NATURAL SUMS AND ABELIANIZING

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Introduction. It is a well known result, essentially due to Mac Lane [13], that the addition in an abelian category is determined by the multiplication, i.e. every categorical isomorphism is additive. The sum $f + g$ of two mappings $f: A \to B$, $g: A \to B$ is defined—in fact, overdefined—by the following diagram involving natural isomorphisms of free sums $A \vee A$ and direct products $A \times A$.

$$\begin{tikzcd}
 & A \vee A \arrow[r, "f \vee g"] \arrow[d, leftrightarrow] & B \vee B \arrow[r, "\nabla"] \arrow[d, leftrightarrow] & B \\
A \arrow[r, "\Delta"] & A \times A \arrow[r, "f \times g"] & B \times B &
\end{tikzcd}$$

In many non-abelian categories, especially in the category of all groups, one has not isomorphisms but naturally distinguished mappings $A \vee A \to A \times A$. The definition of a sum $f + g$ becomes a problem, which for groups may be best posed in the form

$$\begin{tikzcd}
 & A \vee A \arrow[r, "f \vee g"] \arrow[d, two heads] & B \vee B \arrow[r, "\nabla"] & B \\
A \arrow[r, "\Delta"] & A \times A \arrow[urr, dotted] & &
\end{tikzcd}$$

This form is chosen [6] because $A \vee A \to A \times A$ is an epimorphism $\twoheadrightarrow$, so that there is at most one mapping $\cdots\cdots>$ making the diagram commutative; f and g are called *summable* if this mapping exists, and then their sum is the composed mapping $A \to A \times A \to B$. This partially defined addition, called the *natural sum*, turns out to be the same as Fitting's [2] pointwise multiplication of homomorphisms with commuting values; it has been used in extensive investigations of direct decompositions by Kuroš [9] and Livšic [10, 11, 12].

Of course a dual diagram applies in categories which have naturally distinguished monomorphisms $A \vee A \rightarrowtail A \times A$. In many categories there is a naturally distinguished mapping $A \vee A \to A \times A$ which is neither epimorphic nor monomorphic. This is the situation in the category of homotopy types of topological spaces with base point, and the theory of homotopy operations, despite some analogies, apparently requires the apparatus currently in use there [1] which is not related to Mac Lane's fundamental diagram. However, Mac Lane's approach

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can be pushed further when there is a natural factorization $A \vee A \twoheadrightarrow M \rightarrowtail A \times A$. I indicated this development in [6], assuming certain axioms and asserting that the resulting partial sum satisfies the axioms of Kuroš [9]. This is wrong. The present paper defines the partial sum more generally and determines conditions for the distributive law (Axiom II of [9]). The decomposition theory as developed by Livšic seems to require the distributive law. Without it—for example, in non-associative binary systems with zero—we have a theory of *left abelian* and *right abelian* objects, which behave like suspension spaces and H-spaces, also, there is a right abelianization of any object, like a commutator quotient group (and dually); and the two-sided abelian objects form a full subcategory (a retract of the given category) having $f + g$ always defined, commutative, associative, and distributive. Such a category we call *maclanian.*

Groups form a left maclanian category, i.e. every object is left abelian. There is a curious result to the effect that in every left maclanian algebraic category, the distributive law holds; here "left" cannot be replaced by "right".

The development depends on a distinguished factorization $A \vee A \twoheadrightarrow M \rightarrowtail A \times A$, i.e. on a bicategorical structure. Such a structure is unique for groups but not in general, so that a given category may have more than one operation of this type; non-uniqueness occurs at least in some artificial algebraic examples and it occurs in a natural way in the category of Banach spaces and norm-decreasing operators.

1. Maclanian categories. We shall be interested in certain categories $\mathscr{C}$ having zero mappings and finite free sums and direct products. Note that there exist sum and product functors $\vee : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, $\times : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, unique up to natural equivalence; for instance, $\vee$ takes each pair of objects (A, B) to a free sum $A \vee B$, with the obvious behavior on mappings. If $\vee$ and $\times$ are naturally equivalent, we call $\mathscr{C}$ *maclanian.*

The basic facts on maclanian categories are indicated in [13], and details added here (found by me at various times between 1958 and 1962) have been described by two referees as "largely in the folklore". The first few details lead up to the proposition (1.5) that if there is some natural equivalence between $\vee$ and $\times$ then there is a natural equivalence of the precise form assumed in Mac Lane's paper [13]. Then we prove the uniqueness of $+$ and the representation theorem, results which are almost fully given in [13]; and we conclude with two results to the effect that every operation is a sum of unary operations. I do not know how much is folklore.

We define an n-ary *partial operation* in a category $\mathscr{C}$ as a function Q whose domain is some class of ordered n-tuples $(f_1, \cdots, f_n)$ of coterminal mappings of $\mathscr{C}$ and whose value $Q(f_1, \cdots, f_n)$ is coterminal with the f_i. Q is *natural* if $Q(gf_1, \cdots, gf_n) = gQ(f_1, \cdots, f_n)$ and $Q(f_1e, \cdots, f_ne) = Q(f_1, \cdots, f_n)e$ whenever the expressions on the right of these equations are defined; Q is *total* if its domain is the class of all coterminal n-tuples.

To simplify notation we state 1.1 and 1.2 only for $n = 2$. The generalizations to $n > 0$ for 1.1 and $n > 1$ for 1.2 will be left as (easy) exercises, and will be invoked in 1.3 and 1.9. We still need notation for projections $p_1: A \times B \to A$, $p_2: A \times B \to B$; a product $A \times A$ also has a diagonal map $\Delta_A: A \to A \times A$. With similar notation (i_1, i_2, ∇_A) for sums, we have

1.1. *In any category having finite sums and products there is a one-to-one correspondence between natural total binary operations Q and natural transformations Φ: $\times \to \vee$, established by the following formulas*:

$$\begin{aligned} \Phi_{(A,B)} &= Q(i_1p_1, i_2p_2) ; \\ Q(f_1, f_2) &= \nabla_B \vee (f_1, f_2)\Phi_{(A,A)}\Delta_A \\ &= \nabla_B\Phi_{(B,B)} \times (f_1, f_2)\Delta_A \end{aligned}$$

for $f_1, f_2: A \to B$.

The routine verification is omitted.

For the next lemmas we want the matrix notation; a matrix (f_{jk}) of mappings $f_{jk}: X_k \to Y_j$ denotes the mapping f from the free sum of the X_k to the direct product of the Y_j whose coordinates p_jfi_k are f_{jk}. In particular, a row vector is a free sum mapping; a column vector may be conveniently printed as $(g_1, \cdots, g_m)^\times: A \to B_1 \times \cdots \times B_m$. When there are zero mappings there is also the notion of a *diagonal matrix* $D(h_1, \cdots, h_n)$, which is square with zeros off the diagonal and h_k down the diagonal.

1.2. *If $\mathscr{C}$ has free sum and direct product functors $\vee$, $\times$, and there exists a natural transformation from $\vee$ to $\times$, then $\mathscr{C}$ has zero mappings.*

Proof. Given $\Psi: \vee \to \times$. For any objects A and B, consider $\Psi_{(A,B)}: A \vee B \to A \times B$, which may be written as a 2 by 2 matrix (f_{jk}). We claim $f_{21}: A \to B$ is a zero mapping. For any $h: B \to B$, by naturality of Ψ, $(f_{jk}) \vee (1_A, h) = \times(1_A, h)(f_{jk})$; and taking coordinates, $f_{21} = hf_{21}$. Similarly (in fact, dually), for every $e: A \to A$, $f_{21}e = f_{21}$.

1.3. Any natural transformation $\Psi: \vee \to \times$ is determined by a collection of matrices which, by the proof of 1.2, must be diagonal matrices $D(d_1(A_1, A_2), d_2(A_1, A_2))$. More: *the mapping* $d_k(A_1, A_2): A_k \to A_k$ *is determined by the object* A_k *and the index* k. The proof is a trivial computation; and the result holds also for n-fold sum and product functors on $\mathscr{C} \times \cdots \times \mathscr{C}$ to $\mathscr{C}$.

Thus a natural transformation from $\vee$ to $\times$ is determined by an (arbitrary) ordered pair of unary natural total operations. Note that in the maclanian case we have a more or less natural one-to-one correspondence between n-ary natural total operations and n-tuples of unary ones. The reservation "more or less" is removed by 1.5 and the correspondence is exhibited in 1.9.

1.4. *If* Φ *is a natural transformation from the identity to the identity and* $C = A_1 \times A_2$ *then* $\Phi_C = \times(\Phi_{A_1}, \Phi_{A_2})$.

Proof. The coordinate functions p_k satisfy $p_k\Phi_C = \Phi_{A_k}p_k$.

Let Π denote the natural transformation from $\vee$ to $\times$ made up by all the identity matrices $D(1_A, 1_B)$.

1.5. THEOREM. *For any category having the sum and product functors* $\vee$ *and* $\times$ *and some natural equivalence from* $\vee$ *to* $\times$, *the category is maclanian* (*by* 1.2) *and* Π *is a natural equivalence.*

Proof. Consider a natural equivalence Φ given by matrices with diagonal entries $d_k(A)$. The inverse transformation Φ^{-1} corresponds (by 1.1) to a binary operation Q; then the inverse of Π is given by the operation R defined $R(f, g) = Q(fd_1(A), gd_2(A))$ for $f, g: A \to B$. To check this, using 1.4 and its dual, is routine.

1.6. THEOREM. *In a category having zero mappings and finite sums and products, a binary natural total operation* $+$ *satisfying* $f + 0 = 0 + f = f$ *identically must be unique, commutative, and associative; it corresponds* (*by* 1.1) *to the inverse of* Π.

To prove this, use the formulas of 1.1 to write out the conditions $\Phi\Pi = 1$, $\Pi\Phi = 1$, and check that although Π and its inverse are unique only relative to a choice of $\vee$ and $\times$, $+$ is absolutely unique.

The operation $+$ in a maclanian category which exists by 1.5 and 1.1 and is characterized by 1.6 is called the *natural sum.*

The next result (Mac Lane's representation theorem) will be

1.7. THEOREM. *Every small maclanian category is isomorphic* *with a full category of commutative semigroups with operators, under an isomorphism preserving direct products.*

The smallness assumption is weakened later (3.6); it probably cannot be altogether removed, though I have no example. Mac Lane [13] used a much weaker assumption, omitted the operators, and did not get a full representation. However, the operators are by now a standard device; 1.7 as stated is widely known; and I outline the proof only for reference in the proof of 3.6.

For any object A of the small maclanian category $\mathscr{C}$, let A' be the weak direct sum of all the additive semigroups of mappings Map (X, A), X an object of $\mathscr{C}$. For any mapping $f: A \rightarrow B$, let $f': A' \rightarrow B'$ be the homomorphism which takes each mapping $e: X \rightarrow A$ to fe (extended linearly over the rest of A'). This represents $\mathscr{C}$ isomorphically in semigroups. Introduce a semiring R of operators: the weak direct sum of all Map (X, Y), with multiplication $f^*g = fg$ if fg exists in $\mathscr{C}$, $f^*g = 0$ if fg does not exist in $\mathscr{C}$, extended bilinearly. Then each A' is a right ideal of R; let R operate by right multiplication. A routine check establishes 1.7.

The representation is far from unique. However, there are essentially no other algebraic representations except those which arise from cutting down R. Precisely, we define a *maclanian category of algebras* as a full category of algebras (not necessarily small) which is a maclanian abstract category and is closed under finite algebraic (Cartesian) direct product.

1.8. *In any maclanian category of algebras the natural sum is defined pointwise by an operation* $+$ *on elements which makes the algebras commutative semigroups. Every* n*-ary operation of these algebras has the form* $e_1(x_1) + \cdots + e_n(x_n)$, *where all* e_i *are* $+$*-endomorphic operations.*

Proof. Under these hypotheses we may identify free sum and Cartesian product. Then $x + y$ is defined as $\nabla((x, y))$. From 1.1, the natural sum is defined pointwise by $+$; hence the algebras are commutative semigroups under $+$. For an n-ary algebraic operation W, define $e_i(x) = W(0, \cdots, x, \cdots, 0)$; we evaluate $W(x_1, \cdots, x_n)$ in A by considering $z_i = (0, \cdots, x_i, \cdots, 0)$ in A^n. The same device shows that every unary algebraic operation is $+$-endomorphic.

I do not know whether there is a maclanian category of algebras for which $+$ is not expressible in terms of the algebraic operations and 0. If 0 is not allowed, the abelian groups with the sole operation $x + y + z$ provide an example.

1.9. *Every natural total operation on a maclanian category is a sum of unary operations, and the unary operations are* $+$*-endomorphic.*

This can be deduced from the proofs of 1.7 and 1.8, or less artificially, from 1.1, 1.2 and 1.3.

2. Addition in bicategories.

A category with a distinguished factorization of mappings into epimorphisms and monomorphisms, satisfying suitable axioms, is called a *bicategory*. The usual axioms (which contain some redundancy) are as follows [5]. Certain epimorphisms are called *projections*; certain monomorphisms are called *injections*. Both the projections and the injections are closed under multiplication; isomorphisms, but no other mappings, are at once projections and injections; every mapping has the form jp, where j is an injection and p a projection, and this factorization is unique up to an isomorphism z (which would give $(jz^{-1})(zp)$).

When dealing with bicategories, as we shall through most of the rest of this paper, we specialize the two-headed arrow notation; $A \twoheadrightarrow B$ denotes a projection, $A \rightarrowtail B$ an injection. In a bicategory with zero mappings, finite sums and finite products, any coterminal pair of mappings f, g from A to B determines a diagram

$$\begin{tikzcd} A \vee A \arrow[rr, "(f\,g)"] \arrow[dr, two heads] & & B \\ & M \arrow[ur, dotted] \arrow[dr, tail] & \\ A \arrow[ur, dotted] \arrow[rr, "(f\,g)^{\times}"'] & & B \times B \end{tikzcd} \tag{2.0}$$

in which $A \vee A \twoheadrightarrow M \rightarrowtail B \times B$ is a distinguished factorization of $D(f, g)$. We call f and g *summable* if the diagram can be completed, commutatively, with mappings $w: A \to M$, $k: M \to B$; then the sum $f + g$ is defined to be kw. The sum is uniquely defined because projections are epimorphic and injections monomorphic. This defines a binary partial operation $+$, not necessarily natural, which we call the *partial sum*.

It is obvious that $f + g = g + f$ and $f + 0 = f$. To support the Kuroš-Livšic decomposition theory, the partial sum should also be natural and satisfy certain restricted associativity conditions. As to naturality, the basic lemmas are

2.1. *For a projection* $p: P \twoheadrightarrow A$ *and any mappings* $f: A \to B$, $g: A \to B$, $fp + gp$ *exists if and only if* $f + g$ *exists.*

2.2. *Whenever both $fe + ge$ and $(f + g)e$ exist, they are equal.*

Thus the problem of naturality reduces to the case of $fe + ge$ where e is an injection, and the dual. This is not entirely unmanageable; we establish naturality for an extensive collection of algebraic bicategories in 4.1.

As for associativity, we can prove essentially all of Kuroš' Axiom I [9; as changed in the note of correction]. The missing portion, like the missing Axiom V (see my previous paper [6]) is not used in the Kuroš-Livšic applications [9, 10].

Toward the proofs, we introduce the notation $f(A)$ for the middle space M of a factorization $A \twoheadrightarrow M \rightarrowtail B$ of $f: A \to B$. Especially for any two objects C, D, the natural transformation Π gives us an object $\Pi_{(C,D)}(C \vee D)$ which we call the *distinguished weak product* and designate $C\sigma D$. The coordinate functions into $C \vee D$ and on $C \times D$ determine four coordinate functions for $C\sigma D$. We remark that σ is functorial. In fact, the factorization of all $\Pi_{(C,D)}$ yields a factorization of $\Pi: \vee \to \sigma \to \times$; from this it follows readily that σ is commutative and associative in the same sense that $\vee$ and $\times$ are.

We want the diagram

$$\tag{2.0*}\begin{tikzcd}
A \vee A \arrow[r, two heads] \arrow[dr, "q"'] & (fg)(A \vee A) \arrow[r, tail] & B \\
 & f(A)\sigma g(A) \arrow[u, dotted, "k_0"'] \arrow[dr, tail, "i"] & \\
A \arrow[r, two heads] & (fg)^{\times}(A) \arrow[u, dotted, "w_0"'] \arrow[r, tail] & B \times B
\end{tikzcd}$$

First we show that the middle space $M = D(f, g)(A \vee A)$ in (2.0) is just $f(A)\sigma g(A)$. Clearly this will follow if we justify the factorization

$$A \vee A \twoheadrightarrow f(A) \vee g(A) \twoheadrightarrow f(A)\sigma g(A) \rightarrowtail f(A) \times g(A) \rightarrowtail B \times B\,.$$

The middle of this, from $f(A) \vee g(A)$ to $f(A) \times g(A)$, is correct. We need the following lemma and its dual.

2.3. *In a bicategory having free sums, $f \vee g$ is a projection whenever f and g are projections.*

Proof. In connection with $f: A \twoheadrightarrow C$ and $g: B \twoheadrightarrow D$ consider the coordinate functions $i_1: A \to A \vee B$, $i_2: B \to A \vee B$, $j_1: C \to C \vee D$, $j_2: D \to C \vee D$. Take distinguished factorizations $f \vee g = kp$, $j_1 = lq$, $pi_1 = mr$. Now $lqf = j_1 f = (f \vee g)i_1 = kpi_1 = kmr$; this gives us two distinguished factorizations of the same mapping, and applying an isomorphism, we may assume $l = km$, $qf = r$. In particular, j_1 has the form ky_1, where y_1 is mq. Similarly j_2 is a multiple ky_2. Consider next $(y_1\ y_2)\colon C \vee D \to (f \vee g)(A \vee B)$. For $i = 1, 2$, the coordinate $k(y_1\,y_2)j_i = ky_i = j_i$. On a free sum this means $k(y_1\,y_2)$ is the identity. Since k is a monomorphic left factor of an identity, it is an isomorphism, and $f \vee g$ is a projection, as we wanted to prove.

Consequently (2.0*) arises from (2.0), as follows. Factor (fg) and $(fg)^{\times}$ as indicated. Factor k to $f(A)\sigma g(A) \twoheadrightarrow M \rightarrowtail B$, and observe that the two distinguished factorizations of (fg) may be identified; and treat w in the same way.

The proof of 2.1 now reduces to attaching projections $(p\ p)$ and p on the left of (2.0*), and looking. For 2.2, draw the appropriate diagram $(E \to A \to B)$ and see that all paths from E to $B \times B$ give the same mapping $(fg)^{\times}e$; since $f(A)\sigma g(A)$ goes into $B \times B$ monomorphically, the mapping $E \to B$ is also independent of path.

Now consider any pair of objects C, D, and the coordinate retractions $r_1\colon C\sigma D \to C \to C\sigma D$, $r_2\colon C\sigma D \to D \to C\sigma D$. We call $\{C, D\}$ a *right distributive pair* if for every injection $h\colon H \rightarrowtail C\sigma D$, h factors through the distinguished injective factor of $(r_1h\ r_2h)$. Defining *left distributive* dually, we have

2.4. THEOREM. *In a bicategory having finite free sums and direct products and zeros, the partial sum is natural if and only if every pair of objects is both left and right distributive.*

The proof is a routine computation; we omit it, using the space instead to restate the definitions more intuitively. The distinguished weak product $A\sigma B$ is the "subobject" of $A \times B$ "generated by $A \times 0$ and $0 \times B$". Precising these terms offers no difficulty once we note that equivalence of monomorphisms may be defined in the standard way [4] but subobjects must be equivalence classes of injections (not of arbitrary monomorphisms). These subobjects form at least an upper semi-lattice, the supremum of i and j coming from the distinguished factorization of $(i\,j)$. The requirement for right distributivity is just that every subobject (represented by) h in $C\sigma D$ is contained in the supremum of r_1h and r_2h.

To associativity: Kuroš' associativity axiom is complicated in his own terms [9], and stating it without introducing such terms is not worth considering. Unfortunately this means we must introduce (inter alia) a ternary operation $\Sigma(f_1, f_2, f_3)$ not definable in terms of binary $+$. The n-ary operations are essential for the decomposition theory [9, 10 11, 12], and infinitary operations occur also.

For a coterminal family $(f_1, \cdots, f_n)$ of mappings, one constructs a diagram like (2.0) around the corresponding diagonal matrix and row and column vectors; if the diagram can be completed, the family is called *summable* and the sum Σf_i is defined as the composed mapping coterminal with (f_i). 2.1–2.4 generalize trivially. There is an important detail; the problem of 2.4 is not magnified by the introduction of n-ary operations. *If every pair of objects is left* (*right*) *distributive, so is every n-tuple of objects.* The proof is straightforward.

Clearly a summable family $\{f_i\}$ is associatively summable; that is, every subset is summable, and any grouping of the whole family adds up to the same sum.

2.5. *Let $\{f_i: i \in I\}$ be a family of mappings $f_i: A \to A$. Suppose the index set I is partitioned into subsets $I_s(s \in S)$ having sums $g_s = \Sigma(f_i: i \in I_s)$; and $g_s f_i = f_i g_s = f_i$ for $i \in I_s$. If also Σg_s exists, then the whole family $\{f_i\}$ is summable.*

Proof. Let $w_s: A \to M_s$, $k_s: M_s \to A$ be the mappings accomplishing the addition of $\{f_i: i \in I_s\}$. Let $w^*: A \to M^*$, $k^*: M^* \to A$ add $\{g_s\}$. From the latter addition we want the distinguished factors of $D((g_s)) : ((q_s)) : A \vee \cdots \vee A \to M^*$, and $((i_s))^\times: M^* \to A \times \cdots \times A$. More fully, M^* is the distinguished weak product of the objects $g_s(A)$. The object M we must map through is the distinguished weak product of all $f_i(A)$, which we may regard as the product of the partial products M_s of $\{f_i(A): i \in I_s\}$. Then we map $g_s(A)$ to M_s by $w_s i_s$, and A to M by $w = \sigma(w_s i_s: s \in S)w^*$. We define $k: M \to A$ dually.

The distinguished injection from M to the direct product of copies A_i of A indexed by I, composed with w, turns out to have for sth block of coordinates just $((f_i: i \in I_s))^\times g_s$. The hypotheses say that this is $((f_i))$, as we want, and duality completes the proof.

The full axiom [9] incorporates 2.5 for infinite sums, the remark preceding 2.5, and finally, for finite sums, the same conclusion as in 2.5 with the conditions "$f_i g_s = f_i$" omitted from the hypotheses. For infinite sums, of course, we can do everything above provided the bicategory has infinite free sums and direct products. As for the sharpened statement for finite sums, the main point is that it is not used [9, 10, 11, 12]. It occurs in [9] (in the correction) probably because it holds for groups; one can see by analyzing the proof of 2.5 that the sharpened statement holds, more generally, when $\sigma = \times$.

Thus the Kuroš-Livšic decomposition theory applies at least in bicategories with zeros, free sums and direct products in which every pair of objects is left and right distributive. It is a theory of additive decompositions of identity mappings $1: A \to A$, or more generally of idempotent mappings $e: A \to A$. Kuroš and Livšic call it a theory of direct decompositions; but it is not that, if only because it is a self-dual theory (the non-self-dual refinement in Axiom I going unused). It is a theory of decompositions of distinguished weak products $A =$

$A_1\sigma \cdots \sigma A_n$. Precisely, an *additive decomposition* of $1: A \to A$ is defined as a representation $1 = \Sigma f_i$, where $f_i f_i = f_i$ and $f_i f_j = 0$ for $i \neq j$.

2.6. *Additive decompositions $\Sigma f_i = 1: A \to A$ correspond precisely to distinguished weak decompositions $A = f_1(A)\sigma \cdots \sigma f_n(A)$.*

The proof (omitted) does not require naturality.

3. **Abelianizing.** The addition of 1 and 1 (as noted in [6]) is fraught with significance. In fact, to add $1 + 1$ on A we need two mappings: $w: A \to A\sigma A$, dividing the diagonal $\Delta: A \to A \times A$ ($iw = \Delta$, where i is the distinguished injective factor of $\Pi_{(A,A)}$), and $k: A\sigma A \to A$ dividing the codiagonal. We call A *right abelian* if merely k exists, *left abelian* if w exists, *two-sided abelian* if both exist.

3.1. *If A is right abelian then for every n there is a mapping $k_n: A\sigma \cdots \sigma A \to A$ dividing the codiagonal ∇_n. Every finite family of mappings from a left abelian to a right abelian object has a sum (commutative, associative and distributive).*

Proof. Define k_3 as $k(k\sigma 1): (A\sigma A)\sigma A \to A\sigma A \to A$. One checks readily that this has the required property, and concludes by induction, $k_{n+1} = k(k_n\sigma 1)$. With the dually defined mappings w_n for left abelian objects, one adds $f_1, \cdots, f_n$ on left abelian C to right abelian D by factoring $f_1\sigma \cdots \sigma f_n$ as, say, rs, and checking $\Sigma f_i = (k_n r)(s w_n)$. The operation is commutative and associative by previous remarks and distributive by 2.2 and its dual.

3.2. *Every direct product of right abelian objects is right abelian; every subobject of a right abelian object is right abelian. This is not generally true for free sums; however, in the full subcategory of all right abelian objects, σ becomes a free sum functor.*

If $A\sigma A$ is already $A \vee A$ then A is right abelian.

Proof. Let P be a direct product of objects A_α with coordinate projections p_α; let each A_α be right abelian with $k_\alpha: A_\alpha\sigma A_\alpha \to A_\alpha$ dividing ∇_α. Map $P\sigma P$ to the product P by the mapping k whose αth coordinate is $k_\alpha(p_\alpha\sigma p_\alpha)$. We omit the check.

If $h: H \to A$ is an injection into a right abelian object, with $k: A\sigma A \to A$, we map $H\sigma H$ to A by $k(h\sigma h)$. Introduce the projective factor of $\Pi_{(H,H)}$, $z: H \vee H \to H\sigma H$. Evidently

$$k(h\sigma h)z = (1_A\ 1_A)(h \vee h) = h(1_H\ 1_H) .$$

The last factorization is distinguished; so $k(h\sigma h)$ is divisible by the required mapping from $H\sigma H$ to H.

Right abelian objects are not closed under free sum, for example, in the category of groups. However, whether A and B are right abelian or not, every mapping from $A \vee B$ to a right abelian object C factors uniquely over $A\sigma B$. For this, write the mapping as $(f\, g)$ and introduce $f\sigma g$; there is no difficulty. The assumption that A and B are right abelian serves (by the preceding) to make $A\sigma B$ right abelian.

The last assertion of 3.2 is trivial; $k = 1$.

Under additional conditions on the bicategory, we can associate to every object a "nearest" right abelian quotient and left abelian subobject. The conditions are that (1) every family of objects has free sums and direct products, and that (2) each object has only a set (not a proper class) of subobjects and a set of quotients. We call a bicategory which satisfies (1) a *complete* bicategory; we call a bicategory satisfying (2) *well-founded.*

The use of "complete" requires some explanation. Two slightly different definitions of completeness for categories have been advanced by Freyd [3] and me [8]. Each requires more than products. Freyd's definition is shown in [8] to require, besides products, the following condition: for each object X, for any set of ordered pairs (f_α, g_α), with $f_\alpha: X \to Y_\alpha$, $g_\alpha: X \to Y_\alpha$, there exists a monomorphism $m: M \to X$ such that the class of all mappings e into X satisfying $f_\alpha e = g_\alpha e$ for all α is the class of right multiples of m and (of course) the dual condition. My definition extends the conditions to classes of pairs (f_α, g_α).

By straightforward arguments one can establish several relations among these concepts; one gets a fuller picture if one introduces suitable categorical notions of well-foundedness. Here it will suffice to note the following.

3.3. *Every complete well-founded bicategory is a complete category in the strong sense of* [8]. *Moreover, the subobjects of any object form a complete lattice in the natural partial ordering.*

Now an idempotent functor R from a category $\mathscr{C}$ to itself, retracting $\mathscr{C}$ upon a subcategory $\mathscr{R}$, is called a *reflector* if for every object X there exists a mapping $r: X \to R(X)$ such that every mapping $f: X \to S$, S in $\mathscr{R}$, can be expressed in a unique way as gr where $g: R(X) \to S$ is in $\mathscr{R}$. The mappings r are called *reflection mappings.* A subcategory $\mathscr{R}$ is *reflective* if there exists a reflection upon $\mathscr{R}$. The dual concepts are *coreflector, coreflection, coreflective.* (Reflectors and coreflectors are left and right adjoints of embeddings.) Freyd has a characterization [3] of reflective subcategories of categories which are, in a suitable sense, complete and wellfounded. The following is not a special case because we are using a weaker notion of well-foundedness; but except for quibbles, it is a result of Freyd.

3.4. (Freyd's Theorem) *A full subcategory $\mathscr{R}$ of a complete well-founded bicategory is reflective if it is closed under formation of direct products and subobjects. The reflection mappings associated with any reflector upon $\mathscr{R}$ are projections.*

Proof. To construct $r: X \to R(X)$, take a set of projections $p_\alpha: X \to Y_\alpha$ representing all quotients of X in $\mathscr{R}$. Form the mapping p of X into the direct product of the Y_α which has coordinates p_α, and let ir be its distinguished factorization. This r at any rate is a projection. It is clear that every mapping from X to an object of $\mathscr{R}$ is a left multiple of some p_α, hence of r; and the representation is unique since r is a projection. Then the definition of R is completed as follows. For a mapping $f: X \to Z$ having $r: X \to R(X)$ and $r': Z \to R(Z)$, $R(f)$ is the mapping solving $R(f)r = r'f$. One readily verifies (cf. [3]) that R is a functor. Finally, the properties of reflection mappings assure that any two choices of them (for the same or different reflectors) are related by isomorphisms.

Note also that any two reflectors upon the same subcategory are naturally equivalent. By 3.2 and 3.4, there is a reflector R upon the right abelian objects. Dually, there is a coreflector L upon the left abelian objects. R and L need not commute (even up to natural equivalence). However, since the reflection mappings are projections, R takes the image of L into itself; and dually. Therefore:

3.5. THEOREM. *In a complete well-founded bicategory with zero mappings there exist (unique up to natural equivalence) a reflector R upon the right abelian objects and a coreflector L upon the left abelian objects. Both RL and LR are idempotent functors retracting the category upon the subcategory of two-sided abelian objects.*

We note that a third retraction upon the two-sided abelian objects can be constructed by means of factorizations of the mappings $rl: L(X) \to X \to R(X)$. Note also that the subcategories of right, left, or two-sided abelian objects are again complete well-founded bicategories.

If we started with a full category of algebras, these subcategories are again full categories of algebras. But the maclanian subcategory of two-sided abelian objects is not likely to be a maclanian category of algebras as defined just before 1.8; the categorical product in it is not the Cartesian product, but the distinguished weak product σ. So we cannot say that it is a full category of semigroups with operators, until we establish and apply a suitable generalization of Mac Lane's Theorem 1.7.

We need the notion of a *left adequate* set S of objects in a category [7]. Let $\mathscr{S}$ be the full subcategory on the objects of S. Each object X of the whole category $\mathscr{C}$ determines a contravariant functor Map $(\mathscr{S}, X)$ on $\mathscr{S}$ by $\mathrm{Map}(\mathscr{S}, X)(A) = \mathrm{Map}(A. X)$, $\mathrm{Map}(\mathscr{S}, X)(f)(g) = gf$. Now $\mathrm{Map}(\mathscr{S}, X)$ is an object of the category $\mathscr{M}$ of all contravariant functors from $\mathscr{S}$ to the category of sets; and $X \to \mathrm{Map}(\mathscr{S}, X)$ determines a covariant functor $V: \mathscr{C} \to \mathscr{M}$. The condition for S to be left adequate is that V embeds $\mathscr{C}$ as a full subcategory of $\mathscr{M}$.

The leading example of a left adequate set in a large category is the set of all free algebras on finite numbers of generators in any quasi-primitive category of algebras [8].

3.6. *Every maclanian category having a left adequate set of objects is isomorphic with a full category of commutative semigroups with operators, under an isomorphism preserving direct products.*

The proof is a routine modification of the proof of 1.7. Let $\mathscr{C}$ be the category and S a left adequate set of objects. The semigroup A' associated to an object A will be the weak direct sum of all Map (X, A) for $X \in S$. The semiring of operators is the weak direct sum of all $\mathrm{Map}(X, Y)$ for X and Y in S. The rest is defined as before. We omit the verification.

Half of the desired application is immediate. One may see easily that *reflectors* $R: \mathscr{C} \to \mathscr{R}$ *take left adequate sets in* $\mathscr{C}$ *to left adequate sets in* $\mathscr{R}$. One may see as easily that this is not true for coreflectors, considering the coreflector from abelian groups to their torsion subgroups. We fall back on the following lemma.

3.7. *Let* $\mathscr{C}$ *be a full category of algebras and* S *a set of algebras in* $\mathscr{C}$ *such that every* X *in* $\mathscr{C}$ *is a union of subalgebras isomorphic with members of* S*, every two such subalgebras of any* X *in* $\mathscr{C}$ *are contained in a single subalgebra of* X *isomorphic with a member of* S*, and every homomorphic image in* $\mathscr{C}$ *of a member of* S *is isomorphic with a member of* S*. Then* S *is left adequate in* $\mathscr{C}$.

This is so stated as to make the proof as trivial as possible. It should be noted that the noncategorical notion "union" is essential. If we try using direct limits, we run afoul of the dual of the category of all sets (which is a full category of algebras, unless measurable cardinals exist [8]); dualizing back, every set is an inverse limit of finite quotient sets, any two such quotients are common quotients of a third, and subsets of finite sets are finite, but finite sets are not right adequate.

For the proof of 3.7, we must show that, associating to algebras X the functors $\mathrm{Map}(\mathcal{S}, X)$ they induce on the full subcategory with the set of objects S, the obvious function from mappings $f: X \to Y$ to natural transformations $\Phi: \mathrm{Map}(\mathcal{S}, X) \to \mathrm{Map}(\mathcal{S}, Y)$ is one-to-one and onto. If $f: X \to Y$ and $g: X \to Y$ are unequal, they differ on some element of X and hence have different compositions with some embedding from $Z \in S$ to X; this shows that our function is one-to-one. Next suppose Φ given. To define $f: X \to Y$ on an element x of X, choose an embedding $e: Z \to X$ $(Z \in S)$ with $x = e(z) \in e(Z)$, and put $f(x) = \Phi_Z(e)(z)$. To see that another embedding $e': Z' \to X$ would give the same result, use a subalgebra containing $e(Z) \cup e'(Z')$. Similarly (using finite unions; since algebraic operations are finitary) we see that f is a homomorphism. The natural transformation corresponding to f is Φ, since every mapping $d: U \to X$ $(U \in S)$ factors across the embedding of $d(U) \in S$ into X. This completes the proof.

3.7 applies if we started with a quasi-primitive category of algebras, made into a bicategory $\mathcal{B}$ by the usual factorization, projections being homomorphisms onto and injections one-to-one homomorphisms. The condition for A to be left abelian is that every diagonal element (a, a) of $A \times A$ is generated (finitely) by $A \times 0$ and $0 \times A$. This requires some $a_1, \cdots, a_n$ in A for the generating. In case there are an uncountable number $\mathfrak{m}$ of algebraic operations, we may need an $\mathfrak{m}$-element subalgebra to include all we want; but clearly every left abelian algebra is a union of left abelian subalgebras of at most $\mathfrak{m}$ elements. By 3.7, there is a left adequate set of $2^{\mathfrak{m}}$ or fewer left abelian objects.

For summarizing the result, we may reasonably define a *bicategory of algebras* as a category of algebras and homomorphisms made into a bicategory with the usual factorization into one-to-one and onto factors.

3.8. *In a quasi-primitive bicategory of algebras with zero, the subcategories of left abelian and two-sided abelian objects, as well as the whole category and the subcategory of right abelian objects, have left adequate sets. Thus the two-sided abelian objects can be represented by commutative semigroups.*

4. **Examples.** We call a bicategory *left maclanian* if every object is left abelian; by the dual of 3.2, this means exactly that σ is the direct product. We have

4.1. THEOREM. *In a left maclanian quasi-primitive bicategory of algebras with zero, addition is natural.*

But

4.2. EXAMPLE. *There is a right maclanian quasi-primitive bicategory of algebras with zero in which addition is not natural.*

In this connection note that, by 3.2, the right abelian objects in a quasi-primitive bicategory of algebras themselves form a quasi-primitive bicategory of algebras; but this is not true for the left abelian objects. So we have not established naturality either in $\mathscr{R}$ or in $\mathscr{L}$ in this generality.

Proof of 4.1. In any left maclanian bicategory we have the right distributive law. Consider $h: H \to C \times D$ (which is $C\sigma D$). The mapping $(r_1h\ r_2h)$ is $(p_1h \times p_2h)\pi_{(H,H)}$. The diagonal $\Delta: H \to H \times H$ gives us $(p_1h \times p_2h)\Delta = h$ (check coordinatewise). Since $\pi_{(H,H)}$ is a projection, the distinguished factorization of $p_1h \times p_2h$ will show us h factored through $(r_1h\ r_2h)\ (H \vee H)$.

Next consider a free algebra F on one generator x. Since $\pi_{(F,F)}$ is onto $F \times F$, the element (x, x) is algebraically generated by $F \times 0$ and $0 \times F$. This means $(x, x) = W((x, 0), (0, x))$ for some algebraic operation W. It follows that in every algebra $A \times A$ in the category, $(a, b) = W((a, 0), (0, b))$ identically; for there are homomorphisms f, g of F to A taking x to a and to b, and $f \times g$ takes $W((x, 0), (0, x))$ to $W((a, 0), (0, b))$.

For left distributivity we must consider projections $q: C \times D \to Q$; each must factor across the projection upon $qj_1(C) \times qj_2(D)$. That is, for those congruence relations α on $C \times D$ which determine a quotient algebra in the category, inducing congruence relations α_1 on C, α_2 on D, we need the following inference: if $(c, c') \in \alpha_1$ and $(d, d') \in \alpha_2$, then $((c, d), (c', d')) \in \alpha$. This is a sound inference, since $W((c, 0), (0, d))$ and $W((c', 0), (0, d'))$ must be congruent.

Construction of 4.2. The algebras have (besides the necessary 0-ary operation) just one binary operation, which we write as juxtaposition though it is not associative. Beginning with the primitive category of algebras defined by $00 = 0$ and $xy = yx$, we pass to the subcategory of right abelian objects (with respect to the usual bicategorical structure); by 3.2, this is a quasiprimitive bicategory of algebras. In it we note some algebras A, B, and $H = A \times A$. A is the zero semigroup $\{0, a\}$ with $xy = 0$ identically. B contains A, and has another element b, with $b0 = ba = bb = a$. In $A \times A$, $\{(0, 0), (0, a), (a, 0)\}$ is a subalgebra; so it is $A\sigma A$, and A is indeed right abelian. ($k: A\sigma A \to A$ takes $(0, 0)$ to 0 and the other elements to a.) $B\sigma B$ consists of $B \times 0$, $0 \times B$, and the element (a, a); B is right abelian, with $k: B\sigma B \to B$

defined in the obvious way. (In particular, $k(a, a) = a$.) The notation gives us the embedding $h: H \to B\sigma B$. But $r_1 h$ and $r_2 h$ have images $A \times 0$ and $0 \times A$; the image of $(r_1 h\ r_2 h)$ is only $A\sigma A$, and h does not factor through it.

4.3. *In every right maclanian bicategory, addition as we have defined it can be extended to a natural operation.*

This result we stated, in different language, in [6]. Livšic has since given a proof [12].

Extension to a natural operation is not possible in general. In particular, where I claimed to have defined a natural operation on the last page of [6], no natural operation exists satisfying the normalization condition 2.6. The example:

4.4. EXAMPLE. Consider the primitive category of algebras introduced in 4.2. Besides B there is another right abelian algebra B' containing A, $B' = \{0, a, b'\}$, with $b'0 = b'a = a$, $b'b' = 0$. Form the amalgamated product C (which is not right abelian) of B and B' along A, i.e. the quotient of $B \vee B'$ by the smallest congruence relation making $a \in B$ congruent to $a \in B'$. The notation indicates a unique embedding $i: A \to C$. The two mappings $(i\ 0)^\times$, $(0\ i)^\times$ of A into $C \times C$ both factor through $C\sigma C$, giving us mappings $i_1: A \to C\sigma C$, $i_2: A \to C\sigma C$.

Assuming 2.6 and naturality, one can compute from $1 = r_1 + r_2$ on $B\sigma B$ to $i_1 + i_2$, which turns out to have the same values (in $C\sigma C \subset C \times C$) as $(i\ i)^\times$. But a similar computation using $B'\sigma B'$ gives $i_1 + i_2 = 0$.

To conclude, I know almost nothing about the change in $+$ when one changes the bicategorical structure on a fixed category. Many artificial algebraic examples and natural topological examples have more than one bicategorical structure. There is at least one interesting example, the category of all Banach spaces and all linear operators of norm at most 1. It is easy to see that there are exactly two bicategorical structures, one being left maclanian and the other right maclanian. In either case abelianizing is trivial, 0 being the only two-sided abelian object. The decomposition theory is not trivial, but the uniqueness theorems are stronger than the Remak or Krull-Schmidt theorems (as I shall show in a forthcoming paper), and the present approach seems entirely inadequate for these bicategories.

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