A REPRESENTATION OF THE BERNOULLI NUMBER B_n

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The function $\sigma_n(\nu)$ and the polynomial $\phi_n(\nu)$ have been defined in [2] and [3] respectively. Let $J_\nu(z)$ be the Bessel function of the first kind, and $j_{\nu,m}$ be the zeros of $z^{-\nu}J_\nu(z)$, then

$$(1) \qquad \sigma_n(\nu) = \sum_{m=1}^{\infty} (j_{\nu,m})^{-2n} , \qquad n = 1, 2, 3, \cdots,$$

$$(2) \qquad \phi_n(\nu) = 4^n \prod_{k=1}^{n} (\nu + k)^{[n/k]} \sigma_n(\nu) ,$$

where $[x]$ is the greatest integer $\leqq x$.

$\sigma_n(\nu)$ is a rational function of ν with rational coefficient. $\phi_n(\nu)$ is a polynomial in ν with positive integral coefficients, and has degree $1 - 2n + \sum_{k=1}^{n} [n/k]$. All real zeros of $\phi_n(\nu)$ lie in the interval $(-n, -2)$. These polynomials also satisfy certain congruences [3].

Let B_n and G_n be the Bernoulli and Genocchi numbers:

$$(3) \qquad B_n = \sum_{k=0}^{n} \binom{n}{k} B_k , \qquad n \neq 1 ,$$

$$(4) \qquad G_n = 2(1 - 2^n) B_n .$$

The symmetric function $\sigma_n(\nu)$ can be expressed in terms of the Bernoulli and Genocchi numbers by means of the following formulas:

$$(5) \qquad \sigma_n\left(\frac{1}{2}\right) = (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} B_n ,$$

$$(6) \qquad \sigma_n\left(-\frac{1}{2}\right) = (-1)^n \frac{2^{2n-2}}{(2n)!} G_n ,$$

where by B_n and G_n we understand the even-suffix numbers B_{2n} and G_{2n} [2].

In a previous paper [4] a structure of $\phi_n(\nu)$ has been given. This in turn leads, through (2), to a corresponding structure of $\sigma_n(\nu)$. And since for $\nu = 1/2$, $\sigma_n(\nu)$ is expressible in terms of the Bernoulli number B_n it is natural to enquire about a structure of B_n corresponding to that of $\sigma_n(\nu)$.

Three formulas from a previous paper [4, (8), (15), (18)] will be used here. They are written down as formulas (7), (8) and (9).

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(7) $$\phi_n(\nu) = \sum_{k=1}^{[n/2]} \alpha_k \Omega_k(\nu)\phi_k(\nu)\phi_{n-k}(\nu) ,$$

where $\alpha_k = 2$, $k < [n/2]$, and for $k = [n/2]$,

$$\alpha_k = \begin{matrix} 2 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even,} \end{matrix}$$

$$\Omega_k(\nu) = \prod_{s=1}^{n-1} (\nu + s)^{\varepsilon(s,k,n)},\ \varepsilon(s, k, n) = \left[\frac{n}{s}\right] - \left[\frac{k}{s}\right] - \left[\frac{n-k}{s}\right].$$

(8) $$\phi_n(\nu) = \sum_{i=1}^{c(n)} 2^{n_i} \prod_{j=2}^{n-1} (\nu + j)^{n_{ij}} ,$$

where (i) $c(n)$ is the number of components of $\phi_n(\nu)$,

(ii) at most one $n_i = 0$,

(iii) $\sum_{i=1}^{c(n)} 2^{n_i} = n^{-1} \binom{2n-2}{n-1}$,

(iv) $\sum_{j=2}^{n-1} n_{ij} = 1 - 2n + \sum_{s=1}^{n} \left[\frac{n}{s}\right]$, for all i, and

(v) given an integer s, $1 < s < n$, $n > 3$, there exists i such that $0 < n_{is} \leqq [n/s]$.

(9) $$c(n) = \sum_{k=1}^{[n/2]} c(k)c(n-k) , \quad c(1) = 1 .$$

We shall obtain specific information about certain components of $\phi_n(\nu)$ which will be used later on. We begin with

(10) For $2 < s < n$, $(\nu + s)^{[n/s]}$ is a factor of some component of $\phi_n(\nu)$, and if $s = 2$, $(\nu + s)^{[n/s]-1}$ is a factor of a component of $\phi_n(\nu)$, $n > 3$.

Consider the first part of the statement. We observe that if $2 < s < n$, the statement is true for $n = 4, 5, 6, 7$ (see [3]). Assume the statement to be true for $k = 4, 5, \cdots, n-1$. Take the kth term of (7), $T_k = \alpha_k \Omega_k(\nu)\phi_k(\nu)\phi_{n-k}(\nu)$, $k \geqq 4$, $n \geqq 8$. Then some component of $\phi_k(\nu)\phi_{n-k}(\nu)$ has a factor $(\nu + s)^{[k/s]+[(n-k)/s]}$. However, $\Omega_k(\nu)$ has a factor $(\nu + s)$ if and only if $\varepsilon(s, k, n) = 1$. Therefore, some component of T_k which is a component of $\phi_n(\nu)$ has a factor

$$(\nu + s)^{[k/s]+[(n-k/s)]+\varepsilon(s,k,n)} = (\nu + s)^{[n/s]} .$$

The second part of the statement may be proved by a similar method.

The following may be obtained from (10)

(11) $$\begin{aligned} \max (n_{ij}) &= [n/j],\ 2 < j < n, \\ &= [n/2] - 1,\ j = 2 . \end{aligned}$$

(12) For $s > 2$, and m such that $(2m + 1)s + m < n$, the product

$$\prod(n, m) \equiv \prod_{\lambda=0}^{m} \{\nu + (2\lambda + 1)s + \lambda\}^{[n/(2\lambda+1)s+\lambda]}$$

is a factor of some component of $\phi_n(\nu)$.

Proof. We shall use induction. Define the set of integers

$$I_m = \{\text{integers } x\colon (2m + 1)s + m < x < (2m + 3)s + m + 1\},\quad m = 0, 1, 2, \cdots .$$

If $n \in I_0$, $\Pi(n, 0) = (\nu + s)^{[n/s]}$ and $(\nu + s)^{[n/s]}$ is a factor of some component of $\phi_n(\nu)$ by (10). Assume that for $k \leqq m - 1$, $n \in I_k$ implies $\Pi\ (n, k)$ is a factor of some component of $\phi_n(\nu)$. Let $n \in I_m$, and suppose $n = (2m + 1)s + m + i$, $1 \leqq i \leqq 2s$. Then $n - 2i = (2m + 1)s + m - i \in I_{m-1}$. Take formula (7), and consider the $(2i)$-th term,

$$T_{2i} = \alpha_{2i}\Omega_{2i}(\nu)\phi_{2i}(\nu)\phi_{n-2i}(\nu) .$$

By induction hypothesis there are components V_1 of $\phi_{2i}(\nu)$ and V_2 of $\phi_{n-2i}(\nu)$ such that Π_1 and Π_2 are factors of V_1 and V_2 respectively, where

$$\Pi_1 = \prod_{\lambda=0}^{p} \{\nu + (2\lambda + 1)s + \lambda\}^{[2i/(2\lambda+1)s+\lambda]} ,$$

$$\Pi_2 = \prod_{\lambda=0}^{m-1} \{\nu + (2\lambda + 1)s + \lambda\}^{[n-2i/(2\lambda+1)s+\lambda]} ,$$

and $(2p + 1)s + p < 2i$, $(2m - 1)s + m - 1 < n - 2i$. Since the term T_{2i} yields a component of $\phi_n(\nu)$, we have that $\alpha_{2i}\Omega_{2i}(\nu)\ \Pi_1\Pi_2$ is a factor of $\alpha_{2i}\Omega_{2i} V_1 V_2 = V$, where V is a component of $\phi_n(\nu)$. However,

$$\Omega_{2i}(\nu) = \prod_{r=1}^{n-1} (\nu + r)^{\varepsilon(r,2i,n)} .$$

Hence after a simplification, we obtain

$$\alpha_{2i}\Omega_{2i}(\nu)\Pi_1\Pi_2 = P(\nu)\Pi(n, m) ,$$

where $P(\nu)$ is a polynomial in ν of degree $\geqq 0$. Thus the term T_{2i} yields a component V of $\phi_n(\nu)$ such that $\Pi(n, m)$ is a factor of V.

(13) $V(n) \equiv 2^{n-2} \prod_{r=2}^{[n/2]} (\nu + r)^{[n/r]-1}$, $n \geqq 2$, is a component and the only component of $\phi_n(\nu)$ with the greatest numerical factor 2^{n-2}.

Proof. First we shall show that $V(n)$ is a component of $\phi_n(\nu)$. Observe that for $n = 2, 3, 4$, $V(n)$ is a component of $\phi_n(\nu)$. Assume: $V(m)$ is a component of $\phi_m(\nu)$, $2 \leqq m \leqq n - 1$. Consider the first term

T_1 of (7): $T_1 = 2\Omega_1(\nu)\phi_{n-1}(\nu)$. There is a component $V(n-1)$ of $\phi_{n-1}(\nu)$ such that

$$V(n-1) = 2^{n-3} \prod_{r=2}^{[n-1/2]} (\nu + r)^{[n-1/r]-1} .$$

Hence $2\Omega_1(\nu)V(n-1)$ is a component of $\phi_n(\nu)$. Substituting the expression for $\Omega_1(\nu)$, we obtain

$$2\Omega_1(\nu)V(n-1) = 2^{n-2} \prod_{r=2}^{[n/2]} (\nu + r)^{[n/r]-1} = V(n) .$$

The second part of the statement that $V(n)$ is the only component of $\phi_n(\nu)$ with the greatest numerical factor 2^{n-2} may be proved by induction.

(14) $$V_1(n) \equiv \frac{(\nu + 3)V(n)}{4(\nu + 2)}, \quad n \geqq 4, \text{ is a component of } \phi_n(\nu) .$$

This may be proved by considering the first term T_1 of (7) and using induction.

(15) For $\nu = 1/2$, the value of $V_1(n)$ is less than the value of any other component of $\phi_n(\nu)$.

Proof. Take the kth term T_k of (7),

$$T_k = \alpha_k\Omega_k(\nu)\phi_k(\nu)\phi_{n-k}(\nu) .$$

$V_1(n)$ is obtained from T_1. For $k = 2, 3$ and $\nu = 1/2$, the smallest components of T_k correspond to the smallest components of $\phi_{n-k}(\nu)$, because $\alpha_k\Omega_k(\nu)\phi_k(\nu)$ is constant. We observe that for $n = 4, 5, 6, 7$, $V_1(n)$ is less than any other component of $\phi_n(\nu)$, $\nu = 1/2$. Assume that for $\nu = 1/2$, $V_1(m)$ is less than any other component of $\phi_m(\nu)$, $4 \leqq m < n$. Using the induction hypothesis it is seen that for $\nu = 1/2$, $V_1(n)$ is less than any component obtained from T_k, $k = 2, 3$. For $k \geqq 4$, $n \geqq 8$, $\alpha_k\Omega_k(\nu)V_1(k)V_1(n-k)$ is a component of $\phi_n(\nu)$ and its value at $\nu = 1/2$ is less than the value of any other component obtained from T_k. Thus among all components of $\phi_n(\nu)$ there is a set S of exactly $[n/2]$ minimum components

$$S = \left\{\alpha_k\Omega_k(\nu)V_1(k)V_1(n-k) : 1 \leqq k \leqq \left[\frac{n}{2}\right]\right\} .$$

Obviously $V_1(n) \in S$. We claim: $V_1(n)$ is less than any other element of S. It suffices to show that

$$\lim_{\nu \to 1/2} \frac{V_1(n)}{\alpha_k\Omega_k(\nu)V_1(k)V_1(n-k)} < 1 , \quad k \neq 1 .$$

A verification of this inequality is left to the reader.

Let (8) be multiplied by $2^{2-n}(\nu+2)^{1-[n/2]}$. Then considering (7), induction yields the following

(16) $$n - [n/2] - 1 \geqq n_i - n_{i2} .$$

THEOREM. *The Bernoulli number B_n has the following representation*:

(17) $$B_n = \frac{(-1)^{n-1}(2n)!}{20\cdot 6^n\cdot(2n+1)} \sum_{i=1}^{c(n)} (2^{r_i}a_i)^{-1} ,$$

where 1. $\sum_{i=1}^{c(n)} (2^{r_i}a_i)^{-1} \equiv$ 30 *if* $n=1$, 5 *if* $n=2$, 1 *if* $n=3$; *for* $n>3$,

2. $a_i = \prod_{m=1}^{n-2} (2m+3)^{i_m}$, $\sum_{m=1}^{n-2} i_m = n-3$, $0 \leqq i_m < \left[\frac{n}{2}\right]$,

3. $2^{r_1}a_1 = \frac{4}{7}\cdot 5\cdot 7\cdot 9\cdots\cdots(2n-1)$,

$\frac{4}{7}\cdot 5\cdot 7\cdot 9\cdots\cdots(2n-1) > 2^{r_i}a_i > 7^{n-3}$, $i>1$,

4. $r_1 = 2$, $r_2 = 0$; $r_i \neq 0$, $i \neq 2$,

5. $\sum_i 2^{-r_i} = 2^{2-n}\cdot n^{-1}\binom{2n-2}{n-1}$,

6. *the g.c.d.* $(2^{r_1}a_1, 2^{r_2}a_2, \cdots) = 1$, *and*

7. *given an odd integer* s, $5 < s \leqq 2n-1$, *there is* i *such that* $s^{[2n/s-1]}$ *divides* a_i; *if* $s=5$ *then* $s^{[2n/s-1]-1}$ *divides* a_i, *for some* i.

Proof. Substitute (2) in (8) and let $\nu = 1/2$, then in view of (5) the following is obtained after some simplification

$$B_n = \frac{(-1)^{n-1}(2n)!}{20\cdot 6^n\cdot(2n+1} \sum_{i=1}^{c(n)} \left\{2^{r_i}\cdot 5^{-1}\prod_{k=2}^{n-1}(2k+1)^{[n/k]-n_{ik}}\right\}^{-1} ,$$

where $r_i = n-2-n_i \geqq 0$ by (13). Note that

$$\prod_{k=2}^{n-1}(2k+1)^{[n/k]-n_{ik}}$$

is divisible by 5 for each i, because by (11) $[n/2]-n_{i2} \geqq 1$. And $-1+\sum_{k=2}^{n-1}\{[n/k]-n_{ik}\} = n-3$ by (8, (iv)). Therefore, we may write

$$a_i \equiv 5^{-1}\prod_{k=2}^{n-1}(2k+1)^{[n/k]-n_{ik}} = \prod_{m=1}^{n-2}(2m+3)^{i_m} ,$$

where $\sum_{m=1}^{n-2} i_m = n - 3,\ 0 \leqq i_m < [n/2]$ by (11).

$$\text{and}\quad i_1 = [n/2] - 1 - n_{i2}\,,$$
$$i_m = [n/h] - n_{ih}\,,\quad h = m + 1\,,\quad m > 1\,.$$

Thus property 2 is verified.

By (13) and (14), $V(n)$ and $V_1(n)$ are components of $\phi_n(\nu)$. If the components of $\phi_n(\nu)$ are ordered in such a way that $V_1(n)$ is the first and $V(n)$ is the second component, then for $\nu = 1/2$, the values of $V_1(n)$ and $V(n)$ correspond to $2^{r_1}a_1$ and $2^{r_2}a_2$. By actual calculation it is seen that $2^{r_1}a_1 = 4/7\cdot5\cdot7\cdot9\cdot\cdots\cdot(2n-1)$, $r_1 = 2$, $r_2 = 0$. Therefore, by (15) $2^{r_i}a_i < 4/7\cdot5\cdot7\cdot9\cdot\cdots\cdot(2n-1)$, $i > 1$. Since $r_i = n - 2 - n_i$, it follows from (13) that $r_i \neq 0$, if $i \neq 2$. By (16),

$$r_i = n - 2 - n_i \geqq [n/2] - 1 - n_{i2} = i_1\,.$$

Hence for each i,

$$2^{r_i}a_i = 2^{r_i}\sum_{m=1}^{n-2}(2m+3)^{i_m}$$
$$= 2^{r_i - i_1}10^{i_1}\sum_{m=2}^{n-2}(2m+3)^{i_m} > 7^{n-3}\,.$$

Properties 3 and 4 are proved. Property 5 is derived from (8, (iii));

$$\sum_i 2^{-r_i} = \sum_i 2^{2-n+n_i} = 2^{2-n}\sum_i 2^{n_i} = 2^{2-n}n^{-1}\binom{2n-2}{n-1}\,.$$

Concerning property 6, in view of 4, it suffices to prove that g.c.d. $(a_1, a_2, \cdots) = 1$. Note that each a_i is a product of odd integers. By (12), $\prod(n, m)$ is a factor of a component, say V_p, of $\phi_n(\nu)$. However,

$$V_p 2^{2-n}\prod_{k=2}^{n-1}(\nu + k)^{-[n/k]} = \{P(\nu)\}^{-1}\,,$$

where $P(\nu)$, a product of linear factors, is a polynomial in ν of degree >0. $P(\nu)$ is not divisible by any factor of $\Pi(n, m)$. For $\nu = 1/2$, $\Pi(n, m)$ is divisible by all odd factors $q(2s+1)$, $q = 1, 3, 5, \cdots$, which are less than n. Therefore, for $\nu = 1/2$, $P(\nu)$ is not divisible by any factor $q(2s+1)$. Since $P(\nu)$, for $\nu = 1/2$, corresponds to some a_i the latter does not contain any factor $q(2s+1)$. Thus for each $s > 2$, there is a_i which is not divisible by $q(2s+1)$, $q = 1, 3, 5, \cdots$. Hence the g.c.d. $(a_1, a_2,\ \ ) = 1$.

Suppose $s = 2m + 1$. Take a component V' of $\phi_n(\nu)$ which does not have the factor $(\nu + m)$. It may be shown that there exists such a component V'. Then

$$V'2^{2-n}\prod_{k=2}^{n-1}(\nu+k)^{[n/k]}=\{Q(\nu)\}^{-1},$$

where the polynomial $Q(\nu)$ has a factor $(\nu+m)^{[n/m]}$, $m>2$. For $\nu=1/2$, $Q(\nu)$ corresponds to some a_i and $(\nu+m)^{[n/m]}$ corresponds to the factor $(2m+1)^{[n/m]}$ of a_i. However, if $m=2$ than $5^{[n/2]-1}$ is a factor of a_i for some i. This completes the proof of the theorem.

We remark that the Genocchi number G_n and the numbers defined by L. Carlitz (see [1]).

$$a_r=2^{2r}r\,!\,(r-1)!\,\sigma_r(0)\,,$$

may be expressed in a manner similar to (17). In fact, for the numbers a_r we have the following

$$a_r=\{(r-1)\,!\}^2\sum_{i=1}^{c(r)}2^{ri}\prod_{k=2}^{r-1}k^{k_{ik}-[r/k]}\,. \tag{18}$$

A list of first few Bernoulli numbers expressed according to the theorem is given below.

$$B_1=\frac{2!}{20\cdot 6\cdot 3}(30)\,,$$

$$B_2=-\frac{4!}{20\cdot 6^2\cdot 5}(5)\,,$$

$$B_3=\frac{6!}{20\cdot 6^3\cdot 7}(1)\,,$$

$$B_4=-\frac{8!}{20\cdot 6^4\cdot 9}\left(\frac{1}{2^2\cdot 5}+\frac{1}{7}\right),$$

$$B_5=\frac{10!}{20\cdot 6^5\cdot 11}\left(\frac{1}{2^2\cdot 5\cdot 9}+\frac{1}{7\cdot 9}+\frac{1}{2\cdot 5\cdot 7}\right),$$

$$B_6=-\frac{12!}{20\cdot 6^6\cdot 13}\left(\frac{1}{2^2\cdot 5\cdot 9\cdot 11}+\frac{1}{7\cdot 9\cdot 11}+\frac{1}{2\cdot 5\cdot 7\cdot 9}+\frac{1}{2\cdot 5\cdot 7\cdot 11}+\frac{1}{2^2\cdot 5\cdot 7^2}+\frac{1}{2^3\cdot 5^2\cdot 9}\right),$$

$$B_7=\frac{14!}{20\cdot 6^7\cdot 15}\left(\frac{1}{2^2\cdot 5\cdot 9\cdot 11\cdot 13}+\frac{1}{7\cdot 9\cdot 11\cdot 13}+\frac{1}{2\cdot 5\cdot 7\cdot 9\cdot 11}+\frac{1}{2\cdot 5\cdot 7\cdot 11\cdot 13}+\frac{1}{2\cdot 5\cdot 7\cdot 9\cdot 13}+\frac{1}{2\cdot 5\cdot 7^2\cdot 9}+\frac{1}{2^2\cdot 5^2\cdot 7\cdot 11}+\frac{1}{2^2\cdot 5\cdot 7^2\cdot 13}+\frac{1}{2^3\cdot 5^2\cdot 7\cdot 9}+\frac{1}{2^3\cdot 5^2\cdot 9\cdot 11}+\frac{1}{2^3\cdot 5^2\cdot 9\cdot 13}\right).$$

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