ON LOCAL PROPERTIES AND G_δ SETS

EDWARD E. GRACE

1. Introduction. For complete metric spaces, R. H. Bing has shown [2, Theorem 2] that being not connected im kleinen at each point of a dense G_δ set implies the existence of an open set that contains uncountably many components. The proof, in fact, shows that there is an open set no component of which contains an open set. For Baire topological continua, the author has shown [4, Lemma 2] that being not connected im kleinen, at each point of a dense-domain intersection set, implies the existence of an open set no component of which contains an open set.

For a certain class of II_ϕ spaces [3, p. 642], including complete metric spaces and Baire topological spaces, there is a general theorem about local properties which has both of these theorems as corollaries. For simplicity, the theory is presented here only as it applies to the special case of complete metric spaces. Generalization to II_ϕ spaces, which have "developments" [1, p. 180] consisting of ϕ (perhaps not $\aleph_0$) collections of open sets, is rather straightforward. The pattern of this generalization is indicated to some extent by [4].

This theory applies to those local properties which the space has at a point x if and only if x is a distinguished point in the following sense. There is a relation "is a *distinguished subset* of" having the following two properties. (1) D' is a distinguished subset of D only if D' is open and is a subset of D. (2) If D contains D' and D'' is a distinguished subset of D', then D'' is a distinguished subset of D. A point x is said to be a *distinguished point* if each open set containing x contains a distinguished subset containing x.

Several of the corollaries given here have been known for some time.

2. Theorems. The proof of Theorem 1 is complicated slightly in order that it will be clearer how it generalizes to apply to the general class of spaces referred to above.

THEOREM 1. *If M is a complete metric space each open set of which contains a distinguished open subset, then the set of distinguished points of M is a dense G_δ set.*

Received July 9, 1962, and in revised form September 26, 1963. Presented to the American Mathematical Society, November 18, 1961, under the title Local properties and Z category. This research was done while the author was on leave of absence from Emory University as an NSF Science Faculty Fellow at the University of Wisconsin.

Proof. For each $t < \aleph_0$, let U_t be the collection of open sets of diameter less than $1/t$ and let V_t be the union of all distinguished open subsets of members of U_t. The set V_t is dense and open since $U_t^* = \{x \mid x \in N \in U_t\}$ is dense and each open set contained in a member of U_t contains a distinguished open subset of that member. Let x be a point of the dense G_δ set $J = \bigcap_{t<\aleph_0} V_t$ and let D be an open set containing x. Then for some $t < \aleph_0$, the open set D contains $U_t^*(x)$. However, $x \in V_t$ and consequently x is contained in a distinguished open subset of some member N of U_t (containing x). But any distinguished open subset of N is a distinguished open subset of D, since $D \supset N$. It follows that J is a subset of the desired G_δ set. Clearly J contains each of the desired points. Hence, J is the desired dense G_δ set.

COROLLARY 1.1. *If each open set in a complete metric space M has a component with a nonvoid interior, then M is connected im kleinen at each point of some dense G_δ set.*

Proof. Let D' be a distinguished subset of D if D' is an open subset of a component of D.

COROLLARY 1.2. *If f is a function (not necessarily continuous) on a complete metric space and each open set D contains an open set D' such that $D \supset f(D')$, then $f(x) = x$ for each x in some dense G_δ set.*

Proof. Let D' be a distinguished subset of D if D' is an open subset of D and $D \supset f(D')$.

COROLLARY 1.3. *If M is a connected, compact metric space and each open set D contains a point p such that M is aposyndetic* [5, p. 138] *at p with respect to each point of $M - D$, then M is aposyndetic at each point of some dense G_δ set.*

Proof. Let D' be a distinguished subset of D if D' is open and D contains the intersection of a finite number of continua, each of which contains D'.

COROLLARY 1.4. *If M is a connected, complete metric space and the complement of each open set is contained in the union of a finite number of continua the union of which is not M, then M is semi-locally-connected* [7, p. 19] *at each point of a dense G_δ set.*

Proof. Let D' be a distinguished subset of D if $D \supset D'$ and the complement of D' is the union of a finite number of continua.

Theorem 2 follows from (part of) the proof of Theorem 1.

THEOREM 2. *In a metric space, the set of distinguished points is a G_δ set.*

COROLLARY 2.1. *The set of points at which a metric space is connected im kleinen is a G_δ set.*

There are also corollaries to Theorem 2 corresponding of Corollaries 1.2, 1.3 and 1.4.

THEOREM 3. *In a complete metric space, if some dense G_δ set contains no distinguished point then there is an open set which contains no distinguished subset.*

Proof. Assume the conclusion is false. Then by Theorem 1 the set of distinguished points is a dense G_δ set. But any two dense G_δ subsets of a complete metric space have a nonvoid intersection. This contradicts the hypothesis.

COROLLARY 3.1. *If a complete metric space is nonconnected im kleinen at each point of a dense G_δ set then it contains an open set no component of which contains an open set (and which, therefore, has uncountably many components).*

There are also corollaries to Theorem 3 corresponding to Corollaries 1.2, 1.3 and 1.4.

Theorem 4 is proved by repeated application of Theorem 3 to open subspaces.

THEOREM 4. *In a complete metric space, if (1) some dense G_δ set contains no distinguished point and (2) the union of any collection of disjoint open sets contains a distinguished subset only if one of the members of the collection contains a distinguished subset, then some dense open subset contains no distinguished subset.*

COROLLARY 4.1. *If a complete metric space is nonconnected im kleinen at each point of a dense G_δ set then there is a dense open set no component of which contains an open set.*

COROLLARY 4.2. *Let M be a complete, metrizable, open subspace of a connected topological space T such that, at each point of some* *dense G_δ subset of M, the space T fails to be semi-locally-connected. Then there is an open set D, dense in M, such that the complement of each open subset of D has infinitely many components.*

Proof. Let V be called a distinguished subset of an open subset W of M, if V is an open subset of W and the complement of V, in T, consists of a finite number of components. Here the union of a collection of disjoint open subsets of M is a distinguished subset of an open subset U of M containing it only if *each* member of the collection is a distinguished subset of U. Hence the corollary follows from Theorem 4.

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ARIZONA STATE UNIVERSITY