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COLLINEATION GROUPS OF SEMI-TRANSLATION PLANES

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This paper consists in an investigation of the collineations of a class of planes constructed by the author. The construction consists of replacing the lines of a net embedded in a given plane by subplanes of the same net.

For the case in question, the given plane is the dual of a translation plane. The full collineation group of the new plane is isomorphic to a subgroup of the collineation group of the original plane. The main point of the argument is to show that the new planes admit no collineations displacing the line at infinity.

I. In [2], the author introduced a new class of affine planes. These new planes were obtained by a construction which consists of starting with a plane which is the dual of a translation plane and modifying some of the lines. By the very process of construction, a part of the collineation group of the original plane is carried over to the new plane.

However, the full collineation group for these new planes has not been previously determined; in particular, it has not been known whether there are any collineations displacing the line at infinity. In this paper, we show that (with mild restrictions on the nature of the original plane) the full collineation group on each new plane is precisely the group "inherited" from the original plane.

II. **Preliminary definitions and summary of previous results.** We shall be using Hall's ternary [4] and certain slight modifications of the ternary as coordinate systems for planes. The point at infinity on the line $y = xm$ will be denoted by (m); the point at infinity on $x = 0$ will be denoted by (∞).

In any case where the coordinate system contains a subfield $\mathfrak{F}$ it should be understood that small Greek letters (with the exception of ρ and σ) denote elements of $\mathfrak{F}$.

For any affine plane Π and any set $\mathfrak{S}$ of parallel classes, the system consisting of the points of Π and lines belonging to the parallel classes in $\mathfrak{S}$ will be called a net N embedded in Π. If (m) is the point at infinity corresponding to some parallel class in N, we shall

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find it convenient to speak of (m) as "belonging to N."
A quasifield (Veblen-Wedderburn system) will be said to be a left quasifield if the left distributive law, $a(b+c) = ab + ac$, holds.

Let $\mathfrak{T}$ be a coordinate system with associative and commutative addition. If $\mathfrak{T}$ contains a subfield $\mathfrak{F}$ such that

$$(1) \qquad a(\alpha + \beta) = a\alpha + a\beta$$

$$(2) \qquad (a\alpha)\beta = a(\alpha\beta)$$

$$(3) \qquad (a+b)\alpha = a\alpha + b\alpha \ ,$$

for all a, b in $\mathfrak{T}$ and all α, β in $\mathfrak{F}$, we shall say that $\mathfrak{T}$ is a right vector space over $\mathfrak{F}$.

If lines whose slopes are in $\mathfrak{F}$ can be represented by equations of the type $y = x\alpha + b$, we shall say that $\mathfrak{T}$ is linear with respect to $\mathfrak{F}$.

Now let $\mathfrak{T}$ be a left quasifield of order q^2 $(q > 4)$. Suppose that $\mathfrak{T}$ is a right vector space over a subfield $\mathfrak{F}$ of order q. Let Π be the affine plane coordinatised in the usual sense by $\mathfrak{T}$. (Note: The line of slope m through the origin is written $y = xm$ rather than with m on the left.)

We can then define another plane $\bar{\Pi}$ [2] whose points are identical with the points of Π. The lines of $\bar{\Pi}$ are of two kinds:

(1) Lines of Π which have finite slopes not in $\mathfrak{F}$

(2) Sets of points (x, y) such that $x = a\alpha + c$, $y = a\beta + d$,

where $a \neq 0$, c, d are fixed elements of $\mathfrak{T}$ while α and β vary over $\mathfrak{F}$.

Now the lines of type (2) may be identified with subplanes (of order q) of Π. If a permutation σ on the points of Π induces a collineation of either Π or $\bar{\Pi}$ which carries lines of type (1) into lines of type (1), then σ induces collineations of both planes. If σ is a translation (elation with axis L_∞) of either plane, then σ is a translation of both planes [3].

Now let t be a fixed element of $\mathfrak{T}$ which is not in $\mathfrak{F}$. Each element of $\mathfrak{T}$ can be written uniquely in the form $t\alpha + \beta$. The lines of $\bar{\Pi}$ can be written in a more convenient form if each point is assigned new coordinates as follows:

$$\text{If} \quad (x, y) = (t\xi_1 + \xi_2, t\eta_1 + \eta_2), \text{ let}$$
$$(\bar{x}, \bar{y}) = (t\xi_1 + \eta_1, t\xi_2 + \eta_2).$$

Define a new operation $*$ such that

$$(t\xi_1 + \eta_1)*(t\lambda_1 + \lambda_2) = t\xi_2 + \eta_2 \qquad \text{is equivalent to}$$
$$(t\xi_1 + \xi_2)(t\mu_1 + \mu_2) = t\eta_1 + \eta_2; \ (t\xi_1 + \xi_2)*\lambda_2 = t\xi_1\lambda_2 + \eta_1\lambda_2 \ ,$$

where $\lambda_1 \neq 0$ and $\lambda_1(t\mu_1 + \mu_2) = t + \lambda_2$.
See reference [3].

Then the lines of $\bar{\Pi}$ can be represented by equations of the following forms:

Type (1): $\bar{y} = (\bar{x} - \alpha)*m + \beta, m \notin \mathfrak{F}$

Type (2): $\bar{y} = \bar{x}\delta + b$ or $x = c$.

Let Π_0 denote the affine subplane of Π which is coordinatised by $\mathfrak{F}$ in $\mathfrak{T}$; let $\bar{\Pi}_0$ be the affine subplane of $\bar{\Pi}$ which is coordinatised by $\mathfrak{F}$ in $\bar{T}$. Then Π_0 is the set of points for which $\bar{x} = 0$; $\bar{\Pi}_0$ is the set of points for which $x = 0$.

The plane Π admits all translations of the form $(x, y) \to (x, y + b)$. The points of $\bar{\Pi}_0\,(x = 0)$ are in a single transitive class under this group of translations—which also acts as a group of translations on $\bar{\Pi}$. There will be further translations if and only if there is some element c such that $(x + c)m = xm + cm$ for all x and all m. If there are no further translations, $\bar{\Pi}$ is what we call a strict semi-translation plane; we shall say that T is a strict left quasifield.

III. The collineation group. It is well known that a net can be coordinatised in much the same fashion as a plane. If the net is embedded in a plane, a coordinate system for the plane induces a coordinate system for the net, provided the lines $x = 0$, $y = 0$ and $y = x$ all belong to the net. Conversely, any coordinate system for the net can be extended to form a coordinate system for the whole plane.

LEMMA 1. *Let N be a net with $q + 1$ parallel classes. Let N be coordinatised by a system $\mathfrak{C}$, let F be the subset of $\mathfrak{C}$ such that $x\alpha$ is defined for all x in $\mathfrak{C}$, all α in $\mathfrak{F}$. Suppose that*

(1) *Addition in $\mathfrak{C}$ is associative.*

(2) *F is a field of order q with respect to addition and multiplication in $\mathfrak{C}$.*

(3) *The additive group in $\mathfrak{C}$ is a right vector space over F.*

(4) *$\mathfrak{C}$ is linear.*

Then N can be embedded in a Desarguesian plane.

Proof. The additive group is isomorphic to the additive group of a field $\mathfrak{K}$ which is a quadratic extension of $\mathfrak{F}$. For instance, if q is odd, multiplication in $\mathfrak{K}$ may be defined as follows

$$(t\xi_1 + \xi_2) \circ (t\lambda_1 + \lambda_2) = t(\xi_1\lambda_2 + \xi_2\lambda_1) + (\delta\xi_1\lambda_1 + \xi_2\lambda_2)\,,$$

where δ is a fixed nonsquare element of $\mathfrak{F}$ and t is a fixed element not in $\mathfrak{F}$. Then the net N will be embedded in the Desarguesian plane coordinatised by $\mathfrak{K}$.

LEMMA 2. *Let $\mathfrak{T}$ be a left quasifield coordinatising a plane Π of order q^2. Suppose that* (1) *$\mathfrak{T}$ is a right vector space over a subfield $\mathfrak{F}$ of order q and* (2) *$\mathfrak{T}$ is linear with respect to $\mathfrak{F}$. Let $\mathfrak{T}'$ be any other coordinate system for Π subject to the following condition* (a). *The point* (∞) *is the same for both $\mathfrak{T}$ and $\mathfrak{T}'$,* (b) *$\mathfrak{T}'$ is an extension of a coordinate system for the net N consisting of those parallel classes whose slopes in $\mathfrak{T}$ are infinite or belong to $\mathfrak{F}$.*

Then $\mathfrak{T}'$ is also a left quasifield satisfying conditions (1) *and* (2).

Proof. The plane Π is a dual translation plane with special point (∞). This implies that $\mathfrak{T}'$ is a left quasifield.

It follows from Lemma 1 that any coordinate system for N must have properties (1) and (2). These properties will carry over to $\mathfrak{T}'$.

We now return to the construction discussed in part II. It is to be understood that $\mathfrak{T}$ is a left quasifield of order q^2 which is a right vector space over a subfield of order q, that T is linear with respect to $\mathfrak{F}$, and that $\bar{\Pi}$ is the new plane introduced in part II.

Since we shall ultimately be concerned with collineations which might displace the line at infinity, we shall want to deal with the projective version of $\bar{\Pi}$. We modify our previous notation so that (m) denotes the point at infinity on $y = x*m$.

THEOREM 1. *If $\mathfrak{T}$ is a strict left quasifield, then the affine collineations of $\bar{\Pi}$ are precisely those which it shares with Π.*

Proof. For each α in $\mathfrak{F}$, there are exactly q translations of $\bar{\Pi}$ with center (α). Likewise, there are q translations with center (∞). If $\mathfrak{T}$ is a strict left quasifield, so that Π and $\bar{\Pi}$ admit exactly q^2 translations, we have exhausted the translations in $\bar{\Pi}$.

This implies that no affine collineations of $\bar{\Pi}$ carry a line of type (1) into a line of type (2). Hence every affine collineation of $\bar{\Pi}$ is a collineation of Π.

LEMMA 3. *Suppose that $\bar{\Pi}$ admits a collineation which carries the line at infinity into some line L. Then, without loss of generality, we may take L to be $\bar{x} = 0$.*

Proof. By Lemma 2 of [3], L is some line of type 2, hence L consists of the set of points of an affine subplane of Π. By Lemma 2, we can choose a new coordinate system $\mathfrak{T}'$ for Π such that this subplane is coordinatised by a field of order q and $\mathfrak{T}'$ is a left quasifield satisfying (1) and (2) of Lemma 2. If $\mathfrak{T}$ is initially chosen in this way, L has the equation $\bar{x} = 0$. Since the basic construction consists of replacing lines by subplanes (see [3]), the change of coordinate system for Π does not alter the nature of $\bar{\Pi}$.

LEMMA 4. *If $\bar{\Pi}$ admits a collineation carrying L_∞ into $\bar{x} = 0$, multiplication in $\bar{T}$ takes the form*

$$\begin{aligned}(t\alpha_1 + \beta_1) * (t\alpha_2 + \beta_2) = t[&h(\alpha_1, \alpha_2) - \beta_1\alpha_2 + \alpha_1\beta_2] \\ &+ [\beta_1\alpha_1^{-1} h(\alpha_1, \alpha_2) + k(\alpha_1, \alpha_2) \\ &- \beta_1^2 \alpha_1^{-1} \alpha_2 + \beta_1 \beta_2] \ \alpha_1 \neq 0\end{aligned}$$

and

$$\beta_1 * (t\alpha_2 + \beta_2) = t\alpha_2 \beta_1 + \beta_1 \beta_2 + R(\beta_1, \alpha_2)$$

where h, k, and R are functions from $\mathfrak{F} \times \mathfrak{F}$ into $\mathfrak{F}$.

Proof. By Lemma 2 of [3], (∞) is the center of q elations with axis $\bar{x} = 0$. These collineations act on Π in such a way as to leave Π_0 pointwise fixed. Since $x = 0$ is fixed in Π, $\bar{\Pi}_0$ is fixed (not pointwise) in $\bar{\Pi}$. Thus we have a group of elations of $\bar{\Pi}$ which is transitive on the q points of $\bar{\Pi}_0 \cap L_\infty - (\infty)$.

There is a similar group of elations in Π which has center (∞), axis $x = 0$, and is transitive on the points at infinity of Π_0 (excluding the point at infinity of $x = 0$). These collineations carry over into $\bar{\Pi}$, appearing as collineations which leave $\bar{\Pi}_0$ pointwise fixed. The collineations leaving $\bar{\Pi}_0$ pointwise fixed impose automorphisms of $\bar{T}$ which fix each element of F. The elations of $\bar{\Pi}$ with center (∞) and axis $x = 0$ impose the "partial distributive law" $a*(b + \alpha) = ab + a\alpha$ $a, b \in \bar{\mathfrak{T}}$, $\alpha \in \mathfrak{F}$, on $\bar{T}$. Lemma 4 then follows from Theorem 2 and 3 of [1].

LEMMA 5. *Under the conditions of the previous Lemmas, $\bar{\mathfrak{T}}$ has the property that if $b * a = -1$, then $b * (a * m) = (-1) * m$ for all m in $\bar{\mathfrak{T}}$.*

Proof. The proof is essentially the same as the proof of Theorem 11 in [1].

LEMMA 6. *Under the conditions of the previous lemmas, there exist functions f and g such that $h(\alpha_1, \alpha_2) = f(\alpha_1)\alpha_2$, $k(\alpha_1, \alpha_2) = g(\alpha_1)\alpha_2$.*

Proof. Given $t\alpha_1$ $(\alpha_1 \neq 0)$, let $t\alpha_2 + \beta_2$ be determined so that $t\alpha_1 * (t\alpha_2 + \beta_2) = -1$. By Lemma 4, $h(\alpha_1, \alpha_2) + \alpha_1\beta_2 = 0$

$$k(\alpha_1, \alpha_2) = -1.$$

By Lemma 5, we have $t\alpha_1 * (t\alpha_2\gamma + \beta_2\gamma) = -\gamma$, which is equivalent to the pair of equations

$$h(\alpha_1, \alpha_2\gamma) + \alpha_1\beta_2\gamma = 0$$
$$k(\alpha_1, \alpha_2\gamma) = -\gamma .$$

Now, by Theorem 11 of [3], $\bar{T}$ is a right vector space over $\mathfrak{F}$. In particular, $(t\alpha_1 + \beta_1) * \beta_2 = t\alpha_1\beta_2 + \beta_1\beta_2$. From this, and our definition of α_2, we know that $\alpha_2 \neq 0$. We easily get $k(\alpha_1, \alpha_2\gamma) = k(\alpha_1, \alpha_2)\gamma$ for each nonzero α_1 and γ in F, where α_2 depends on α_1. Letting $\alpha_2\gamma = \alpha$, we get $k(\alpha_1, \alpha) = k(\alpha_1, \alpha_2)\alpha_2^{-1}\alpha = g(\alpha_1)\alpha$. Moreover, $k(\alpha_1, 0) = 0$. This establishes the part of our Lemma that pertains to k. A similar argument works for h.

THEOREM 2. *Under the hypotheses of Theorem 1 and the additional requirement that $q > 4$, $\bar{\Pi}$ admits no collineations displacing L_∞; the full collineation group of $\bar{\Pi}$ is the group of affine collineations which it shares with Π.*

Proof. The relations between the multiplications in T and $\bar{T}$ is reciprocal, i.e.

$$(t\xi_1 + \eta_1)(t\lambda_1 + \lambda_2) = t\xi_2 + \eta_2 \iff$$
$$(t\xi_1 + \xi_2) * (t\mu_1 + \mu_2) = t\eta_1 + \eta_2 \qquad \text{if } \lambda_1 \neq 0 ,$$

where
$$\lambda_1 * (t\mu_1 + \mu_2) = t + \lambda_2 .$$

Let us assume that $\bar{\Pi}$ does admit a collineation displacing L_∞. We shall show that we must have $q \leqq 4$. Now let $\lambda_2 = 0$, $\xi_1 \neq 0$, $\bar{\lambda}_1 \neq 0$. We have:

$$(t\xi_1 + \eta_1)(t\lambda_1) = t\xi_2 + \eta_2$$

is equivalent to

$$(t\xi_1 + \xi_2) * (t\lambda_1^{-1} - \lambda_1^{-1} R(\lambda_1, \lambda_1^{-1})) = t\eta_1 + \eta_2 ,$$

which is in turn equivalent to the pair of equations (by Lemmas 4 and 6)

$$\eta_1 = f(\xi_1)\lambda_1^{-1} - \xi_2\lambda_1^{-1} - \xi_1\lambda_1^{-1}R(\lambda_1, \lambda_1^{-1})$$
$$\eta_2 = \xi_2\xi_1^{-1} f(\xi_1)\lambda_1^{-1} + g(\xi_1)\lambda_1^{-1} - \xi_2^2\xi_1^{-1}\lambda_1^{-1} - \xi_2\lambda_1^{-1}R(\lambda_1, \lambda_1^{-1}).$$

Let $R(\lambda_1, \lambda_1^{-1}) = S(\lambda_1)$. Solving for ξ_2 and η_2, we get

$$(t\xi_1 + \eta_1)(t\lambda_1) = t[f(\xi_1) - \xi_1 S(\lambda_1) - \eta_1\lambda_1]$$
$$+ [g(\xi_1)\lambda_1^{-1} + f(\xi_1)\eta_1\xi_1^{-1} - \eta_1 S(\lambda_1) - \eta_2^2\xi_1^{-1}\lambda_1] .$$

By hypotheses, $\mathfrak{T}$ is a left quasifield which is a right vector space over $\mathfrak{F}$. Hence

$$(t\xi_1 + \eta_1)[t(\lambda + \mu)] = (t\xi_1 + \eta_1)(t\lambda) + (t\xi_1 + \eta_1)(t\mu) .$$

Carrying out the multiplications in the above equation and separating the components, we get the two equations

$$\begin{aligned} f(\xi_1) - \xi_1 S(\lambda + \mu) - \eta_1(\lambda + \mu) = [f(\xi_1) - \xi_1 S(\lambda) - \eta_1\lambda] \\ + [f(\xi_1) - \xi_1 S(\mu) - \eta_1\mu] , \end{aligned}$$

$$\begin{aligned} &g(\xi_1)(\lambda + \mu)^{-1} + f(\xi_1)\eta_1\xi_1^{-1} - \eta_1 S(\lambda + \mu) - \eta_1^2\xi_1^{-1}(\lambda + \mu) \\ &= [g(\xi_1)\lambda^{-1} + f(\xi_1)\eta_1\xi_1^{-1} - \eta_1 S(\lambda) - \eta_1^2\xi_1^{-1}\lambda] \\ &\quad + [g(\xi_1)\mu^{-1} + f(\xi_1)\eta_1\xi_1^{-1} - \eta_1 S(\mu) - \eta_1^2\xi_1^{-1}\mu]. \end{aligned}$$

Eliminating $f(\xi_1)$, we find that the terms involving S also drop out and we get

$$g(\xi_1)(\lambda + \mu)^{-1} = g(\xi_1)\lambda^{-1} + g(\xi_1)\mu^{-1}.$$

Now if $g(\xi_1) = 0$, then $(t\xi_1)(t\lambda) = t[f(\xi_1) - \xi_1 S(\lambda_1)]$. But the solution of any equation of the type $(t\xi)x = t\beta$ is $x = \xi^{-1}\beta$, which is in $\mathfrak{F}$.

Since $t\lambda \notin \mathfrak{F}$, we have a contradiction. We conclude that $g(\xi_1) \neq 0$. Hence we must have $(\lambda + \mu)^{-1} = \lambda^{-1} + \mu^{-1}$ for all λ, μ in $\mathfrak{F}$ except in the cases that λ, μ, or $\lambda + \mu$ is zero.

With $\mu = 1$, this equation is equivalent to

$$\lambda^2 + \lambda + 1 = 0,\ \lambda \neq 0, -1 .$$

Hence $\mathfrak{F}$ can contain at most 4 elements. Since we assumed $q > 4$, the theorem is proved.

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