 \wedge C_2 \wedge \neg F_2 \models \neg D \vee G$. In the statement of Lemma 7 replace $H(\theta, \sigma)$ by $C_1 \wedge \neg F_1$, $H(\phi, \sigma)$ by $C_2 \wedge \neg F_2$ and $J(\sigma)$ by $\neg D \vee G$. Then, with $\langle C_1, F_1\rangle$ and $\langle C_2, F_2\rangle$ as indicated above, the generalization of Lemma 7 obtained by replacing each of θ, ϕ and σ with a set of propositional variables (here taken as the atomic formulas of N'') would show that $Q'N''$ does not have a minimum number of main disjuncts of $Q'N'$. Here any common atomic formulas of F_1 and F_2 appear in corresponding positions in F_1 and F_2 since any two main disjuncts of $Q'N'$ have an individual variable in common if and only if it is the individual variable of the same order in both main disjuncts. Thus no two members of $\mathscr{T}$ are congruent.

To determine an uppor bound $b(i)$ for the number of elements in $\mathscr{T}$ (and thus for the number of $\bar{\rho}_i$ classes included in $\mathscr{C}_{i+1}$) we observe that the number of non-congruent formulas F that are the disjunctions of the main disjuncts in a $\bar{\rho}_i$ subclass of $\mathscr{C}_{i+1}$ is $\leqq p(\bar{m}(i))^{i\bar{m}(i)}$. Suppose F_1 is the disjunction of all main disjuncts in a $\bar{\rho}_i$ class $\mathscr{C}_i^1$ and F_2 is the disjunction of all main disjuncts in another $\bar{\rho}_i$ class $\mathscr{C}_i^2$. Then F_1 and F_2 are congruent if and only if $\mathscr{C}_i^1$ and $\mathscr{C}_i^2$ are isomorphic with respect to the relations $\bar{\rho}_u$ for $u < i$ (that is, if and only if $\mathscr{C}_i^1$ and $\mathscr{C}_i^2$ include the same number of $\bar{\rho}_{i-1}$ classes which can be paired up in such a way that a $\bar{\rho}_{i-1}$ subclass of $\mathscr{C}_i'$ includes the same number of $\bar{\rho}_{i-2}$ classes as the corresponding $\bar{\rho}_{i-1}$ subclass of $\mathscr{C}_i^2$ etc.) There are $\leqq p(\bar{m}(i))$ ways of partitioning the set of $\leqq \bar{m}(i)$ main disjuncts in a $\bar{\rho}_i$ class into distinct $\bar{\rho}_{i-1}$ classes. If $u \leqq i$ then $\bar{m}(u) \leqq \bar{m}(i)$ and the number of $\bar{\rho}_u$ classes included in a $\bar{\rho}_i$ class is $\leqq \bar{m}(i)$. It follows that the number of ways of first partitioning the set of main disjuncts in a $\bar{\rho}_i$ class into $\bar{\rho}_{i-1}$ classes, then partitioning these $\bar{\rho}_{i-1}$ classes into

$\bar{\rho}_{i-2}$ classes, $\cdots$ and then partitioning these $\bar{\rho}_2$ classes into $\bar{\rho}_1$ classes, is $\leqq p(\bar{m}(i))^{i\bar{m}(i)}$. To take account of non-congruent elements $\langle C_1, F_1\rangle$, $\langle C_2, F_2\rangle \in \mathscr{T}$ where F_1 and F_2 are congruent, we observe that there are $\leqq \bar{m}(i)(h_1 + \cdots + h_i) \leqq \bar{m}(i)h$ atomic formulas of order $\leqq i$ occurring in any formula F that is a disjunction of no more than $\bar{m}(i)$ main disjuncts. Each of these $\leqq \bar{m}(i)h$ atomic formulas may either not occur, occur positively, or occur negatively in such a conjunction C. We need not consider the case of an atomic formula occurring both positively and negatively in C since, for our applications, C is a subconjunction of the conjunction C' appearing in the statement of the lemma and the conclusion of the lemma follows immediately if C' is a contradiction. Thus for the upper bound $b(i)$ we may take $p(\bar{m}(i))^{i\bar{m}(i)}3^{h\bar{m}(i)}$. By the definition of $\bar{m}(i)$, this implies that

$$\bar{m}(i+1) \leqq \bar{m}(i)b(i) = \bar{m}(i)p(\bar{m}(i))^{i\bar{m}(i)}3^{h\bar{m}(i)} ,$$

for $1 \leqq i < k$. The number of main disjuncts of $Q'N'$ that remain in $Q'N''$ is $\leqq \bar{m}(k)b(k)$. The formulas of any two main disjuncts that are ρ_1 related are identical so $\bar{m}(1) = 1$. From the inequalities established above relating $\bar{m}(i+1)$ and $\bar{m}(i)$ and from the definition of $m_h(i)$ given above it follows that $\bar{m}(k)b(k) \leqq g(QN)$.

THEOREM 2. *There exists an effective procedure for deciding for any prenex formula Γ the matrix of which is a conjunction of signed atomic formulas and any ND formula Δ whether $\Gamma \rightarrow \Delta$ is valid.*

Proof. Let Γ and Δ be as prescribed and for convenience we also assume Δ has no vacuous quantifier occurrences. First determine whether the matrix of Δ is valid. If it is then $\Gamma \rightarrow \Delta$ is valid. If not proceed as follows. First, determine the effectively calculable number $g(\Delta)$. Second, determine a set $\mathscr{S}$ of ordered pairs of formulas $\langle C, QN\rangle$ with the following properties:

(1) For any element $\langle C, QN\rangle$ of $\mathscr{S}$, C is a conjunction of signed atomic formulas the atomic formulas of which occur in N, C has no more than two occurrences of any one atomic formula, and $C \models N$.

(2) For any element $\langle C, QN\rangle$ of $\mathscr{S}$, Δ can be obtained from QN by applications of disassembling operations and N has no more than $g(\Delta)$ main disjuncts (relative to Δ).

(3) There are no two members of $\mathscr{S}$ of the form $\langle C', Q'N'\rangle$ and $\langle C, QN\rangle$ such that C is similar C' and QN is congruent to $Q'N'$ and such that $\langle C', Q'N'\rangle$ is obtained from $\langle C, QN\rangle$ by alphabetic changes of individual variables in such a way that the same individual variable occurs at two places in $\langle C', Q'N'\rangle$ if and only if the individual variables occurring at the two corresponding places in $\langle C, QN\rangle$ are the same.

(4) $\mathscr{S}$ is a maximal set with respect to set inclusion satisfying

conditions (1), (2) and (3) above. It is easy to see that there is an effective procedure for determining such a set $\mathscr{S}$. If a formula QN occurs in a element $\langle C, QN\rangle$ of $\mathscr{S}$ then it must have no vacuous quantifier occurrences and its matrix must be a disjunction of $1, 2, \cdots$, or $g(\Delta)$ matrices each similar to the matrix of Δ. One can first obtain a maximal set $\mathscr{U}$ of pairs of formulas $\langle C, QN\rangle$ satisfying these two conditions and conditions (1) and (3) above. Then the required set $\mathscr{S}$ can be obtained from $\mathscr{U}$ by deleting those members of $\mathscr{U}$ for which condition (2) is not satisfied. One can effectively decide whether Δ can be obtained from a given prenex formula QN by applications of disassembling operations since in disassembling operations quantifiers must be imported in their right to left order. Third, decide for each element $\langle C, QN\rangle$ of $\mathscr{S}$ whether there is a modified symmetric L-deduction of QC from Γ. By Lemma 5 this step is effective. We conclude that $\Gamma \rightarrow \Delta$ is valid if and only if the answer is yes in step three for at least one member of $\mathscr{S}$.

It is easy to see that if the answer is yes in part three for one member of $\mathscr{S}$, say for $\langle C_1, Q_1N_1\rangle$, then $\Gamma \rightarrow \Delta$ is valid. In this case $\Gamma \models Q_1C_1$, and $C_1 \models N_1$ so $Q_1C_1 \models Q_1N_1$. Hence $\Gamma \models Q_1N_1$. By the definition of $\mathscr{S}$, $Q_1N_1 \models \Delta$ so $\Gamma \models \Delta$. That is, $\Gamma \rightarrow \Delta$ is valid.

Conversely, suppose that $\Gamma \rightarrow \Delta$ is valid. Then the hypotheses of Lemma 4 are satisfied for Γ and Δ. Let $\langle \Gamma, \cdots, Q'C', Q'N', \cdots, \Delta\rangle$ be the L-deduction whose existence is asserted by the application of Lemma 4 to Γ and Δ. By Lemma 8 and 6 we may delete all but $\leqq g(\Delta)$ main disjuncts of $Q'N'$ and all quantifier occurrences thereby made vacuous to obtain a formula $Q''N''$ such that Δ can be obtained from $Q''N''$ by applications of disassembling operations and such that if C'' is the subconjunction of C' formed with conjuncts the atomic formulas of which occur in $Q''N''$ then $C'' \models N''$. Thus $\langle C'', Q''N''\rangle$ or an ordered pair congruent to it is in $\mathscr{S}$. The L-deduction $\langle \Gamma, \cdots, Q'C'\rangle$ can be extended to a modified symmetric L-deduction $\langle \Gamma, \cdots, Q'C', \cdots, Q''C''\rangle$ so that the answer is yes in step three for one member of $\mathscr{S}$.

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UNIVERSITY OF CALIFORNIA, DAVIS